SUPERSTRING σ-MODELS FROM BOSONIC ONES

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ABSTRACT

We consider the derivation of the background field heterotic σ-model starting from the corresponding bosonic σ-model compactified on the group manifold G[\(G\times E_8\times E_8\) or Spin32/\(\mathbb{Z}_2\)]. It is necessary not only to identify the SO(8) \(\subset G\) gauge potentials with the space-time SO(8) spin connections, but also to identify the Kaluza-Klein scalars with the \(G_L\) Yang-Mills field strengths. Similar remarks apply to Type II. This explains the origin of the four-fermion interactions on the worldsheet.

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It has been suggested that superstrings have their origin in the bosonic string\(^1\)–\(^5\). In fact, it has now been proved that the heterotic and Type II superstrings are modular invariant truncations of the bosonic string\(^6\). The aim of this paper is to derive the background-field Lagrangian for the heterotic string in ten dimensions with background fields \(g_{\mu\nu}, B_{\mu\nu}\) and \(A^{ij}_\mu\)

\[
L_{\text{heterotic}} = \partial_+ X^\mu \partial_- X^\nu \left( g_{\mu\nu} + B_{\mu\nu} \right) + i \lambda^\alpha \partial_+ \lambda^\beta + i \psi^I \partial_+ \psi^J + i \lambda^\alpha \left( \omega^{\alpha\beta}_\mu + \frac{1}{2} H^\beta_{\mu\nu} \right) \lambda^\beta \partial_+ X^\nu \\
+ i \psi^I \partial_+ \psi^J + i \psi^I A^{i j}_\mu \psi^J \partial_+ X^\mu + \frac{1}{2} F^{\alpha\beta ij}_\mu \lambda^\lambda \lambda^\beta \psi^I \psi^J
\]

(1)

starting from the Lagrangian for the bosonic string in D dimensions with background fields \(\hat{\mathbf{g}}_{\mu\nu}^M\) and \(\hat{\mathbf{B}}_{\mu\nu}^M\)

\[
L = \partial_+ X^M \partial_- X^N \left( \hat{g}_{MN} + \hat{B}_{MN} \right)
\]

(2)

We shall work in light-cone formalism so that \(\mu, \nu = 1, \ldots, 8\) and \(M, N = 1, \ldots, D-2\). In (1) \(I, J = 1, \ldots, 32\), and the background Yang-Mills potentials \(A_{\mu}^{ij}\) take values in the Lie algebras \(SO(16)\times SO(16)\) or \(SO(32)\), corresponding to the \(G = E_8\times E_8\) or \(G = \text{Spin}(32)/Z_2\) heterotic strings.

The derivation of (1) from (2) proceeds as follows. The theory described by Eq. (2) is known to be conformally invariant when \(\hat{g}_{MN}\) and \(\hat{B}_{MN}\) are chosen so that the Lagrangian reduces to that of a non-linear \(\sigma\)-model on \(\mathbb{H}^*\times G\) with Wess-Zumino term\(^7\), where \(\mathbb{H}^d\) is d-dimensional Minkowski space (\(d= D-\dim G\)) and \(G\) is a simply-laced non-Abelian group manifold with

\[
D - 26 = \dim G \cdot \frac{C_A}{\dim G - \text{rank } G} = \dim G \cdot \frac{C_A}{\dim G - \text{rank } G}
\]

(3)

