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E R R A T U M

ANOMALIES AND A POSSIBLE SOLUTION OF PROBLEMS OF ZERO-CHARGE

AND INFRA-RED INSTABILITY

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ERRATUM:

In Eqs. (2), (20) and (21), $\frac{1}{2}$ has to be replaced by 2.

In Eq. (3), $\sqrt{2}$ should be $1/\sqrt{2}$.

In Eq. (6) $1/11$ should be $4/11$.

In Eq. (7), $\sqrt{11}$ should be $\frac{1}{2}\sqrt{11}$.

In Eq. (15) a factor ρ_r is missing: instead of $-\alpha^0/3\pi\log r^2 r_0^2$
read : $-\alpha^0/3\pi\log r^2 r_0^2 \rho_r$.

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A B S T R A C T

We discuss a possible solution of the problems of zero-charge in QED and of infra-red instability in QCD. The solution is based on the introduction of extra currents which compensate the Pauli-Villars subtraction in the amplitudes of photon-photon or gluon-gluon scattering.

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1. - INTRODUCTION

One of the main hypotheses of modern field theory is a hypothesis that the theory can be defined by a Lagrangian with an ultra-violet cut-off λ . Practically, this means that when calculating any loops we can neglect all contributions of particles of momenta larger than $\sqrt{\lambda}$. The only manifestation of momenta beyond $\sqrt{\lambda}$ is that the coupling constant must depend on λ . This hypothesis is commonly accepted as a very natural one. Theories which do not satisfy this condition, such as those which contain an axial anomaly, are considered as not self-consistent. But even in a usual theory like QED or QCD, the existence of anomalies in the axial current and the energy-momentum tensor shows that fluctuations with frequencies higher than $\sqrt{\lambda}$ can be essential. In this paper we shall show that in usual gauge theory (QED, QCD) the vector current generally contains an "anomalous" part at third order in an external field, which comes from particles with momenta larger than $\sqrt{\lambda}$. Unlike the case of an axial current, contributions to the vector current from momenta above $\sqrt{\lambda}$ and below $\sqrt{\lambda}$ are separately conserved, so that the vector current coming from above $\sqrt{\lambda}$ could have been neglected. Another reason why this "anomalous" part of the current has hitherto been neglected is that it is impossible to write it down in a gauge invariant form in the framework of the conventional Lagrangian or Feynman formulations of the field theory. This is due to the fact that this current could not be ascribed, in perturbation theory, to some degrees of freedom known a priori. It is not a current of real particles of the theory, because all the particles have a momentum less than $\sqrt{\lambda}$. It could be a current of some condensate, but condensates do not show up in perturbation theory.

The existence of this anomalous part of the current means that the bare vacuum is unstable and that inclusion of interactions will lead to the creation of some condensates and, maybe, of some new physical states. If we knew these condensates and states, we could in principle formulate the theory in terms of new fields in the usual Lagrangian way and write the current down in a gauge invariant form. In perturbation theory we have no choice at the moment, but to look at the development of the vacuum state as a function of time in the process of switching on the interaction or some external fields.

In this time-dependent formulation of the theory the anomalous part of the current can be written down in a gauge invariant form.

In this paper we shall present the arguments for the existence of an anomalous current in QED and QCD and propose a well-defined expression for

calculating it. Qualitatively, the result is very simple. Calculations of the vacuum polarization to third order in the external field require a Pauli-Villars subtraction in order to maintain gauge invariance. Adding in the anomalous current cancels exactly the regulator contribution and the total current is really defined by a Feynman diagram without regularization. The anomalous current corresponds to the diagrams of Fig. 1 with Pauli-Villars loops, but is of opposite sign. Using this result we have been able to calculate the change in charge renormalization which is due to the anomalous current.

In QED we find, instead of the usual formula discovered by Landau, Abrikosov and Khalatnikov,

$$\alpha_L(q^2) = \frac{3\pi}{\ln \frac{L^2}{q^2}} \quad (1)$$

as new expression for the "running charge"

$$\alpha(q^2) = \frac{\alpha_L(q^2)}{\sqrt{\frac{1}{2}\alpha_L^2(q^2) + 1}} \quad (2)$$

at $q^2/L^2 < 1$, and

$$\alpha(q^2) = \sqrt{2}, \quad q^2 > L^2 \quad (3)$$

where $\alpha_L(q^2)$ is parametrized in terms of the position of the Landau pole L^2 , it is assumed that $q^2 \gg m^2$, m^2 is the electron mass. This new expression has no pole as a function of q^2 and $\alpha(q^2)$ is of the order of unity in the vicinity of the Landau pole, $q^2 \sim L^2$. Of course, $\alpha(q^2)$ does not depend on the initial ultra-violet cut-off λ , because contributions from momenta both below and above $\sqrt{\lambda}$ have been taken into account. Here L^2 is playing the rôle of a physical cut-off. At this cut-off, contrary to the usual Lagrangian formulation, the interaction does not vanish, but becomes strong.

