CERN-TH.4697/87

GROUP THEORETIC APPROACH TO THE PERTURBATIVE STRING S-MATRIX

A. Neveu and P. West^{*)}

CERN - Geneva

A B S T R A C T

A new approach to the computation of string scattering is given. From duality, unitarity and a generic overlap property, we determine entirely the N-string amplitude, including the integration measure, and its gauge properties. The techniques do not use any oscillator algebra, but the computation is reduced to a straightforward exercise in conformal group theory. This can be applied to fermionic trees and multiloop diagrams, but in this paper it is demonstrated on the open bosonic tree.

^{*)} Permanent address: Mathematics Department, King's College, London, U.K.

CERN-TH.4697/87

March 1987

String theory began with a guess for the scattering, at tree level, of four tachyonic states, which was soon generalized to an arbitrary number of tachyonic states^{*}). The first step taken to understand these results was the development of the operator formalism. Using the latter, the tree level scattering of N open string states was computed, the final results being the Lovelace²⁾ and Olive³⁾ oscillator vertices. Multiloop contributions were also computed⁴⁾. The effect of properly removing the propagation of negative norm states was only known at one loop⁵⁾.

It was then realized that the dual model was equivalent to a sum over Riemann surfaces weighted with the Nambu action⁶⁾. This led to the Polyakov approach, which has recently received much attention. Computations in this approach have been simplified with the application of conformal field theory techniques.

An alternative development, which also recovers the known tree and loop amplitudes was the construction of a second-quantized field theory in the light-cone gauge⁷⁾. More recently, there has been the construction of gauge covariant interacting second-quantized string theories. The latter formulations agree with the known string scattering at tree level, but they also give in principle a method for calculating non-perturbative quantities as can be done for point particle theories. The gauge covariant string theories have introduced several new features and it will require new techniques before their content can be extracted.

These historical remarks show that the S-matrix for string scattering has a conceptual simplicity which is absent for point particle scattering. The latter is best computed from a Feynman path integral approach. One reason for this simplicity is duality which greatly reduces the number of graphs. One may suspect that string theory contains further specific simplifying features yet to be discovered, which, when properly encoded, allow for a more direct deduction of the string S-matrix and its properties. In a series of papers, we shall indeed show that combining duality with another property specific to strings, namely the existence of generalized overlap conditions, together with unitarity, we can completely determine the perturbative S-matrix for the string and many of its properties. This paper treats the open bosonic string at tree level. Superstring amplitudes and loops will be treated in subsequent publications.

^{*}) For a review and extensive list of references, see Ref. 1).

In a past paper⁸⁾, we had noticed already that the Ganeschi-Schwimmer-Veneziano three-string vertex⁹⁾ satisfies simple overlap and "cycling" properties. We had used these properties to determine the covariant vertex for three Neveu-Schwarz strings¹⁰⁾, which reproduces the known answers at low levels. In this paper, we show that the features found for these three-point vertices generalize the bosonic string N-point functions, and that the latter can indeed be derived from these properties, together with unitarity. This new approach reduces the computation of string amplitudes to an exercise in conformal group theory, and allows one to deduce their general properties without having to know their explicit form. For example, we shall derive in this paper as an application the explicit form of the Virasoro gauge conditions satisfied by the N-point vertex.

Let us consider an N-point oscillator vertex $V(z_1, z_2, \dots, z_N)$ which is labelled by the N Koba-Nielsen variables z_i ($i=1, \dots, N$). The on-shell scattering of N strings will be given by an expansion of the generic form

$$\int d\mu(z_i) V(z_1, \dots, z_N) \prod_{j=1}^N |\psi_j\rangle, \quad (1)$$

where $|\psi_j\rangle$ are the N on-shell string states which obey

$$L_n |\psi_j\rangle = 0, \quad n \geq 1; \quad (L_0 - 1) |\psi_j\rangle = 0, \quad (2)$$

and $d\mu(z_i)$ is the measure which will be determined. We set this discussion in the framework of the open bosonic string trees, but it will be quite clear that it applies to any string with obvious generalizations. For example, for the superstring, z will be replaced by z and θ , and for loops one requires additional parameters (moduli).

