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CONSTRUCTION OF AN EFFECTIVE COLOUR DIELECTRIC LATTICE ACTION
FROM QCD BY BLOCKING

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A B S T R A C T

We define a colour dielectric field by averaging over gauge field configurations in pure gluon SU(2). We determine the coupling constants of the effective colour dielectric lattice action in a Monte Carlo renormalization group (MCRG) calculation. The feasibility of mapping out the full renormalization group trajectory is discussed.

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1. - INTRODUCTION

Colour dielectrics¹⁾⁻³⁾ is based on regarding the QCD vacuum as a medium characterized by a dielectric field. It is a theoretical approach which promises to construct an effective action for a bag model directly from the QCD Lagrangian. The role of the polarizable medium is played by the high frequency parts of the gauge fields. Since this effective theory has the same appearance as electrodynamics of dielectric media, it is called colour dielectrics. From the macroscopic Hamiltonian $\mathcal{H}_{\text{eff}} = \int d^3x \frac{1}{2\epsilon} [\vec{D}_1^2 + \vec{H}_1^2]$ it is easily seen that colour displacement fields $\vec{D}_1$ and magnetic fields $\vec{H}_1$ cannot subsist in a medium where $\epsilon = 0$, because the energy associated with them would be infinite. The dynamics of confinement is reduced to find an action which makes ϵ fluctuate around zero. For heavy quark charges space will divide into two regions. The region with $\epsilon = 0$ forms a bag around the $\epsilon \neq 0$ region inside the hadron, where colour is present and suppresses the strong vacuum fluctuations. This philosophy has been very successful in bag models^{4),5)}.

Especially for a description of nuclear physics at high energies we need a description which at the same time contains the long-range features of confinement without giving up the elementary quark and gluon degrees of freedom. It is this search for an understanding of the QCD degrees of freedom which motivates the experimental work about the EMC effect, Drell-Yan production on nuclear targets and relativistic nucleus-nucleus collisions. Studying the transition from asymptotically free QCD at short distances to confining hadron dynamics presents the main theoretical challenge.

For this problem the method of block spin transformations is ideal. This method defines a sequence of lattice theories with increasing lattice size. More precisely, the high momentum variables are integrated out on each stage and put into the effective colour dielectric field and its interactions. Previously Nielsen and Patkos¹⁾ calculated one block spinning step perturbatively. Mack²⁾ has introduced the form of the effective action and a definition of the colour dielectric field which will be used by us.

In this paper the first Monte Carlo renormalization group (MCRG) calculation of the coupling constants in the colour dielectric action is reported by performing two blocking steps from a 8^4 to a 4^4 lattice and from 4^4 to 2^4 . Using a gauge fixing in each second hypercube on the lattice we determine the colour dielectric field by an arithmetic mean over the link variables connecting such

hypercubes. The expectation value of the so-defined dielectric constant $\epsilon = \langle \chi^4 \rangle$ shows a cross-over transition at $\beta \approx 2.0$. Translational invariance in the dielectric field and gauge invariance of the effective gauge field lead to Schwinger-Dyson equations which determine the coupling constants of the dielectric theory numerically. These results can be checked in the strong coupling limit. We find that the effective interaction of the dielectric field χ changes from a potential localizing χ around $\chi = 1$ to a potential where the dielectric field fluctuates around zero as is necessary for confinement. The behaviour of the plaquette coupling constant can be compared to MCRG studies⁶⁾ in the character expansion.

The outline of the paper is as follows. In Section 2 we introduce the definition of the colour dielectric field. In Section 3 we derive the Schwinger-Dyson equations. Section 4 is devoted to a presentation and discussion of the numerical results.

2. - TRANSFORMATION OF PURE SU(2) LATTICE THEORY INTO THE COLOUR DIELECTRIC LAGRANGIAN

We consider pure SU(2) lattice gauge theory with Wilson action on a four-dimensional lattice with lattice spacing a . The gluon fields are represented by SU(2) 2×2 matrices $A_\mu(x) = \frac{1}{2} \sum_{i=1}^3 A_\mu^i \sigma^i$ and the action has the form

$$S(U) = \sum_{\text{plaquettes}} \left[1 - \frac{1}{2} \text{Tr} (U_{il} U_{ek} U_{kj} U_{ji}) \right] \quad (1)$$

with the link variables $U_{ij} = \exp i \int_{x_j}^{x_i} A_\mu dx_\mu$. The partition function is given by the sum over all possible link configurations weighted with the "Boltzmann factor"

$$Z = \int \mathcal{D}U e^{-\beta S(U)}$$

Here the functional integral is to be thought of as a multiple integration over the constrained vector variables $a_0, \vec{a}$ for $U = a_0 + i \vec{\sigma} \vec{a}$ with $U^\dagger U = 1$

$$\int \mathcal{D}U = \int \prod_e da_0^e d\vec{a}_e \delta(a_0^e{}^2 + \vec{a}_e^2 - 1) \quad (2)$$