If one tried to write down the effective Lagrangian in the region of momenta of the order of L , it would be a complicated and non-local one. This is the region where the condensates and new degrees of freedom can show up. If, by virtue of strong interactions in the region of momenta $q^2 \sim L^2$, effective confinement takes place, in the sense that at $q^2 \gg L^2$ there will be no charged states, then there will be essentially no current associated with particles with

momenta above L . Then, we will have an effective Lagrangian theory with physical cut-off L , which would differ from the usual one only at $q^2 \sim L^2$.

In QCD, the answer for the "running charge" $\alpha(q^2)$ has a similar structure, but quite a different meaning. If we introduce $\alpha_\rho(q^2)$ in the form found by Politzer, Gross and Wilczek which, can be written as, at $q^2 > \lambda_s^2$,

$$\mathcal{L}_\rho(q^2) = \frac{4\pi}{b \ln \frac{q^2}{\lambda_s^2}} \quad (4)$$

where in QCD without quarks

$$b = \frac{11}{3} n_c \quad (5)$$

n_c the number of colours, and then the new "running charge" would be

$$\alpha(q^2) = \frac{\mathcal{L}_\rho(q^2)}{\sqrt{\frac{1}{11} \mathcal{L}_\rho^2(q^2) + 1}} \quad q^2 > \lambda_s^2 \quad (6)$$

$$\alpha(q^2) = \sqrt{11} \quad q^2 < \lambda_s^2 \quad (7)$$

This charge is small at large q^2 and approaches a constant value ($\sqrt{11}$) of the order of unity at $q^2 = \lambda_s^2$.

This means that the usual infra-red instability is stabilized, if we add the "anomalous" current to the usual one. Again, as in QED, we come to the region $q^2 \sim \lambda_q^2$ where the coupling is strong and where new states (presumably hadrons and condensates such as $\langle G^2 \rangle$, $\langle \bar{\psi}\psi \rangle$) can be created. If it is true, we can associate "anomalous" currents with the hadronic states and these condensates.

2. - VACUUM POLARIZATION IN EXTERNAL COULOMB FIELD

In order to understand the reason for introducing the anomalous current, let us consider the vacuum polarization by an external field in the usual Feynman way. The vacuum polarization current is defined by the set of diagrams shown in Fig. 2, with one-electron loops.

In the standard formulation of the theory, only the first-order diagram leads to charge renormalization. The third- and higher-order diagrams are supposed to give no contribution to the charge renormalization. This means that the contribution of this diagram to the current in the momentum representation $j_\mu(q)$ must vanish at $q = 0$. However, if we calculate explicitly the third-order diagram we will find, as is well known, that it does not vanish at $q = 0$. It can be seen clearly, if one writes down the contribution of the third-order diagram to the zeroth component at the current j_0 at $q \rightarrow 0$ in the form

$$j_0(q) = \frac{\text{diagram}}{q \rightarrow 0} = \int d^4k \frac{\partial}{\partial k_0} \frac{\text{diagram}}{q \rightarrow 0} \equiv \int d^4k_0 \frac{\partial}{\partial k_0} f(k_0, q_1, q_2) \quad (8)$$

and notices that $f(k_0, q_1, q_2)$ at $k_0 \gg q_1, q_2$ does not depend on k_0 by dimensional considerations. Therefore $f(k_0, q_1, q_2)$ as a function of k_0 , has the structure shown in Fig. 3, with a derivative as depicted in Fig. 4.

As a result, from (8) one gets

$$j_0(q) = \int \frac{d^3q_1 d^3q_2}{(2\pi)^9} \frac{f(\infty)}{q_1^2 q_2^2 (q - q_1 - q_2)^2} \quad (9)$$

In order to get an expression for $j_0(q)$ which vanishes at $q = 0$ we can make a subtraction of the Pauli-Villars type or, more generally, introduce a cut-off which would force $f(k_0, q_1, q_2)$ to vanish at $k_0 \rightarrow \infty$. Instead of Fig. 3, we would then have Fig. 3', and the counterpart of Fig. 4 will be Fig. 4'. This cut-off can be introduced, for example, if we consider the current as the limit at $\varepsilon \rightarrow 0$ of the expression

$$j^\varepsilon(x) = \bar{\psi}(x-\varepsilon) \gamma_\mu e^{-i \int_{x-\varepsilon}^{x+\varepsilon} A_\mu dx'} \psi(x+\varepsilon) \quad (10)$$

Of course, we can proceed this way, but it is clear that this procedure is a very unnatural one. The quantity $f(\infty)$ is defined by the scattering amplitude of vacuum fluctuations on an external field and by the density of particle states in the vacuum. In order for $f(k_0, q_1, q_2)$ to start vanish as $k_0 \rightarrow \infty$, some physical interaction must change the number of states, or the scattering amplitude.