where \(C_A\) is the second-order Casimir in the adjoint representation. In the language of Kaluza-Klein, the Einstein-matter field equations for the metric \(\hat{g}_{MN}\) and the antisymmetric tensor \(\hat{A}^M_{\mu
u}\) (obtained by required vanishing trace of the worldsheet stress tensor) admit a spontaneous compactification on \(G\) from \(D\) to \(d\) dimensions. The massless states in \(d\) dimensions are given by the metric \(\hat{g}_{\mu\nu}\), the antisymmetric tensor \(\hat{A}^M_{\mu
u}\), the Yang-Mills gauge potentials \(A^{ij}_\mu\) of \(G_L\times G_R\) and scalars \(S^{ij}_I\) in the \((\text{adjoint } G_L, \text{adjoint } G_R)\) representation of \(G_L\times G_R\). Here \(G_L\) and \(G_R\) correspond to the left and right actions of \(G\) on the group manifold \((i, i'=1, \ldots, \dim G)\).
In principle, we now substitute the Kaluza-Klein ansatz for the above massless states into (2), noting that the truncation of the massive Kaluza-Klein modes is, contrary to generic Kaluza-Klein theories 8), completely consistent 4). Unfortunately, although the ansatz for \( g_{\mu\nu} \), \( H_{\mu\nu} \), \( A_{\mu} \) and \( \tilde{A}^{i}_{\mu} \) is known exactly, the ansatz for \( S^{ij}_{\mu} \) is known only to linearized order 4) in the case where \( G \) is non-Abelian. To begin with, therefore, we shall see how far we can get by ignoring these Kaluza-Klein scalars and then return to the problem of including them. The ansatz for \( \hat{g}_{MN} \) is then 4)

\[
\hat{g}_{MN}(x,\gamma) = g_{MN}(x) + (A_{\mu}^{i}(x) L^{\mu i} + \tilde{A}_{\mu}^{i}(x) R^{\mu i})(A_{\nu}^{j}(x) L^{\nu j} + \tilde{A}_{\nu}^{j}(x) R^{\nu j})g_{MN}(\gamma)
\]

\[
\hat{g}_{\mu\nu}(x,\gamma) = (A_{\mu}^{i}(x) L^{\mu i} + \tilde{A}_{\mu}^{i}(x) R^{\mu i}) g_{\mu\nu}(\gamma)
\]

\[
\hat{g}_{\mu\nu}(x,\gamma) = g_{\mu\nu}(\gamma)
\]

and for \( \hat{B}_{MN} \):

\[
\hat{B}_{MN}(x,\gamma) = B_{MN}(x) + (A_{\mu}^{i}(x) L^{\mu i} \tilde{A}_{\nu}^{j}(x) R^{\nu j} - A_{\nu}^{i}(x) L^{\nu i} \tilde{A}_{\mu}^{j}(x) R^{\mu j}) g_{MN}(\gamma)
\]

\[
\hat{B}_{\mu\nu}(x,\gamma) = (A_{\mu}^{i}(x) L^{\mu i} - \tilde{A}_{\mu}^{i}(x) R^{\mu i}) g_{\mu\nu}(\gamma)
\]

\[
\hat{B}_{\mu\nu}(x,\gamma) = B_{\mu\nu}(\gamma)
\]

where \( g_{MN} \) is the \( G_L \times G_R \) invariant metric on \( G \), \( B_{MN} \) is given by

\[
H_{\mu\nu\rho} = 3 \partial_{[\mu} B_{\nu\rho]} = C_{\mu\nu\rho}
\]

where \( C_{\mu\nu\rho} \) is the \( G_L \times G_R \) invariant metric on \( G \). Here \( L_{\mu} \) and \( R_{\mu} \) are respectively the left and right Killing vectors on \( G \), satisfying \([L_{\mu}, L_{\nu}] = c_{\mu\nu}^{\rho} L_{\rho}, [R_{\mu}, R_{\nu}] = -c_{\mu\nu}^{\rho} R_{\rho}\). In terms of the corresponding one-forms, the ansatz (4) becomes

\[
\hat{g}_{MN} = 2 A_{M}^{A} B_{N}^{A} \gamma_{AB}
\]