where the product runs over all links l on the lattice. The QCD coupling constant g enters into the expression $\beta = 4/g^2$. We use the heat-bath method⁷⁾ to generate the configurations for our simulations. Starting from the initial cold configuration with all $U = 1$ we work our way through the lattice and successively update each link with the probability $d^4a \delta(a^2-1) \exp \beta k a_0$, where k is the norm of the sum of six products made up from the three-neighbouring links $k = |\sum_{\alpha=1}^6 \tilde{U}_\alpha|$. We choose the heat bath method to reach equilibrium quickly. We stay with the continuous SU(2) group because we are taking arithmetic averages of link variables later. For this feasibility study we use an 8^4 lattice and typically make 100 iterations for $\beta < 2.5$ to reach equilibrium. Then we take expectation values of the relevant operators every fifth configuration from configuration 105 up to 600. To check equilibrium, we calculate the Polyakov loop

$$\langle L(x) \rangle = \left\langle \text{Tr} \prod_{t=1}^{t_n=8} U_t(x, t_n) \right\rangle$$

and find that it averages to zero. $|\langle L \rangle|$ is smaller than 0.006 for $\beta < 2.2$. For these values of β , the lattice size is large enough to incorporate large scale fluctuations, for larger β we would need a bigger lattice size. The resulting value of the dielectric constant for the large β values really corresponds to a finite temperature expectation value.

We transform the QCD Wilson action into the colour dielectric form by a blocking procedure. For a general introduction to the Monte Carlo renormalization group, see Ref. 8). We define block variables for each second hypercube of size a^4 . The real size of the lattice is thus reduced from $(8a)^4$ to $(4a)^4$. First, we do a gauge fixing in each hypercube x which is to be eliminated by maximizing the average trace of all its link variables, i.e.,

$$\text{av } \frac{1}{2} \text{Tr} \sum U_{m_x n_x} \rightarrow \max \quad (3)$$

Numerically, this is achieved by going through all sites n_x of this hypercube x and maximizing the trace of the sum of the adjacent four links. These are the links emerging from the site n_x which point into the hypercube

$$\sum_{m_{x_i}=1}^4 U_{m_{x_i} n_x} = U_{\text{sum}} \quad (4)$$

Recall that the links from site n to site m (U_{mn}) transform under a gauge transformation as

$$U_{mn} \rightarrow U'_{mn} = G_m U_{mn} G_n^{-1} \quad (5)$$

with the unitary matrices G_m, G_n . Therefore the transformation G_{n_x} needed to maximize U' is given as:

$$U'_{sum} = \left(\sum_{m_{x_i}} U_{m_{x_i} n_x} \right) G_{n_x}^{-1} \quad (6)$$

with

$$G_{n_x}^{-1} = \frac{\sum_i U_{m_{x_i} n_x}^+}{\left(\text{Det} \sum_i U_{m_{x_i} n_x}^+ \right)^{1/2}} \quad (7)$$

In our numerical calculations⁹⁾ we go through each selected hypercube 10 times. The average value of $\frac{1}{2} \text{tr} U$ saturates with the number of iterations as is shown in Fig. 1.

Of course the same gauge transformation also acts on the external links pointing outward from the sites of the chosen hypercube x . Finally all the information is in the links connecting the different blocked hypercubes $x, y, z, \dots$. In Fig. 2 we illustrate the blocking procedure in a two-dimensional picture. In four dimensions there are eight different links connecting two neighbouring blocks x and y . We take the average of these eight links to define the new block-link variable V and the scalar colour dielectric field χ

$$\Phi = \chi \cdot V = \frac{1}{8} \sum_{\substack{i=1 \\ \text{connecting}}}^8 U_i \quad (8)$$

The scalar field χ is necessary to make the new block link variables V unitary. It can be easily shown that for unitary matrices $V = b_0 + i \vec{\sigma} \cdot \vec{b}$ and $U^i = a_0^i + i \vec{\sigma} \cdot \vec{a}^i$

$$\begin{aligned} \chi^2 &= \text{Det} \left(\frac{1}{8} (\sum a_0^i + i \vec{\sigma} \cdot \sum \vec{a}^i) \right) = \\ &= \left(\frac{1}{8} \sum a_0^i \right)^2 + \left(\frac{1}{8} \sum \vec{a}^i \right)^2 \leq \max_{V_i} [(a_0^i)^2 + (\vec{a}^i)^2] = 1 \quad (9) \end{aligned}$$

Therefore the dielectric field takes values between 0 and 1. Using this definition of the new fields on the lattice with lattice constant $2a$ we write the expression for the effective action depending on $\phi = \chi \cdot V$

$$e^{-S_{\text{eff}}(\phi)} = \int \mathcal{D}U \prod_{x,y} \delta(\phi_{yx} - \frac{1}{8} \sum U_{n_y n_x}) \cdot \Delta(\text{F.P.}) \cdot \prod_x \delta(\text{gauge conditions at sites } n_x) \cdot e^{-S(U)} \quad (10)$$

Since we do not need the explicit form of the gauge condition and the necessary Fadeev-Popov determinant we have left them out in Eq. (10).