We note that $V(z_1, \dots, z_N)$ has the generic form

$$V(z_1, \dots, z_N) = \langle 0_1 | \langle 0_2 | \dots \langle 0_N | f(\alpha_n^{r(n)}, z_i), n \geq 0, (3)$$

where

$$\alpha_n^{r(n)} | 0_n \rangle = 0, \quad n \geq 1. \quad (4)$$

In fact we will see that f is always an exponential bilinear in the oscillators $\alpha_n^{\mu(r)} (2), (3)$.

The $T^{(r)}$ transformations are of the form of the exponential of $L_n^{(r)}$'s and implement the cycling conformal transformation. Essentially, A1 ensures that the amplitudes will satisfy duality.

Since the amplitudes describe string scattering, one must have at least one overlap taking R as the energy-momentum density of the string

$$P^r(z) = \sum_{n=-\infty}^{+\infty} \alpha_n^r z^{-n} \quad (10)$$

which has conformal weight one. This overlap generalizes the conservation of energy and momentum. From the point of view of second-quantized string field theory, where the interaction vertices are overlap δ functions, it is clear that overlap identities exist for N point functions. Technically, overlap conditions result from pushing the creation operators in R to the left to the vacuum and cancelling the resulting commutator. Integrating with respect to z, we expect an overlap identity for

$$Q^r(z) = - \sum_{n \neq 0} \frac{\alpha_n^r}{n} z^{-n} + \alpha_0^r \ln z + q^r \quad (11)$$

which has conformal weight zero. Given the P^μ overlap we can compute one for

$$L(z) = \sum_{n=-\infty}^{+\infty} L_n z^{-n} = : P^2(z) : \quad (12)$$

which has conformal weight 2. We recall that an operator R has conformal weight d if

$$U(g) R(z) U^{-1}(g) = \left[\frac{z}{g(z)} \frac{dg(z)}{dz} \right]^d R(g(z)) \quad (13)$$

where $U(g)$ implements in terms of L_n 's the conformal transformation g. In particular, $U(g) = 1 + a_n L_n$ corresponds to $g(z) = z + a_n z^{n+1}$, with a_n infinitesimal.

As we shall see, A1, A2 and A3 allow us to determine $V(z_1, \dots, z_N)$ as well as the measure in Eq. (1). Thus they determine the string scattering amplitudes uniquely. Perhaps more importantly, they allow us to derive their properties without having to know the explicit form of $V(z_1, \dots, z_N)$ nor of the measure. This includes the behaviour of the amplitudes under general Virasoro gauge transformations.

The assumptions from which the string scattering amplitudes follow are:

A1) There exists a conformal cycling transformation which permutes the z_i variables, that is

$$V(z_1, z_2, \dots, z_N) U^{\text{cyc}} = V(z_2, z_3, \dots, z_N, z_1), \quad (5)$$

where

$$U^{\text{cyc}} = \prod_{\alpha=1}^N [T_{\alpha+1}^{\alpha}]^{-1} \quad (6)$$

In this equation, as in Eq. (4), the upper index specifies the external line on which the transformation acts. The T transformations must obey the equation

$$T(z_N, z_1, z_2, \dots, z_{N-1}) \dots T(z_2, z_3, \dots, z_1) T(z_1, \dots, z_N) = 1. \quad (7)$$

In the following, we shall often use the notation

$$T_i = T(z_i, z_{i+1}, \dots, z_N, z_1, \dots, z_{i-1}),$$

suppressing the upper index when not required for clarity.

A2) The vertex $V(z_1, \dots, z_N)$ satisfies overlap identities whose generic form is

$$V(z_1, \dots, z_N) \left\{ \sum_{i=1}^N A_i(z) R^{(i)}(T_i, T_{i-1}, \dots, T_1(z)) \right\} = 0 \quad (8)$$

where the $A_i(z)$ are functions subject to a condition of the form

$$\sum_{i=1}^N \frac{A_i(z)}{f_i(z)} = 0 \quad (9)$$

$f_i(z)$ being themselves functions of z , and R is an object of a given conformal weight.