In fact, we find it more convenient to work in a vielbein formulation, for which \( \hat{g}_{MN} = 2 A_{M}^{A} B_{N}^{A} \eta_{AB} \). In terms of the corresponding one-forms, the ansatz (4) becomes
\[ \hat{\epsilon}^\alpha(X, \gamma) = e^\alpha_\mu(X) dX^\mu \]
\[ \hat{\epsilon}^\alpha(X, \gamma) = e^\alpha_\mu(\gamma) dy^\mu - A^i_\mu(X) L^i_\alpha dX^\mu - \tilde{A}^i_\mu R^i_\alpha dX^\mu \] (7)

where \( \alpha \) and \( a \) are space-time and internal tangent space indices respectively. In vielbein formulation the bosonic equation of motion which follows from (2) is

\[ \partial_+ J_{-A} + \tilde{\omega}(+)_{ABC} J^B_+ J^C_+ = 0 \] (8)

where we have defined the "current"

\[ J_+^A = \hat{\epsilon}^A_m \partial_+ X^m \] (9)

and where \( \tilde{\omega}(\pm)_{ABC} \) is the spin connection with torsion \( \pm \hat{H}_{ABC} \)

\[ \tilde{\omega}(\pm)_{ABC} = \hat{\omega}_{ABC} \pm \frac{1}{2} \hat{H}_{ABC} \] (10)

the connection one-form being \( \hat{\omega}_{AB} = \hat{\omega}_{ABC} \hat{A}^C \). Substituting the above ansatz into (9) we find the space-time current

\[ J_+^\alpha = e^\alpha_\mu \partial_+ X^\mu \] (11)

and the internal current

\[ J_+^a = e^a_\mu \partial_+ \gamma^\mu - A^i_\mu L^i_\alpha \partial_+ X^\mu - \tilde{A}^i_\mu R^i_\alpha \partial_+ X^\mu \] (12)

The corresponding equations of motion from (8) are then

\[ \partial_+ J_\alpha + \omega^{(\gamma)}_{k\beta} J^{\beta}_{-} J_+^\gamma + F_{k\gamma}^{i} J^i_{-} J^\gamma_{+} + \tilde{F}_{k\beta}^{a} J_{-a} J^\beta_{+} J^\gamma_{+} = 0 \] (13)

and

\[ \partial_+ J_{-a} + \omega^{(\gamma)}_{abc} J^{b}_{-} J^{c}_{+} + \omega^{(\gamma)}_{a \beta} L_{c}^{\gamma} J_{-}^{b} J_{+}^{\gamma} + \tilde{A}_{i}^{a \beta} R_{c}^{i} J_{-}^{b} J_{+}^{\gamma} - F_{k\gamma}^{i} L_{a}^{\beta} J_{-}^{b} J_{+}^{\gamma} = 0 \] (14)
In terms of the \( G_L \) and \( G_R \) currents
\[
\begin{align*}
\mathbf{J}^i_- & \equiv J^i_- L^i_- \\
\mathbf{J}^i_+ & \equiv J^i_+ R^i_+ 
\end{align*}
\]  
we have
\[
\begin{align*}
D_+ J^i_- &= F^{i}_{\beta\gamma} J^\beta_- J^\gamma_+ \\
D_- J^i_+ &= \tilde{F}^{i}_{\beta\gamma} J^\beta_+ J^\gamma_- 
\end{align*}
\]  
where \( D_+ J^i_- = \delta_+ J^i_- + c^{-1}_j A^j_{\mu} \delta_+ x^{\mu j} J^i_- \) is the Yang-Mills covariant derivative, with a similar expression for \( D_- J^i_+ \).