To understand this phenomenon it is reasonable to ask what are the bad features of the vacuum polarization charge density, if we calculate it without subtraction. For a point-like external charge, the calculation of the third-order

diagram is very simple, if we are only interested in the charge density at distances from the source which are much larger than the Compton wavelength of the particles in the loops. At such distances, unlike the contribution of the first-order diagram, which decreases exponentially, the third-order vacuum polarization density is equal to

$$\rho(z) = \frac{\alpha_0^3}{3\pi^2 z^3} \quad (11)$$

and exhibits instead a power-like decrease. Here $\alpha_0 = \ell_0^2/4\pi$, ℓ_0 is the charge of the external source, taken equal the electron charge. Such a slow decrease of the charge density means that the bare vacuum has an infra-red instability, because an external field produces infinite charge. Of course, we would not like to live with such a theory. But, it may be that by eliminating this term we go to another extreme and ask for a good theory too early, at a price of encountering the zero-charge problem. As was discussed above, an increase of the charge at large momenta can diminish the number of charged states at momenta of the order of the Landau scale L . That would mean a real physical cut-off. Contributions of various momenta to the current would have the form of Fig. 4' with $\sqrt{\lambda} = L$ and the infra-red instability would be killed.

Now it is clear what we mean when speaking of an anomalous current. If the contributions of various momenta to the current have a structure corresponding to Figs. 3 and 4, but the Lagrangian part of the current has the structure of Figs. 3' and 4', then the difference between these two currents is the anomalous current. One reason for choosing this wording is that an anomaly, i.e., the non-conservation of some charge (axial,...) is due to the same states with infinite momenta we are concerned with.

3. - "RUNNING" CHARGE OF AN EXTERNAL SOURCE

The usual way of calculating the running coupling constant consists of calculating the renormalization of the vertex parts and Green functions in a logarithmic approximation. It can be done in a simpler and more transparent way. Because we are interested in the ultra-violet behaviour of QED, we can neglect the electron mass. In this case, the calculation is especially simple.

As is well known since the old paper of Kroll and Wiechman, all the diagrams, except for the first-order one, give a polarization density $\rho(r)$ which is a delta-function, $\delta(\vec{r})$, for massless electrons and a point-like source, if one makes a Pauli-Villars type subtraction. The polarization density, corresponding to the first-order diagram, can be written in the form

$$\rho_p(z) = - \frac{\alpha_0}{6\pi^2} \int d^3z' \frac{\rho^{ext}(z')}{|\vec{z} - \vec{z}'|^3} \quad (12)$$

where $\rho^{ext}(r) = \delta(\vec{r})$.

In order to find the charge density distribution in the logarithmic approximation it is sufficient to rewrite (12) in a self-consistent equation, putting in (12), instead of $\rho^{ext}(r)$, the total charge density:

$$\rho(z) = \rho^{ext} + \rho_p(z) \quad (13)$$

One then obtains an equation

$$\rho(z) = \rho^{ext} - \frac{\alpha_0}{6\pi^2} \int d^3z' \frac{\rho(z')}{|\vec{z} - \vec{z}'|^3} \quad (14)$$

This equation contains the logarithmic divergence coming from the region $|\vec{r} - \vec{r}'| \rightarrow 0$. If we introduce a small radius of the source r_0 , corresponding to the ultra-violet cut-off $\lambda^2 \sim 1/r_0^2$, and exclude the region $|\vec{r} - \vec{r}'| < r_0$, then in the logarithmic approximation (14) at $r > r_0$ can be rewritten as:

$$\rho(z) = - \frac{\alpha_0}{3\pi} \ln \frac{z^2}{z_0^2} - \frac{\alpha_0}{6\pi^2 z^3} \int_{z' < z} d^3z' \rho(z') \quad (15)$$

This equation can be solved very easily.