A3) The theory is unitary.

As stated above, these assumptions may look quite arbitrary and abstract to the reader, but we hope that their motivation and consequences will be clarified by the rest of this paper.

Let us begin by determining the functions f_i in Eq. (9). Since Eqs. (8) and (9) are homogeneous in A_i , we can choose one of the f_i 's to be one, and we take $f_2 = 1$. We apply U^{cyc} to Eq. (8) and use Eqs. (5) and (13) to obtain

$$V(z_2, z_3, \dots, z_N, z_1) [A_N S_{T_1}(z) R^{(N)}(T_1 z) + A_1 S_{T_2}(T_1 z) R^{(1)}(T_2 T_1 z) \\ + \dots + A_p S_{T_{p+1}}(T_p \dots T_1 z) R^{(p)}(T_{p+1} \dots T_2 T_1 z) + \dots] = 0 \quad (14)$$

where

$$S_g(z) = \left[\frac{z}{g(z)} \frac{dg(z)}{dz} \right]^d \quad (15)$$

This equation is an identity for the cycled V , so it must also obey Eq. (9) appropriately cycled, namely:

$$\frac{1}{\bar{f}_1} \frac{A_N S_{T_1}(z)}{S_{T_2}(T_1 z)} + A_1 + \dots + \frac{A_p S_{T_{p+1}}(T_p \dots T_1 z)}{\bar{f}_{p+1} S_{T_2}(T_1 z)} + \dots = 0 \quad (16)$$

The bar on f indicates that it is the f with the Koba-Nielsen variables cycled. Comparing this relation with the original identity when cycled, we have the result

$$\frac{\bar{f}_{p+1}}{\bar{f}_p} = x S_{T_{p+1}}(T_p \dots T_1 z) \quad (17)$$

where

$$x = \frac{\bar{f}_3}{S_{T_3}(T_2 T_1 z)} \quad (18)$$

Iterating the above equation, we find that

$$\bar{f}_{p+1} = S_{T_{p+1}}(T_p \dots T_1 z) S_{T_p}(T_{p-1} \dots T_1 z) \dots S_{T_3}(T_2 T_1 z) x^{p-1} \\ = S_{T_{p+1} \dots T_3}(T_2 T_1 z) x^{p-1} \quad (19)$$

In the derivation of the first equation, we used $f_2 = 1$, while in the next equation, we used the fact that from its definition as a conformal weight, $S_g(z)$ obeys a one-cocycle condition, namely

$$S_{g_1}(z) S_{g_2}(g_1(z)) = S_{g_2 g_1}(z) \quad (20)$$

We can compute f_N in two ways, namely by the above equation or using $p = 1$ in Eq. (17). Carrying this out, we find that

$$\bar{f}_N = S_{T_N \dots T_3}(T_2 T_1, z) x^{N-2} = \frac{\bar{f}_1}{x S_{T_1}(z)} = \frac{1}{x^2 S_{T_1}(z) S_{T_2}(T_1, z)} \quad (21)$$

Rewriting this result, we find that

$$S_{T_N \dots T_1}(z) x^N = x^N = 1 \Rightarrow \begin{cases} x = 1 & \text{for } N \text{ odd} \\ x = \pm 1 & \text{for } N \text{ even} \end{cases} \quad (22)$$

One may verify that only $x = 1$ provides a solution. As such, we have shown that

$$\bar{f}_p = S_{T_p \dots T_3}(T_2 T_1, z) \quad \text{with } f_2 = 1, f_1 = \frac{1}{S_{T_2}(T_1, z)} \quad (23)$$

The overlap identity (8) must be used in such a way that the infinite sums that occur in their verification in the operator formalism are convergent. Each term is only convergent in a given region of the z plane, and so this places a restriction on how many A 's can be non-zero at the same time. We find that for the cases studied in this paper only two consecutive A 's may be non-zero. For a concrete discussion of this point in the case of the three-point function, see Ref. 8).