The logical definition of the block spin variable ϕ^{new} for the second blocking step is [cf. Eq. (8)]

$$\phi^{\text{new}} = \frac{1}{8} \sum_{i=1}^8 \phi_i^{\text{old}} \quad (11)$$

In the weak field limit of ϕ there are only a finite number of terms¹⁰⁾ contributing to $\delta_{\text{eff}}(\phi)$. To fourth order in ϕ these possible terms are:

$$(1) \quad \frac{1}{2} \text{Tr}(\phi_{em}^+ \phi_{em}) = \chi_{em}^2$$

$$(2) \quad \frac{1}{2} \text{Tr} \phi_{ie} \phi_{ek} \phi_{kj} \phi_{ji}$$

$$(3) \quad \left(\frac{1}{2} \text{Tr} \phi_{em}^+ \phi_{em} \right)^2 = \chi_{em}^4$$

where the fields ϕ^{ji} and χ^{ji} are associated with the links connecting the hypercube j with i as shown in Fig. 2. In $SU(2)$ with $N > 3$ there would be another term $\det \phi$ allowed. We have also investigated χ^4 couplings with the links bordering at an angle in a plaquette but we found no correlation $|\langle \chi_{\parallel}^2 \chi_{\perp}^2 \rangle - \langle \chi^4 \rangle| < 0.001$. Therefore we limit our effective action to the form:

$$S_{\text{eff}}(\phi) = \beta' \sum_{\square} \left(1 - \frac{1}{2} \text{Tr} \phi_{ie} \phi_{ek} \phi_{kj} \phi_{ji} \right) - \sum_{\substack{\text{links} \\ \text{between hypercubes}}} (m^2 \chi^2 + \lambda \chi^4) \quad (12)$$

It is an assumption that such a local form of the action suffices to model the original dynamics correctly. Work has been done in spin models¹¹⁾ by holding the block spin variables fixed to check whether the resulting effective actions have good locality properties. We think that probably as long as the block lattice spacing is smaller than the correlation length one can use a local action of the

above form. But certainly explicit calculations with the method of Ref. 11) are important to verify this assumption.

The chosen form of the effective interaction has been called colour dielectric theory because in the continuum limit the function $\varepsilon = \chi^4$ modifies the gluon energy density

$$\mathcal{H}_{eff}(x) = \frac{1}{2} \sum_i (\vec{E}_i^2 + \vec{B}_i^2) \chi^4 = \frac{1}{2} \sum_i \frac{1}{\varepsilon} (\vec{D}_i^2 + \vec{H}_i^2) \quad (13)$$

which is indeed of the same form as in electrodynamics of macroscopic media. The remaining gauge fields contain only long wave-length modes. All the high-frequency dynamics has been absorbed into the χ -field. As a first result we show the expectation values of $\langle \varepsilon \rangle = \langle \chi^4 \rangle$ as a function of $\beta = 4/g^2$ in Fig. 3. Coming from large values at large β , $\langle \varepsilon \rangle$ changes abruptly at $\beta \approx 2.0$ and becomes small in the strong coupling limit. This is indeed in agreement with the expected confining behaviour of the potential $\mathcal{V}(\chi) = m^2 \chi^2 + \lambda \chi^4$. In the second blocking step the effective dielectric constant of the vacuum $\varepsilon = \langle \chi^4 \rangle / (4a)$ approaches zero already for values of $\beta < 2.2$. This gives the hope that with a few steps one may arrive from the scaling region to a colour dielectric action which confines on the classical level.

In strong coupling the expectation value of $\langle \chi^4 \rangle$ after one blocking can be determined exactly from the random average of eight link variables. Letting $\beta \rightarrow 0$ and using the correct phase space for χ we find for each link the distribution

$$\begin{aligned} \chi^3 e^{-S(\chi)} &= \int \prod_{i=1}^8 d^4 a_i \delta(a_i^2 - 1) \delta\left(\chi - \sqrt{\frac{1}{8} \sum a_i^2}\right) = \\ &= \int f(A) 2\chi \theta(\chi) \delta(\chi^2 - A^2) d^4 A \end{aligned} \quad (14)$$

with

$$f(A) = \int \prod_{i=1}^8 d^4 a_i \delta(a_i^2 - 1) \delta\left(A - \frac{1}{8} \sum a_i\right) \quad (15)$$

The unknown distribution $f(A)$ is well represented by a Gaussian function of the form $f(A) = e^{-16A^2} \text{const}$, which gives for the effective χ -action at $\beta = 0$

$$e^{-S(\chi)} = e^{-16\chi^2} \cdot \text{const}' \quad (16)$$

The factor 16 is directly related to the eight link variables in the average. The average of $\langle \chi^4 \rangle$ is then obtained as

$$\langle \chi^4 \rangle = \int_0^1 \chi^{3+4} e^{-16\chi^2} d\chi / \int_0^1 \chi^3 e^{-16\chi^2} d\chi \approx -0.023 \quad (17)$$

Numerically we find $\langle \chi^4 \rangle = 0.0227 \pm 0.0003$ at $\beta = 0.01$. We have also plotted the effective χ distribution functions $S(\chi)$ obtained from the numerical simulations. For $\beta = 0$ the plaquette term contributes to $S(\chi)$ after integration of $S_{\text{eff}}(\chi, V)$ over V . Otherwise we see already in Fig. 4 the shape of the effective χ part of the action which is calculated in the next section.