The next step in the derivation of the superstring Lagrangian (1) is to specialize to the cases \( G = E_8 \times E_8 \) or \( \text{Spin}(32)/\mathbb{Z}_2 \) for which \( \dim G = 496 \), rank \( G = 16 \) and \( C_A = 60 \). Hence from (3)
\[
D = 506 \quad , \quad d = 10
\]
We then make the group decomposition
\[
G_R \rightarrow SO(8)
\]
with a regular embedding, keeping only the \( SO(8) \) potentials
\[
\mathbf{\tilde{A}}^i_\alpha \rightarrow \mathbf{\tilde{A}}^{\alpha \beta}_\mu , \quad \alpha, \beta = 1, \ldots, 8
\]
and identify these potentials with the space-time \( SO(8) \) spin connection with torsion
\[
\tilde{A}^{\alpha \beta}_\mu = \omega^{(\ast)}_{\mu \alpha \beta}
\]
The identification of spin-connections with gauge potentials \(^9\) is, as discussed in Ref. 4\), just the field-theoretic realization of the identification of the transverse space-time \( SO(8) \) of the superstring with the diagonal subgroup of the transverse space-time \( SO(8) \) of the compactified bosonic string and the internal \( SO(8) \subset G_R \). The equations of motion (13) and (16) now become
\[ \partial_+ J^- \alpha + \omega^{(x)}_{\alpha \beta \gamma} J^- \beta J^+ \gamma + \frac{i}{2} F^{IJ}_{\alpha \beta} J^- I J^+ J^+ \gamma + \frac{i}{2} R^{(n)}_{\alpha \beta} \psi J^+ J_- \alpha \beta = 0 \quad (21) \]

and

\[ D_+ J^- I J = F^{IJ}_{\beta \gamma} J^- \beta J^+ \gamma \]
\[ D_- J^+ \alpha \beta = R^{(n)}_{\gamma \delta} \psi \alpha \beta J^+ J^- \gamma \delta \quad (22) \]

where we have specialized to the case of an orthogonal group with adjoint indices \( IJ \). This accounts for the factors of \( \frac{1}{2} \) in (21). This covers the case of \( \text{SO}(32) \) and the \( \text{SO}(16) \times \text{SO}(16) \) subgroup of \( \text{E}_8 \times \text{E}_8 \), and allows the theory to be cast into a more familiar form using a non-Abelian fermionization of the currents \( J^+ \) and \( J^- \).

To see this, we now note that the equations of motion (21) and (22) are just equivalent to the equations of motion obtained by varying the \( \Lambda_{\text{heterotic}} \) of (1) (excluding the four-fermion term!) provided we make the identifications

\[ J^+ \alpha \beta = \propto \lambda^\alpha \lambda^\beta \]
\[ J^- I J = \propto \psi^I \psi^J \quad (23) \]

and provided we take into account the fermion anomaly when computing \( D_- J^+ \alpha \beta \) and \( D_+ J^- I J \). It is interesting to note that the "anomaly" terms on the right-hand side of (22) which arise from quantum \( \sigma \)-model effects in the fermionized picture \(^{10}\) are present already at the classical level in the bosonized picture. This is why, incidentally, one is able to derive the correct Yang-Mills and Lorentz Chern-Simons terms

\[ d\mathcal{H} = \text{Tr} \left( F^a F_{\alpha \beta} - 2 \Gamma^a_{\alpha \beta} \Gamma^b_{\gamma \delta} \right) \quad (24) \]

in the Kaluza-Klein approach to the heterotic string already at the classical level\(^{2,4}\). [The generalization of Witten's non-Abelian bosonization to the case of non-vanishing background fields and the derivation of the anomalies has also been treated in Ref. 11.] Thus we have succeeded in deriving the heterotic string Lagrangian (1) up to the four-fermion terms.
So far, our analysis has been deficient in two respects: we have ignored the Kaluza-Klein scalars when compactifying the bosonic string on the group manifold and we have failed to account for the four-fermion interaction in the heterotic string Lagrangian. We shall now argue that, despite appearances, the four-fermion problem is in fact solved by the inclusion of these Kaluza-Klein scalars. Our argument will be necessarily incomplete because, as we have already discussed, we do not know the exact non-linear ansatz for these scalars and secondly because we do not yet know how to generalize Witten's non-Abelian bosonization to the case of non-vanishing background scalars. Nevertheless, the following argument seems plausible. After compactification, the bosonic string $\sigma$-model will contain a term
\[
\left[ \hat{A}^\mu_{\alpha\beta}(X,\gamma) + \hat{B}^\mu_{\alpha\beta}(X,\gamma) \right] \partial_\mu \gamma^\alpha \partial_\nu \gamma^\beta
\]
Assuming a scalar ansatz of the form
\[
\hat{A}^\mu_{\alpha\beta}(X,\gamma) + \hat{B}^\mu_{\alpha\beta}(X,\gamma) = A^\mu_{\alpha\beta}(\gamma) + B^\mu_{\alpha\beta}(\gamma) + L^i_{\alpha\beta} \hat{S}^{ij}(X) R^j_{\alpha\beta}
\]
we find the current-current interaction
\[
\mathcal{J}_-^i S^{ij'} \mathcal{J}_+^{j'}
\]
After the truncation of $G_R$ to $SO(8)$ as in (19), this becomes
\[
\frac{1}{4} \mathcal{J}_-^{\alpha\beta} S^{\alpha\beta \gamma\delta} \mathcal{J}_+^{\gamma\delta}
\]
or, on using (23)
\[
- \frac{1}{4} \psi^\alpha \psi^\gamma S^{\alpha\beta \gamma\delta} \lambda^\alpha \lambda^\beta
\]
Comparing (29) with the four-fermion term in (1) we obtain the desired result provided we identify the Kaluza-Klein scalars with the $G_L$ Yang-Mills field strength