It is sometimes more useful to have integrated identities which we now derive. Consider the quantity

$$\oint_{|z| \ll 1} \frac{dz}{z} \varphi(z) R^{(n)}(z) \quad (24)$$

We may deform the contour away from $|z| \ll 1$ to include the rest of the positive real axis. In doing this, one must for convergence in the appropriate regions one must carefully use Eq. (9) taking $A_N = \phi(z)$. Use of Eq. (9) introduces quantities $R(T_p \dots T_z)$ which have poles at $T_p \dots T_z$. The function $\phi(z)$ must not introduce poles at other points but is otherwise arbitrary. One finds the result

$$V \left[\oint_{z=0} \frac{dz}{z} \varphi(z) R^{(N)}(z) + \oint_{\xi_1=0} \frac{d\xi_1}{\xi_1} \varphi(z) \left(-\frac{f_1}{f_N}\right) R^{(1)}(T_1 z) \right. \\ \left. + \dots + \oint_{\xi_p=0} \frac{d\xi_p}{\xi_p} \varphi(z) \left(-\frac{f_p}{f_N}\right) R^{(p)}(T_p \dots T_1 z) + \dots \right] = 0, \quad (25)$$

where

$$\xi_p = T_p \dots T_2 T_1 z, \quad \xi_{N+1} = \xi_1, \quad \xi_N = z. \quad (26)$$

Making the change of variables and using Eq. (22), we find that

$$V \left[\sum_{p=1}^N \oint_{\xi_p=0} \frac{d\xi_p}{\xi_p} \varphi(z) \bar{S}_{T_p \dots T_1}(z) R^{(p)}(\xi_p) \right] = 0, \quad (27)$$

where

$$\bar{S}_{T_p \dots T_1}(z) = \left(\frac{z}{z'} \frac{dz'}{dz} \right)^{d-1}, \quad z' = T_p \dots T_1 z. \quad (28)$$

Historically, such integral form of overlap conditions were first introduced for the fermion emission vertex¹¹⁾ and have been used¹²⁾ more recently in the framework of gauge covariant string field theory.

Let us now illustrate how the overlaps determine V . For the case $R = p^\mu$, we have $\bar{S} = 1$, and so the identity becomes rather simple. If we choose $\phi(z) = \xi_p^{-n}$, the term involving $R^{(q)}(\xi_q)$, $p > q$ includes a factor

$$\varphi = \xi_p^{-n} = \left[\Gamma T_p \dots T_{q+1} \xi_q \right]^n, \quad r(z) = z^{-1}. \quad (29)$$

For any conformal transformation g , we may define an infinite dimensional matrix by¹⁾

$$\frac{[g(z)]^n}{n} = \sum_{m=1}^{\infty} g_{nm} z^m + \frac{[g(z=0)]^n}{n} \quad (30)$$

and so

$$\varphi = n (\Gamma T_p \dots T_{q+1})_{nm} \xi_q^m + n \Gamma T_p \dots T_{q+1}(0). \quad (31)$$

Inserting these results into the identity (27), we find

$$V \left[\dots \alpha_n^{r(p)} + \dots + \sum_{m=0}^{\infty} n (\Gamma T_p \dots T_{q+1})_{nm} \alpha_m^{r(q)} + \dots \right] = 0 \quad (32)$$

As a result:

$$V(z_1, \dots, z_N) = \langle 0, | \langle 0_2 | \dots \langle 0_N | \\ \times \exp \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{1 \leq q < p \leq N} \alpha_m^{r(q)} (\Gamma T_p \dots T_{q+1})_{nm} \alpha_n^{r(p)} \quad (33)$$

This does not determine the $(\alpha_0^\mu)^2$ piece of V which is fixed by taking $R = Q^\mu$.