If we introduce

$$\alpha(z) = \alpha_0 \int_{z' < z} d^3z' \rho(z') \quad (16)$$

$$\rho(z) = \frac{1}{4\pi z^3 \alpha_0} \alpha(z)$$

$$\left(\frac{3\pi}{\alpha_0} + \ln \frac{z^2}{z_0^2} \right) z^2 \alpha'(z) = - \alpha(z) \quad (17)$$

$$\alpha(z) = \frac{3\bar{\pi}}{\frac{3\bar{\pi}}{\alpha_0} + \ln \frac{z^2}{z_0^2}} \quad (18)$$

It is the same expression as (1) with

$$z^2 \rightarrow \frac{1}{qz} \quad L^2 \Rightarrow \frac{1}{z_0^2} e^{\frac{3\bar{\pi}}{\alpha_0}}$$

Up to this moment we have been calculating the current in the usual way, with the Pauli-Villars type regularization which permitted us to throw out all the higher-order diagrams.

As we discussed above, if we do not make any subtractions, the third order would give us the non-zero contribution (11). It can be shown that only this contribution survives at $r > r_0$. This means that we must add (11) to (14), changing again the external charge α_0 to the total charge $\alpha(r)$, as we did when deriving (14). We get, instead of (14),

$$\rho(z) = \rho^{ext} - \frac{\alpha_0}{6\bar{\pi}^2} \int d^3z' \frac{\rho(z')}{|z-z'|^3} + \frac{\alpha^3(z)}{3\bar{\pi}^2 z^3} \quad (19)$$

and, instead of (17), we obtain:

$$\left(\frac{3\bar{\pi}}{\alpha_0} + \ln \frac{z^2}{z_0^2} \right) z^2 \partial_{z^2} \alpha(z) = -\alpha(z) + \frac{1}{2} \alpha^3(z) \quad (20)$$

This equation can be solved easily. If $\alpha_L(r)$ satisfies (17), then

$$\alpha(z) = \frac{\alpha_L(z)}{\sqrt{\frac{1}{2} \alpha_L^2(z) + 1}} \quad (21)$$

satisfies (20).

This is just expression (2) whose properties were discussed in the Introduction.

4. - GAUGE INVARIANCE AND A GENERAL EXPRESSION FOR THE ANOMALOUS CURRENT

While considering the vacuum current in a static field we did not encounter the problem of gauge invariance of the anomalous current (11) explicitly. But, of course, in the general case it exists. Moreover, we do not know how in general to find a definition of this current. What is the underlying principle of this definition? We have not solved this problem. We just guess a general gauge-invariant expression for this anomalous current using the analysis the vacuum current in an explicitly gauge-invariant form and the limiting case of static field. The current (11) can be written in the form

$$\rho(z) = \frac{1}{3\pi^2} \varphi^3(z) \quad (22)$$

where $\phi(r)$ is the Coulomb potential. But the Coulomb potential can always be written as an integral of the field strength along some line

$$\varphi(z) = - \int \mathbf{E}_i dx_i \quad (23)$$

and does not depend on the direction of the line. For a general field, the natural generalization of (24) would be the expression

$$A_\mu(x, z) = \int_{-\infty}^x dx'_\nu F_{\nu\mu}(x') \quad (24)$$

where the integration is performed along some straight line where z_μ is a vector defining the direction of the line. But now, $A_\mu(x, z)$ would depend on z_μ . If we want to write the current of the particles of infinitely large momenta, we can use the fact that the particles propagate along the light cone. It is natural to expect that the A_μ which defines the current would be an integral of $F_{\mu\nu}(x)$ along a straight line on the light-cone. Therefore, the natural generalization of (22) would be

$$\int_\mu d = \frac{1}{3\pi^2} \int \frac{d\Omega}{4\pi} A_\mu(x, z) A^2(x, z) \quad (25)$$

where the angular integration is performed over the directions of null-vectors z_μ , corresponding to different straight trajectories from $-\infty$ to the top of the light cone (see Fig. 5). However, the expression (25) does not have the right Lorentz transformation properties; because if we parametrize z_μ in the form $(1, \vec{z})$ $|\vec{z}|^2 = 1$ and $d\Omega$ is then the integration over the orientations of the three-dimensional unit vector $\vec{z}$, after Lorentz transformation $(1, \vec{z})$ goes to $\gamma(1, \vec{z}')$ and $d\Omega$

goes to $d\Omega'\gamma^2$. If we would write, instead of $A^2(x,z)$ in (25), $[n_\nu A_\nu(x,z)]^2$ where $n = \pm(1, -\vec{z})$ is the space or time reflection of the vector $(1, \vec{z})$. This n_ν is a vector normal to the light-cone which transforms as follows under the Lorentz transformation: $n_\mu \rightarrow 1/\gamma(1, -z')$, and would compensate the change in $d\Omega$. Therefore we would like to rewrite (25) as:

$$j_\mu^4 = \frac{1}{3\pi^2} \int \frac{d\Omega}{4\pi} A_\mu(x, z) (n A(x, z))^2 \quad (26)$$