3. - SCHWINGER-DYSON EQUATIONS AND THE DETERMINATION OF THE NEW COUPLING CONSTANTS

Recently¹²⁾ a very powerful method has been proposed to determine the couplings after a renormalization group transformation. Instead of simulating systems with different couplings in order to match various expectation values, the authors of Ref. 12) propose to solve the Schwinger-Dyson equations for the new action using the set of field configurations generated on the original lattice. We are going to apply the same method to determine β' the new gauge coupling and m^2 and λ , the mass and self-interaction of the colour dielectric field.

From the definition of the block variables ϕ we have [cf. Eq. (10)]

$$e^{-S_{\text{eff}}(\phi)} = \int \mathcal{D}u e^{-S(u)} \pi \delta(\phi - \frac{1}{8} \sum u_i) [\Delta(\text{F.P.}) \pi \delta(\text{gauge fixing})] \quad (18)$$

The new partition function $\int \mathcal{D}\phi e^{-S_{\text{eff}}(\phi)}$ gives the correct χ -measure if we express $\phi = \chi V$ with $V^\dagger V = 1$. Introducing a factor of unity we have for $V = a_0 + i\vec{\sigma} \vec{a}$

$$\int d\phi \delta(\chi - \sqrt{\frac{1}{2} \text{Tr}(\phi^\dagger \phi)}) d\chi = 2 \int d^4 a \delta(a^2 - 1) d\chi \cdot \chi^3 \theta(1-\chi) \theta(\chi) \quad (19)$$

$$\int \mathcal{D}\chi = \int \tilde{\pi}_e d\chi_e \cdot \chi_e^3 \cdot \theta(1-\chi_e) \theta(\chi_e) \quad (20)$$

The integration over χ is restricted between zero and one. In order to use an unlimited range for χ we introduce an extra integration

$$\theta(1-\chi) \theta(\chi) = \int d\alpha \frac{e^{-i(1-(2\chi-1)^2)\alpha}}{-2\pi i (\alpha + i\epsilon)} \quad (21)$$

Then we obtain the first set of Schwinger-Dyson equations from the invariance of the action under field translations $\chi \rightarrow \chi + \delta\chi$

$$\int \prod_e d\alpha_e dV_e d\chi_e \frac{\delta}{\delta\chi_i} \left\{ \prod_e \left[\frac{e^{-i[1-(2\chi_e-1)^2]\alpha_e}}{-2\pi i (\alpha_e + i\epsilon)} \chi_e^3 \right] \theta_p \cdot e^{-\mathcal{V} + \sum_{\square} S_{\square}} \right\} = 0 \quad (22)$$

with

$$\mathcal{V} = \sum_e m^2 \chi_e^2 + \lambda \chi_e^4$$

and

$$S_{\square} = \frac{\beta'}{2} \cdot \text{Tr} \prod_{\ell \in \square} V_{\ell} \chi_{\ell}$$

where Σ_{ℓ} denotes the summation over all block links and $\Sigma_{\square}$ is the sum over all plaquettes on the blocked lattice. The action of the derivative on the curly bracket produces four terms

$$\frac{\delta}{\delta\chi_i} \left\{ \right\} = I_1 + I_2 + I_3 + I_4 \quad (23)$$

$$I_1 = \left[\prod_{\ell \neq i} \chi_{\ell}^3 \theta(1 - (2\chi_{\ell} - 1)^2) \right] \delta(1 - (2\chi_i - 1)^2) (4 - 8\chi_i) \chi_i^3 \cdot \theta_p \cdot e^{-\mathcal{V} + \sum_{\square} S_{\square}} \quad (24)$$

$$I_2 = \left[\prod_{\ell \neq i} \chi_{\ell}^3 \theta(1 - (2\chi_{\ell} - 1)^2) \right] \theta(1 - (2\chi_i - 1)^2) 3\chi_i^2 \cdot \theta_p \cdot e^{-\mathcal{V} + \sum_{\square} S_{\square}} \quad (25)$$

$$I_3 = \left[\prod_e \chi_e^3 \theta(1 - (2\chi_e - 1)^2) \right] \frac{\partial \mathcal{O}_p}{\partial \chi_i} e^{-\mathcal{V} + \sum S_\square} \quad (26)$$

$$I_4 = \left[\prod_e \chi_e^3 \theta(1 - (2\chi_e - 1)^2) \right] \cdot \mathcal{O}_p \cdot \left[-\frac{\partial \mathcal{V}}{\partial \chi_i} + \sum_{\square_i} \left(\frac{\beta'}{2} \prod_{\ell \in \square_i} \text{Tr} (V_\ell \chi_\ell) \frac{1}{\chi_i} \right) \right] e^{-\mathcal{V} + \sum S_\square} \quad (27)$$