We now derive the conformal properties of $V(z_1, \dots, z_N)$. Taking $R = L(z)$, Eq. (27) becomes

$$V \left[\sum_{p=1}^N \oint_{\xi_p=0} \frac{d\xi_p}{\xi_p} \varphi(z) S_{T_p \dots T_1}(z) L^{(p)}(\xi_p) \right] = 0 \quad (30)$$

This identity states that the infinitesimal conformal transformation defined by the term in the bracket leaves $V(z)$ inert. It is more interesting to consider transformations that change the arguments of the vertex. Let us consider a vertex $V(z_1, \dots, z_N)$ and a vertex $V(\hat{z}_1, \dots, \hat{z}_N)$ with cycling transformations T_r and $\hat{T}_r$ respectively. We will now show that

$$V(z_1, \dots, z_N) \prod_{n=1}^N U^{(n)} = V(\hat{z}_1, \dots, \hat{z}_N), \quad (35)$$

where

$$U^{(n)} = [(T_2 \dots T_1) g (\hat{T}_2 \dots \hat{T}_1)^{-1}]^{(n)}, \quad (36)$$

and g is any conformal transformation.

Since $V(z_i)$ and $V(\hat{z}_i)$ are uniquely determined by A1 and A2, with the appropriate cycling transformations, it suffices to show that they satisfy these axioms. To show that $V(\hat{z}_1, \dots, \hat{z}_N)$ satisfies the correct cycling properties, we

apply $\prod_{r=1}^N (\hat{T}_{r+1}^{(r)})^{-1}$ to Eq. (35). We find

$$\begin{aligned} V(\hat{z}_1, \dots, \hat{z}_N) \prod_{r=1}^N (\hat{T}_{r+1}^{(r)})^{-1} &= V(z_1, \dots, z_N) \prod_{r=1}^N [T_2 \dots T_1 g(\hat{T}_{r+1} \dots \hat{T}_1)]^{(r)} \\ &= V(z_2, \dots, z_N, z_1) \prod_{r=1}^N [T_{r+1} \dots T_1 g(\hat{T}_{r+1} \dots \hat{T}_1)]^{(r)} \\ &= V(\hat{z}_2, \dots, \hat{z}_N, \hat{z}_1), \end{aligned} \quad (37)$$

which demonstrates that $V(\hat{z}_i)$ cycles correctly. It is clear that if $V(z_i)$ satisfies the appropriate identities, then, using (35), the identities for $V(\hat{z}_i)$ will be of the appropriate form, and as such A2 will hold. This completes the demonstration of (35).

We note that we may take $\hat{T}_r = T_r$ and $\hat{z}_r = z_r$ and then we find that

$$V(z_1, \dots, z_N) \prod_{r=1}^N [T_2 \dots T_1 g(T_2 \dots T_1)]^{-1} = V(z_1, \dots, z_N). \quad (38)$$

In fact, for infinitesimal transformations g , this is non other than Eq. (34). As a result, the most general shift in the z_i 's can be obtained by considering $g = 1$, and is determined only by the cycling transformations T_r and $\hat{T}_r$.

To illustrate this discussion, let us consider a three-string vertex $V(z_1, z_2, z_3)$. We take as cycling transformations $T^{(r)} = T$ for $r = 1, 2, 3$, where, in the notation of Ref. 2),

$$T = \begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_3 & z_1 \end{pmatrix}. \quad (39)$$

Taking Eq. (22) we find that

$$\bar{f}_1 = [S_T(T_3)]^{-1}, \quad \bar{f}_2 = 1, \quad \bar{f}_3 = S_T(T_2). \quad (40)$$

The P^μ identity is given by Eq. (27) for $N = 3$ and $\bar{S} = 1$. Due to the relationship $\xi_{p+1} = T \xi_p$ for $p = 1, 2, 3$, we find that the vertex is of the form

$$V = \langle \alpha_1 | \langle \alpha_2 | \langle \alpha_3 | \exp \sum_{n=1}^3 \alpha_n^{\mu(n)} \bar{T}_{nn} \alpha_n^{\mu(n)}, \quad (41)$$

where

$$\frac{1}{n}(\Gamma T \xi)^n = \sum_{m=1}^{\infty} \bar{T}_{nm} \xi^m + \frac{\bar{T}_{n0}}{n} \quad (42)$$

Thus, the three-string vertex only depends on T and is otherwise independent of the z_i 's. We also note that, unlike higher vertices, the three-string vertex is cyclic, i.e., $V(1,2,3) = V(2,3,1)$, without z_i integration.