*) It should be borne in mind that the analogous problem for the $S^7$ compactification of $d = 11$ supergravity took five years to solve.
\[ S^{\mu \nu} = -2 \, F^{\mu \nu} \]  

(30)

Although the above argument is only hand-waving, the result of (30) may be substantiated by examining the bosonic string \(\sigma\)-model \(\beta\)-functions \(\beta^{\mu}_{AB}\) and \(\beta^{\nu}_{AB}\) whose ab components yield the scalar field equation

\[ S^{ij} = 2 \, F_{\gamma \delta}^{i} \, F^{\gamma \delta} j + \text{higher order terms} \]  

(31)

Truncating \(G_{\mu}\) to \(SO(8)\) and identifying \(\tilde{A}^{IJ}_{\mu}\) with \(\omega^{IJ}_{\mu}\) yields

\[ S^{\mu \nu} = 2 \, F_{\gamma \delta}^{\mu \nu} \, R^{\gamma \delta} \]  

(32)

On the other hand,

\[ -2 \, F^{\mu \nu} = 2 \, F_{\gamma \delta}^{\mu \nu} \, R^{\gamma \delta} \]  

(33)

on using the Bianchi identity and field equation for \(F^{IJ}_{\mu \nu}\). We see that (32) and (33) are entirely consistent with the identification (30).

Although we have been focussing our attention on deriving the heterotic string \(\sigma\)-model from the bosonic one, the derivation of the Type II \(\sigma\)-model goes through in much the same way. Now, however, we must make the \(SO(8)\) decomposition with both \(G_{L}\) and \(G_{R}\):

\[ G_{L} \rightarrow SO(8), \quad G_{R} \rightarrow SO(8) \]

\[ A^{i}_{\mu} \rightarrow A^{\mu \nu} = \omega^{(\pm)}_{\mu \nu} \]

\[ \tilde{A}^{i}_{\mu} \rightarrow \tilde{A}^{\mu \nu} = \omega^{(\pm)}_{\mu \nu} \]  

(34)

The equations of motion (13) and (16) now become

\[ \partial_{+} J^{- \alpha} + \omega^{(\pm)}_{\alpha \beta \gamma} J^{- \beta} J^{\gamma} + \frac{1}{2} \, R^{(\pm)}_{\alpha \beta \gamma} J^{- \alpha} J^{\beta} J^{\gamma} \]

\[ + \frac{1}{2} \, R^{(\pm)}_{\alpha \beta} J^{\alpha} J^{\beta} \]  

(35)

and
\[ D_+ J_+^{\alpha\beta} = R_{\gamma\delta}^{(-)} \alpha\beta J_- \gamma J_+ \delta \]
\[ D_- J_+^{\alpha\beta} = R_{\gamma\delta}^{(+)\alpha\beta} J_+ \gamma J_- \delta \]

where the fermionized currents are now
\[ J_+^{\alpha\beta} = i \lambda^\alpha \lambda^\beta \]
\[ J_-^{\alpha\beta} = i \psi^\alpha \psi^\beta \]