In the last term the summation $\sum_i$ is over all plaquettes which contain the link i . The first term has to be transformed back into the form of the original integral. We use the trick that

$$\delta(1 - (2\chi - 1)^2) = \frac{d}{d\varrho} \theta(\varrho - (2\chi - 1)^2) \Big|_{\varrho=1} \quad (28)$$

to convert the δ function into a θ function. Then we rescale and shift χ to get rid of ρ in the θ function

$$\tilde{\chi} = \frac{1}{\sqrt{\varrho}} (\chi - 1/2) + 1/2 \quad (29)$$

This yields

$$\begin{aligned} I_1 &= \frac{d}{d\varrho} \int \prod_e dV_e \int \prod_{\ell \in i} \chi_\ell^3 \theta(1 - (2\chi_\ell - 1)^2) d\chi_\ell \cdot \\ &\int d\tilde{\chi}_i \sqrt{\varrho} \left[\sqrt{\varrho} (\tilde{\chi}_i - \frac{1}{2}) + \frac{1}{2} \right]^3 \cdot \mathcal{O}_p(\tilde{\chi}_i, \varrho) \cdot \left[4 - 8[\sqrt{\varrho} (\tilde{\chi}_i - \frac{1}{2}) + \frac{1}{2}] \right] \cdot \\ &\cdot \theta(1 - (2\tilde{\chi}_i - 1)^2) \cdot e^{-\mathcal{V}[\tilde{\chi}_i, \sqrt{\varrho}] + \sum S_\square(\tilde{\chi}_i, \varrho)} \end{aligned} \quad (30)$$

$$\begin{aligned} I_1 &= \int dV d\chi e^{-\mathcal{V} + \sum S_\square} \cdot \\ &\left\{ \left[16 - 20\chi_i - \frac{3}{\chi_i} \right] \cdot \mathcal{O}_p \right. \\ &\quad + \left[4\chi_i - 4\chi_i^2 - 1 \right] \frac{d\mathcal{O}_p}{d\chi_i} \\ &\quad \left. + \left[4\chi_i - 4\chi_i^2 - 1 \right] \cdot \mathcal{O}_p \left[-\frac{\partial \mathcal{V}}{\partial \chi_i} + \frac{\beta'}{2} \sum_{\square_i} \text{Tr} \prod_{\ell \in \square_i} (V_\ell \chi_\ell) \frac{1}{\chi_i} \right] \right\} \end{aligned} \quad (31)$$

and

$$I_2 = \int \mathcal{D}V \mathcal{D}\chi e^{-\mathcal{V} + \sum S_0} \left(\frac{3}{\chi_i} \right) \quad (32)$$

$$I_3 = \int \mathcal{D}V \mathcal{D}\chi e^{-\mathcal{V} + \sum S_0} \left(\frac{d\phi_p}{d\chi_i} \right) \quad (33)$$

$$I_4 = \int \mathcal{D}V \mathcal{D}\chi e^{-\mathcal{V} + \sum S_0} \left(\phi_p \cdot \left(-\frac{dV}{d\chi_i} + \frac{\beta'}{2} \sum_{\square_i} \text{Tr} \prod_{\ell \in \square_i} (V_\ell \chi_\ell)^{\frac{1}{\chi_i}} \right) \right) \quad (34)$$

We simplify the notation by denoting the averages with the Boltzmann weight $\int \mathcal{D}V \mathcal{D}\chi e^{-\mathcal{V} + \sum S}$ by brackets $\langle \rangle$. The first set of Schwinger-Dyson equations reads:

$$\begin{aligned} & \langle (4 - 5\chi_i) \phi_p \rangle + \langle \chi_i (1 - \chi_i) \frac{d\phi_p}{d\chi_i} \rangle + \\ & \langle \chi_i (1 - \chi_i) \cdot \phi_p \cdot \left(-\frac{dV}{d\chi_i} + \frac{\beta'}{2} \sum_{\square_i} \text{Tr} \prod_{\ell \in \square_i} (V_\ell \chi_\ell)^{\frac{1}{\chi_i}} \right) \rangle = 0 \end{aligned} \quad (35)$$

For the numerical analysis we will take the average over all links χ_i in each configuration in addition to the "ensemble" average over configurations. The second set of Schwinger-Dyson equations comes from the invariance of ϕ or V with respect to gauge transformations. Its derivation follows lines similar to Ref. 13) and is summarized in Appendix A. Using a single plaquette with ϕ links one has [cf. Eqs. (A9) and (A10)]

$$\begin{aligned} & \frac{\beta'}{2} \left\langle \sum_{\square} \text{Tr} \phi_1^\dagger \phi_7^\dagger \phi_6^\dagger \phi_5^\dagger \phi_4 \phi_3 \phi_2 \phi_1 - \text{Tr} \phi_5 \phi_6 \phi_7 \phi_1 \phi_4 \phi_3 \phi_2 \phi_1 \right\rangle \\ & = 3 \langle \text{Tr} \phi_4 \phi_3 \phi_2 \phi_1 \rangle. \end{aligned} \quad (36)$$

The first combination of eight links is arranged in the straight window: the second corresponds to the crossed window. Note that $\phi_1 \phi_1^\dagger = \chi_1^2$. Besides this difference the above equation resembles the Migdal-Makeenko equation¹⁴⁾. It alone suffices to determine β' .