Taking $\{z_1, z_2, z_3\} = \{0, 1, \infty\}$, we can immediately compute $\bar{f}_r$ and $\bar{T}_{nm}$ and we recover all the results of Ref. 8).

The relation of Eq. (35) between two three-string vertices with different z_i 's is

$$V(\hat{z}_r) = V(z_r) g^{(3)} (\tau g \hat{\tau}^{-1})^{(1)} (\tau^2 g \hat{\tau}^2)^{(2)}, \quad (43)$$

for all g . In this case, it is not difficult to compute $V(z_i)$ and $V(\hat{z}_i)$, and explicitly verify that they are related by this equation. One particularly simple case is

$$V(0, 1, \infty) = V^{CSV} = V(0, \omega, \infty) \omega^{-(L_0^{(1)} + L_0^{(2)} + L_0^{(3)})} \quad (44)$$

The vertex $V(0, \omega, \infty)$ contains after an appropriate twist the tachyon vertex $\exp ikQ(\omega)$. The well-known conformal properties of this latter vertex are then a very special case of Eq. (43). Another simple case is to take g to be the $SU(1,1)$ transformation which sends $\hat{z}_r$ to z_r , where one finds

$$V(\hat{z}_r) = V(z_r) g^{(3)} g^{(1)} g^{(2)}. \quad (45)$$

One may verify from Eq. (27) that $V(0, 1, \infty)$ is unitary. By this, we mean that if two of the three legs are physical, on shell, and the third is projected on the mass shell, then it also obeys the Virasoro conditions. This was already done in Ref. 8)*

The reader may wonder whether the vertices for arbitrary z_r 's give the same result when applied to on-shell states. In fact this is the case up to a

*) Equation (30) of Ref. 8) contains a type-setting error: the coefficient of L_{n+p}^2 should read $\binom{-n+1}{p}$ instead of $\binom{n+1}{p}$.

multiplicative constant, since L_0 and L_1 in g of Eq. (45) give zero, while L_{-1} on one leg may be swapped for L_0 and L_1 on the other two. As a result, no integration over the z_r 's is required.

To compute an actual N string vertex, we must specify what are the cycling transformations $T^{(r)}$. Since duality is related to $SU(1,1)$ transformations, we look for such transformations which cycle the Chan variables and which reproduce vertices already in the literature^{2),3)}. For this purpose, we take

$$T^{(j)} z = \frac{z-1}{z(1-u_{j-1,j})}, \quad (46)$$

$u_{j-1,j}$ being the Chan variable of the $(j-1, j)$ channel:

$$u_{j-1,j} = \frac{(z_{j+1}-z_{j-2})(z_j-z_{j-1})}{(z_{j+1}-z_{j-1})(z_j-z_{j-2})}. \quad (47)$$

A straightforward computation shows that

$$T^{(j)} = \Gamma \tilde{V}^j (\tilde{V}^{j-1})^{-1} \Gamma, \quad (48)$$

where V^j is the $SU(1,1)$ transformation

$$V^j = \begin{pmatrix} \infty & 0 & 1 \\ z_{j-1} & z_j & z_{j+1} \end{pmatrix}, \quad (49)$$

and

$$\tilde{V} = \Gamma V^{-1} \Gamma \quad (50)$$

Since one can make gauge transformations involving L_0 and L_1 on the external lines and not affect on-shell quantities, and L_{-1} may be turned into L_0 and L_1 , on other legs, there are other cycling transformations which lead to the same S -matrix. The exact form of the class of admissible cycling transformations is beyond the scope of this paper.