Unfortunately, this is covariant but not conserved. A conserved current can be written down in the form

$$j_\mu^4 = \frac{1}{3\pi^2} \int \frac{d\Omega}{4\pi} \left\{ A_\mu(x, z) (n A(x, z))^2 - \int_{-}^x dx'_\nu \partial'_\nu A_\nu(x'/z) (n A)^2 \right\} \quad (27)$$

where the integration over x' is performed along the same line defined by z . This expression is our final hypothesis on the form and magnitude of the anomalous current in QED.

It is easy to show that in the case of a time-independent field, $F_{\mu\nu}$ reduces to (23) because $\partial_\nu A_\nu(n_\nu A_\nu) = 0$ and A_0 and $n_\nu A_\nu$ do not depend on z .

The expression (27) can be written down in an explicitly covariant form if we introduce, instead of (24),

$$A_\mu(x, x') = \int_{x'}^x dx''_\nu F_{\nu\mu}(x'') \quad (28)$$

where the integration is performed along a straight line, in which case (27) is equivalent to

$$j_\mu^4 = \frac{1}{6\pi^2} \int d^4x' \partial'_\nu x'_\nu \frac{\delta((x-x')^2)}{(n, x-x')^2} \left\{ A_\mu(x, x') (n A)^2 - \int_{x'}^x dx''_\nu \partial''_\nu A_\nu(x''x') (n A)^2 \right\} \quad (29)$$

with n_μ a space or time reflection of $(x-x')_\mu$.

Because (29) contains a total derivative, one can reduce the integration to the surface $x'_0 \rightarrow -\infty$ and get (27). If the expressions (27) and (29) are a correct

guess, then the non-Lagrangian theory can be formulated by writing down the Maxwell equation for second-quantized operators in the form

$$\partial_\nu F_{\nu\mu}(x) = e_0^2 \left[(\bar{\psi} \gamma_\mu \psi)_\lambda + j_\mu^a \right] \quad (30)$$

where $(\bar{\psi} \gamma_\mu \psi)_\lambda$ is the current defined in the usual way with a cut-off. The j_μ^a in Eq. (30) is that shown in Eqs. (27) and (29), and is of course non-local, but this should not lead to a violation of causality, because all operators are on the light cone. The expression (29) for the current appears comparatively natural if one is considering the current $\langle \Omega_{in} | j_\mu(x) | \Omega_{in} \rangle$ as a function of time in an external field. But this we would like to postpone for a more lengthy paper. We also will postpone to this more lengthy paper the derivation of Eq. (6) for $\alpha(q^2)$ and the expressions analogous to (27) and (29) in QCD.

In connection with QCD we want only to draw the reader's attention to the following interesting problem. The calculation of the gluonic contribution to the anomalous current is straightforward, if we know the formulae of type (27) and (29) in QCD, because the gluons are asymptotically free. But quarks are not asymptotically free, and have electromagnetic interactions which are strong for large momenta. As we have discussed, the charged one-particle states must be suppressed at large momenta in order to produce an effective Pauli-Villars cut-off for low-momentum physics. But, if one-particle states were suppressed, the vacuum colour current of quarks would also be suppressed at large momenta and we would have no anomalous vacuum colour current of quarks. This means that the formula (6) for the QCD coupling constant could be correct even with the inclusion of quarks, if we change the $b = 11/3 n_c$ to $b = 11/3 n_c - 2/3 n_f$, where n_f is the number of flavours.

Another interesting thing to mention is that in scalar electrodynamics, instead of $\frac{1}{2}$ in Eq. (2) for QED, one has -1 , and the theory is unstable even after the inclusion of the anomalous current.

In conclusion, we would like to point out that the approach we have described here gives us the possibility of considering a theory with axial anomaly. If we add to the usual axial current with non-zero divergence a current of the form

$$j_{\mu}^{\cdot q} = -\frac{\varepsilon^2}{32\pi^3} \int \frac{d\Omega}{(\varepsilon z)^2} F_{\mu\nu}^x(x) A_{\nu}(x, z) \quad (31)$$

where ε_{μ} is an arbitrary vector, similarly to (28), the total current would be conserved, and the theory could make sense.

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- Figure 1 -

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- Figure 2 -

- Figure 3 -

- Figure 3' -

- Figure 4 -

- Figure 4' -

- Figure 5 -