For the χ Eq. (35) we use the operators $\phi_p = 1, \chi_i, \chi_i^2, \chi_i^3$ and the potential $\mathcal{V} = \sum_{\chi} m^2 \chi_{\chi}^2 + \sum_{\chi} \lambda \chi_{\chi}^4$. These four operators generate four equations for the three coupling constants m^2, λ, β' , namely (N_L = number of links)

$$\frac{1}{N_L} \sum_{i=1}^{N_L} \left\{ m^2 \langle 2\chi_i^{2+n} (\chi_{i-1}) \rangle + \lambda \langle 4\chi_i^{4+n} (\chi_{i-1}) \rangle \right. \\ \left. + \frac{\beta'}{2} \left\langle \sum_{\square_i} \pi (V\chi)^4 \chi_i^n (1-\chi_i) \right\rangle - \left\langle (5+n) \chi_i^{n+1} - (4+n) \chi_i^n \right\rangle \right\} = 0 \quad (37)$$

for $n = 0, 1, 2, 3$

Thus the moments of χ and the modified plaquette expectation value determine the coupling constants. The set of equations we use is over-determined. It will consist of the five equations [Eqs. (37)] for $n = 0, 1, 2, 3$ and Eq. (36) taken for each of the 100 configurations in our ensemble. We will then group this ensemble into subgroups of 25 and estimate an error.

Equation (37) can be simply checked in the limit of $\beta = 0$. Using the input distribution $P(\chi) = \chi^3 e^{-m^2 \chi^2 - \lambda \chi^4}$, Eq. (37) is easily derived by partial integration

$$\begin{aligned} \langle (2m^2 \chi + 4\lambda \chi^3) \chi^{n+1} (\chi-1) \rangle &= \int_0^1 d\chi (\chi^3 e^{-m^2 \chi^2 - \lambda \chi^4}) (2m^2 \chi + 4\lambda \chi^3) \chi^{n+1} (\chi-1) = \\ &= \int d\chi \cdot \chi^3 \cdot e^{-m^2 \chi^2 - \lambda \chi^4} [(5+n) \chi^{n+1} - (4+n) \chi^n] = \langle (5+n) \chi^{n+1} - (4+n) \chi^n \rangle \end{aligned} \quad (38)$$

The systematic errors of the method with Schwinger-Dyson equations come from two sources. One is the finite size of the lattice used in the Monte Carlo simulation. The other arises from the finite number of equations of motion which are considered. To limit the finite size effect we are constrained to β values $\beta < 2.5$. The second error was checked by adding or leaving out some Schwinger-Dyson equations. We found no improvement on increasing the number of operators beyond χ^3 .

4. - NUMERICAL RESULTS AND DISCUSSION

We have taken expectation values of the moments of χ and the plaquettes: every fifth sweep from sweep Nr 105 to sweep Nr 500. We solved the Schwinger-Dyson equations [Eqs. (36) and (37)] for the different subensembles of 25 configurations we had stored. The time correlation of the Polyakov loop showed that for $\beta < 2.2$ every 10th sweep is statistically uncorrelated (corr. < 0.05). For the larger β values the decay times are larger $i_t > 25$. The total number of sweeps is really too small for these values. Solving the 100 equations for $\beta'(2a)$, $m^2(2a)$, $\lambda(2a)$ we obtain the values listed in Table 1. The errors are the standard deviations obtained from the four bunches of 25 configurations. They vary between 1% to 5%. One can see quite clearly that at $\beta < 2.0$ the effective χ potential changes from a broken symmetry shape with the minimum near $\chi = 1$ to a potential for χ with an absolute minimum at $\chi = 0$. The shapes of the effective χ potentials are plotted in Fig. 5. Note that at $\chi = 1$ there is the hard wall from the θ function. At the smallest β value, the shape of the numerical solution is identical to the strong coupling $\beta = 0$ solution.

The behaviour of the $\beta'(2a)$ values in the colour dielectric action needs some more thinking to be interpreted. At first sight $\beta'(2a)$ increases in comparison to $\beta(a)$ which contradicts our intuition about the gauge-coupling $g^2 = 4/\psi$, becoming stronger with larger lattice constant. One can, however, transform the colour dielectric action into a form which contains only plaquettes in order to understand the β transformation better. The effective action for the new SU(2) links V [cf. Eq. (12)] is approximately given by a fluctuating coupling constant $\beta'\chi^4$ as:

$$e^{S_{eff}(V)} \approx \int d\chi P(\chi) e^{\beta'\chi^4 \sum \frac{1}{2} \text{Tr} \pi V_e} = \int Z(t, \chi) P(\chi) d\chi \quad (39)$$

with

$$Z(t) = e^{\chi^4 t}$$

and

$$P(\chi) = \chi^3 e^{-\sum \frac{m^2}{\epsilon} \chi^2 - \sum \lambda \chi^4}$$

Now let us make a cumulant expansion for $Z(t)$

$$\ln Z(t) = \ln Z(0) + \left. \frac{d \ln Z(t)}{dt} \right|_{t=0} \cdot t + \dots \quad (40)$$

then we obtain

$$e^{S_{\text{eff}}(V)} = Z_0 e^{\beta' \frac{1}{2} \text{Tr} \sum_a \pi V_e \langle \chi^4 \rangle_{\text{red}} + \beta'^2 \frac{3}{8} [\langle \chi^8 \rangle_{\text{red}} - \langle \chi^4 \rangle_{\text{red}}^2] \text{Tr}_A \pi V_e \cdot \frac{1}{3}} \quad (41)$$