Adopting the cycling transformations $T^{(r)}$ of Eq. (46), it is straightforward from Eq. (33) to recover the oscillator and momentum-dependent part of the Lovelace vertex²⁾, namely

$$V(z_1, \dots, z_N) = \langle 0, 1 | \dots \langle 0, N | \exp \sum_{1 \leq i < j \leq N} \sum_{\substack{n \in \mathbb{Z}_0 \\ m \in \mathbb{Z}_0}} \alpha_n^{(i)} [\Gamma(V^j)^{-1} V^j]_{nm} \alpha_m^{(j)}$$

To complete the computation of the open bosonic tree graphs, we must evaluate the measure in Eq. (1). We observe that if we keep all the z_i 's fixed except a given z_j , then the only cycling transformations that change are $T^{(j-1)}$, $T^{(j)}$, $T^{(j+1)}$ and $T^{(j+2)}$. Using Eq. (35), we then observe that $V(z_1, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_N)$ and $V(z_1, \dots, z_{j-1}, \hat{z}_j, z_{j+1}, \dots, z_N)$ are related by $U^{(r)}$ matrices. We find $U^{(r)} = 1$ for $1 \leq r < j-1$, while

$$U^{(j-1)} = T^{(j-1)} (\hat{T}^{(j-1)})^{-1}, \quad U^{(j)} = T^{(j)} T^{(j-1)} (\hat{T}^{(j-1)})^{-1} (\hat{T}^{(j)})^{-1}$$

$$U^{(j+1)} = T^{(j+1)} U^{(j)} (\hat{T}^{(j+1)})^{-1}, \text{ etc } \dots \quad (51)$$

Explicitly evaluating the conformal transformations, one finds for small $\hat{z}_j - z_j$ that

$$U^{(j-1)}(z) = z + (1 - u_j) \frac{r_j}{u_j}$$

$$U^{(j)}(z) = z + \frac{r_j}{u_j} (1 - z)$$

$$U^{(j+1)}(z) = z + z(z-1)(1 - u_{j+1}) \frac{r_{j+1}}{u_{j+1}}$$

$$U^{(j+2)}(z) = z, \quad (52)$$

where

$$u_j = u_{j-1, j}, \quad r_j = \frac{\hat{u}_j - u_j}{1 - u_j}. \quad (53)$$

In this derivation, use has been made of the equation

$$-\frac{r_j}{u_j} = \frac{r_{j-1}}{1 - u_j} = \frac{r_{j+1}}{u_{j+1}} = \frac{r_{j+2}}{1 - u_{j+1}}. \quad (54)$$

Converting the above conformal transformations into the action of L_n 's and using

$$V(\hat{z}_j) = V(z_j) U^{(j-1)} U^{(j)} U^{(j+1)}$$

one obtains

$$\frac{\partial V}{\partial z_j} = V \left[\frac{z_{j-1} - z_{j-2}}{(z_{j-2} - z_j)(z_{j-1} - z_j)} L_0^{(j-1)} + \frac{z_{j-1} - z_{j+1}}{(z_{j-1} - z_j)(z_{j+1} - z_j)} (L_{-1} - L_0)^{(j)} - \frac{z_{j+2} - z_{j+1}}{(z_j - z_{j+2})(z_{j+1} - z_j)} (L_1 - L_0)^{(j+1)} \right]. \quad (55)$$

The actual vertex is of the form

$$W = \int dz_n f(z_n) V(z_n), \quad (56)$$

and we must insist, because of unitarity, that if all legs but j are physical, i.e., satisfy Eqs. (2), and j is on-shell, i.e., is annihilated by $L_0^{(j)} - 1$, W must be annihilated by $L_{-n}^{(j)}$, in particular, $L_{-1}^{(j)}$. Using Eq. (55), and integrating by parts, we find the differential equation

$$\frac{\partial}{\partial z_j} \ln f = - \frac{1}{z_{j-2} - z_j} + \frac{2}{z_{j-1} - z_j} + \frac{2}{z_{j+1} - z_j} - \frac{1}{z_{j+2} - z_j}. \quad (57)$$

As a result, we find that

$$f(z) = \prod_{j=1}^N \frac{z_j - z_{j+2}}{(z_j - z_{j-1})^2} \quad (58)$$

In the above integration by parts, boundary terms, for which two z_i variables coincide, vanish by the usual cancelled propagator argument.