Note that there are still different anomaly terms on the right-hand sides of (36) even in the Type II case, and hence non-zero contributions to the Chern-Simons term:\(^{14})
\[ dH = T \left( R^{(-)} \cdot R^{(-)} - R^{(+) \cdot R^{(+)}} \right) \]

The Kaluza-Klein scalar contribution to the Lagrangian is now
\[ - \frac{1}{4} \psi^\alpha \psi^\beta \mathcal{S}_{\alpha\beta} \gamma^\delta \lambda^\gamma \lambda^\delta \]

Since \( A_{\mu}^{\alpha\beta} \) has been identified with \( \omega_{\mu}^{(-)\alpha\beta} \) in (34), we have from (30)
\[ \mathcal{S}_{\alpha\beta} \gamma^\delta = - 2 R^{(-)\alpha\beta \gamma^\delta} \]

and hence we recover the Type II \( \sigma \) model
\[ L_{\tau \omega \Pi} = \partial_+ X^\alpha \partial_+ X^\nu (\sigma_{\mu
u} + B_{\mu
u}) + i \lambda^\alpha (\partial_+ \lambda^\beta + \omega_{\mu}^{(-)\alpha\beta} \lambda^\beta \partial_+ X^\mu) \]
\[ + \psi^\alpha (\partial_+ \psi^\beta + \omega_{\mu}^{(-)\psi^\beta} \psi^\beta \partial_+ X^\mu) - \frac{1}{2} \omega_{\mu\nu}^{(-)} \mathcal{S}_{\alpha\beta} \gamma^\delta \lambda^\alpha \lambda^\beta \psi^\nu \psi^\delta \]

The virtue in deriving the heterotic and Type II \( \sigma \) models from the bosonic ones, is that the previous derivation of heterotic and Type II \( \beta \)-functions from the bosonic ones to lowest order in \( \alpha' \) \(^{2,4}\), will now carry through to all orders in \( \alpha' \). (Since these \( \beta \)-functions are scheme-dependent, one can expect equivalence only when the same renormalization scheme is used in each case\(^{15,16}\).) In particular, this would resolve a previously outstanding problem concerning the (curvature)\(^3\) terms in the \( \beta \) function. Briefly, the problem was as follows. The \( \beta \)-functions of the bosonic string are known\(^{17}\) to receive contributions cubic in the Riemann tensor (three loops in \( \beta^g \) or four loops in \( \beta^\phi \)). For example, the scalar invariants \( R_{\mu
u\rho\lambda} R^{\rho\lambda\sigma\tau} \mu\nu \) and \( R_{\mu
u\lambda\sigma} R^{\rho\sigma\lambda\tau} \mu\nu \) are both known to appear.
On the other hand, these terms are absent for both the heterotic and Type II strings on the grounds of supersymmetry\textsuperscript{17}). At first sight, this presents a paradox because after Kaluza-Klein compactification of the bosonic string such terms would be expected to survive even after identifying gauge potentials with spin connections\textsuperscript{*}). Once we allow the presence of Kaluza-Klein scalars, however, the situation changes. Using the identifications (30) and (40), one sees how the required cancellations could in principle come about. Of course, once the equivalence of the theories at the level of the $\sigma$-model has been established, there is no need to sweat the details.

Finally, nothing in our derivation of superstring $\sigma$-models from bosonic ones described in this paper prevents us from extending it to dimensions less than ten. In particular from (3), we could contemplate compactification on group manifolds for which

\begin{equation}
\text{rank } G = 22, \quad d = 4
\end{equation}

This might, in fact, prove the most efficient way of constructing the non-linear $\sigma$-models corresponding to the recently constructed four-dimensional string theories.

We strongly suspect that the connection between Kaluza-Klein scalars and four-fermion terms discussed in this paper is related to the work on Thirring strings discussed recently\textsuperscript{18}).

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