where we used the identity relating the fundamental and adjoint representation:

$$|\text{Tr} \pi_e V_e|^2 - 1 = \text{Tr} \pi_e V_e \quad (42)$$

The effective plaquette coupling $\beta_{\text{PL}} = \beta' \langle \chi^4 \rangle_{\text{red}}$ and adjoint coupling $\beta_A = 3/8 \beta'^2 [\langle \chi^8 \rangle_{\text{red}} - \langle \chi^4 \rangle_{\text{red}}^2]$ can be compared with MCRG calculations of a character expansion of the action^{6),15)}. We show the values of β_{PL} and β_A in Table 2. The MCRG flow is different. Our curve approaches the strong coupling limit from the positive β_A axis. This may be due to a different blocking procedure. The asymptotic freedom value for $\Delta\beta$

$$\Delta\beta = \frac{-11}{6\pi^2} \ln 4 = -0.258$$

is approached at $\beta_{\text{PL}}(a) = 2.55$ with $\beta_{\text{PL}}(a) - \beta_{\text{PL}}(2a) = -0.33$. The trend to smaller β_{PL} is obvious for $\beta < 2.55$ from Fig. 6, where we show the MCRG flow in the $\beta_{\text{PL}}, \beta_A$ plane. $\beta > 2.55$ values are no longer trustworthy, probably because the effective χ action near $\chi = 1$ does not contain sufficient higher order terms in χ . Therefore as $\langle \chi^4 \rangle_{\text{eff}}$ becomes too large, the effective potential should be more repulsive. One possibility for a more faithful effective action in this region would be to include the phase space cut-off explicitly which arises from the average over eight SU(2) matrices which are near the identity. Further work in this region should do that.

To explore the behaviour under blocking we have also included a second MCRG-step. The accuracy is, of course, considerably reduced on a 2^4 lattice. Typical errors in m^2 and λ are 10% in $\beta'(4a)$ even higher, since $\beta' \langle \chi^4 \rangle$ with a very small χ^4 enters in the action. Following the results of Table 1 one now recognizes that after two blocking transformations the $\beta(a) = 2$ starting point gives a confining colour dielectric action. This result is promising since in reality, even with

more computational effort, no more than three steps are realistic. One can even see that the chain starting at $\beta = 2.2(a_0)$ leads after two blockings to $m^2(4a_0) = -5(1)$ which is rather close to the chain starting at $\beta = 2.0(2a)$, which gives after one blocking $m^2(4a_0) = -4.7(2)$. The λ values are more separate. Rewriting the colour-dielectric action in the continuum limit one finds

$$\int \mathcal{L} d^4x = \int d^4x \left[(\lambda - \beta') \chi^4 \frac{1}{a^4} + \frac{m^2}{a^4} \chi^2 + \frac{\beta'}{4a^2} \chi^2 (\partial_\mu \chi)^2 + \chi^4 \frac{1}{2} \text{Tr} F_{\mu\nu} F_{\mu\nu} \right] \quad (43)$$

So an effective mass of the χ field would be obtained by taking $\beta'/4a^2 \langle \chi^2 \rangle_{\text{red}}$ as σ_V^2 and $m^2/a^4 = \frac{1}{2} m^2 \sigma_V^2$. Consequently

$$m_\chi^2 = \frac{8m^2}{a^2 \cdot \beta' \langle \chi^2 \rangle_{\text{red}}} \quad (44)$$

Completing the above chains:

a_0	$\rightarrow$	$2a_0$	$\rightarrow$	$4a_0$	$\rightarrow$	$8a_0$
$\beta = 2.2$		$\beta = 2.0$		$m^2 = -4.7 (2)$		$m^2 = 10$
				$\lambda = 11.9 (2)$		$\lambda = 31$
				$\beta_{\text{PL}} = 1.10(6)$		$\beta' \langle \chi^2 \rangle_{\text{red}} = 0.38$

gives a colour dielectric confining action at $8a_0$ with $m^2 = 10$; $(2a_0)^2 \kappa \approx 1$ where $\sqrt{\kappa}$ is the string tension, this can be interpreted as

$$m_\chi^2 = 13.15 \kappa \quad (45)$$

or $m_\chi = 1.4 \text{ GeV}$ for $\sqrt{\kappa} = 400 \text{ MeV}$.

Phenomenologically $m_\chi = 0.72 \text{ GeV}$ was used in the colour dielectric bag model¹⁶⁾. The σ_V^2 comes out very small which produces very little surface energy for the bag. The above derivation gives an action with a $\chi^2 (\partial_\mu \chi)^2$ kinetic term.

The present study gives a clear picture of how confinement arises in the colour dielectric model. We showed that two block spinning steps are sufficient to reach a confining colour dielectric action from the weak coupling domain. Semi-quantitative results have been obtained with a moderate numerical effort (15 CPU hours IBM). Certainly a more extensive study on a larger lattice is feasible and worthwhile. It may lead the way to an understanding of quark confinement into bag-like objects based on lattice QCD.