It is straightforward to verify that not only L_{-1} but L_{-2} and so L_{-n} annihilate the on-shell N -point function. We first use the identity of Eq. (34) to relate $L_{-n}^{(j)}$ on one leg, j , to terms on the other legs, i , which can be seen to be of the form $\sum_{n=0}^{\infty} a_n^{(i)} L_{n-1}^{(i)}$, $a_n^{(i)}$ being c numbers; since $L_0^{(i)} = 1$ and $L_{n>1}^{(i)} = 0$ on the physical external state i , we are left with a term of the form $a_0 L_{-1}^{(i)} + a_1$ on the i^{th} leg. The $a_0 L_{-1}^{(i)}$ term can itself be traded for a constant by using Eq. (55). It is then a matter of algebra to show that all these constants cancel. Thus, we find no further restriction on the integrand of W from L_{-2} . This is not surprising, as the old dual model bosonic trees are consistent outside the critical dimension.

Finally, we recall that for the three-string vertex no integration was necessary, and we observe that the N-string vertex was found from cycling transformations which only depend on Chan variables, i.e., cross ratios of the z_i 's. Consequently, it is invariant under a $SU(1,1)$ transformation applied to all the z_i 's. Thus, this group must be divided out from the measure, and one ends with the usual $N-3$ integration variable, and exact agreement with Ref. 2).

In the above, a new technique which depends on properties uniquely string-like is used to determine string S-matrix elements and their properties. It did not require any oscillator algebra for the actual computation of tree N-string scattering, which is reduced to an exercise in conformal group theory. As our application, we derived the transmission and reflection behaviour of the Virasoro gauge operators; this calculation when done directly using the explicit form of the S-matrix and the L_n 's in terms of oscillators would have been prohibitively complicated.

Since we have computed an S-matrix directly, we have not needed the ghost oscillators, and, more surprisingly, as will be shown in a future paper, they are not needed either for the deduction of loops. However, for some purposes, their incorporation may be of use. In fact, for the open three-string¹²⁾ and N-string¹³⁾ vertex, the ghost part was found using the present techniques.

We have used these methods to derive the vertex for multi-superstring tree scattering with arbitrary excited bosonic or fermionic excited states, the result being manifestly Lorentz invariant. We have also derived the bosonic loop including its integration measure. We will present these results and some of their consequences shortly.

REFERENCES

- 1) V. Alessandrini, D. Amati, M. Le Bellac and D. Olive - Physics Reports 1C (1971) 269.
- 2) C. Lovelace - Phys.Lett. 32B (1970) 490.
- 3) D. Olive - Nuovo Cimento 3A (1971) 399.
- 4) V. Alessandrini and D. Amati - Nuovo Cimento 4A (1971) 793;
C. Lovelace - Phys.Lett. 32B (1970) 703;
M. Kaku and L.P. Yu - Phys.Lett. 33B (1970) 166; Phys.Rev. D3 (1971) 2992, 3007, 3020.
- 5) L. Brink and D. Olive - Nucl.Phys. B58 (1973) 237.
- 6) S. Mandelstam - Physics Reports 13C (1974) 259.
- 7) M. Kaku and K. Kikkawa - Phys.Rev. D10 (1974) 1110, 1823;
E. Cremmer and J.-L. Gervais - Nucl.Phys. B90 (1975) 410.
- 8) A. Neveu and P. West - Phys.Lett. 179B (1986) 235.
- 9) S. Sciuto - Lett.Nuovo Cimento 2 (1969) 411;
L. Caneschi, A. Schwimmer and G. Veneziano - Phys.Lett. 30B (1969) 351.
- 10) A. Neveu and P. West - Phys.Lett. 180B (1986) 34;
See also: P. Di Vecchia, R. Nakayama, J.L. Petersen and S., Sciuto - NORDITA Preprint 86/8 (1986).
- 11) E. Corrigan and D. Olive - Nuovo Cimento 11A (1972) 749;
L. Brink, D. Olive, C. Rebbi and J. Scherk - Phys.Lett. 45B (1973) 379.
- 12) A. Neveu and P. West - Nucl.Phys. B278 (1986) 601.
- 13) P. Di Vecchia, R. Nakayama, J.L. Petersen, J. Sidenius and S. Sciuto - NORDITA Preprint 86/15 (1986).