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APPENDIX A

We evaluate the expectation value of a closed loop consisting of links $\phi_1 \cdot \phi_2 \dots \phi_i \dots \phi_n$, recall $\phi_i = \chi_i V_i$ with $\phi_i^\dagger \phi_i = \chi_i^2$

$$\begin{aligned} & \int \prod_{\ell \in i} \pi d\phi_\ell d\phi_i \text{Tr}(\phi_1 \dots \phi_i \dots \phi_n) \exp \left\{ -\mathcal{V} + \sum_{\square}' S_{\square} + \right. \\ & \qquad \qquad \qquad \left. + \frac{\beta'}{4} \sum_{\pi \in \Pi_i} \text{Tr}[\phi_i \phi_\pi + \phi_\pi^\dagger \phi_i] \right\} = \\ & = \int \prod_{\ell \in i} \pi d\phi_\ell d\phi_i \text{Tr}(\phi_1 \dots (1 + \theta\sigma + \frac{1}{2}(\theta\sigma)^2 \dots) \phi_i \dots \phi_n) \exp \left\{ -\mathcal{V} + \sum_{\square}' S_{\square} \right\} \\ & \quad \cdot \exp \left[\frac{\beta'}{4} \left(\sum_{\pi \in \Pi_i} \text{Tr}(1 + \theta\sigma + \frac{1}{2}(\theta\sigma)^2 \dots) \phi_i \phi_\pi + h.c. \right) \right] \end{aligned} \quad (A1)$$

We exploit the invariance of the measure under the gauge transformation $d\phi_i = d(e^{\theta\sigma} \phi_i)$. Combine the above equation in second order of θ

$$\begin{aligned} \theta = & \langle \text{Tr} \phi_1 \dots \frac{1}{2}(\theta\sigma)^2 \phi_i \dots \phi_n \rangle \\ & + \frac{\beta'}{4} \langle \text{Tr}(\phi_1 \dots (\theta\sigma) \phi_i \dots \phi_n) \sum_{\pi} [\text{Tr}(\theta\sigma \phi_i) \phi_\pi + h.c.] \rangle \\ & + \frac{\beta'}{8} \langle \text{Tr}(\phi_1 \dots \phi_i \dots \phi_n) \sum_{\pi} [\text{Tr}(\theta\sigma)^2 \phi_i \phi_\pi + h.c.] \rangle \\ & + \left(\frac{\beta'}{4}\right)^2 \langle \text{Tr} \phi_1 \dots \phi_i \dots \phi_n \frac{1}{2} \sum_{\pi_1} \sum_{\pi_2} [\text{Tr}(\theta\sigma) \phi_i \phi_{\pi_1} + h.c.] [\text{Tr} \theta\sigma \phi_i \phi_{\pi_2} + h.c.] \rangle \end{aligned} \quad (A2)$$

with the first-order equation

$$\begin{aligned} & \langle \text{Tr}(\phi_1 \dots \phi_i \dots \phi_n) \text{Tr} \sum_{\pi_1} (\theta\sigma \phi_i \phi_{\pi_1} + h.c.) \rangle = \\ & = \langle \text{Tr}(\phi_1 \dots (1 + \theta\sigma) \phi_i \dots \phi_n) \sum_{\pi_1} \text{Tr}[\theta\sigma (1 + \theta\sigma) \phi_i \phi_{\pi_1} + h.c.] \rangle \\ & \quad \cdot \left(1 + \frac{\beta'}{4} \sum_{\pi_2} (\text{Tr} \theta\sigma \phi_i \phi_{\pi_2} + h.c.) \right) \end{aligned} \quad (A3)$$

Use the terms of order θ^2 in Eq. (A3)

$$\begin{aligned} 0 = & \langle \text{Tr } \phi_1 \dots \theta \sigma \phi_i \dots \phi_n \sum_{\pi_1} [\text{Tr } \theta \sigma \phi_i \phi_{\pi_1} + \text{h.c.}] \rangle \\ & + \langle \text{Tr } \phi_1 \dots \phi_i \dots \phi_n \sum_{\pi_1} [\text{Tr } (\theta \sigma)^2 \phi_i \phi_{\pi_1} + \text{h.c.}] \rangle \\ & + \frac{\beta'}{4} \langle \text{Tr } \phi_1 \dots \phi_i \dots \phi_n \text{Tr } \sum_{\pi_1} (\theta \sigma \phi_i \phi_{\pi_1} + \text{h.c.}) \cdot \text{Tr } \sum_{\pi_2} (\theta \sigma \phi_i \phi_{\pi_2} + \text{h.c.}) \rangle \quad (\text{A4}) \end{aligned}$$

Substitution of (A4) into (A2) eliminates the two last terms of (A2):

$$\begin{aligned} 0 = & \langle \text{Tr } \phi_1 \dots (\theta \sigma)^2 \phi_i \dots \phi_n \rangle \\ & + \frac{\beta'}{4} \langle \text{Tr } \phi_1 \dots (\theta \sigma) \phi_i \dots \phi_n \text{Tr } \sum_{\pi} (\theta \sigma \phi_i \phi_{\pi} + \text{h.c.}) \rangle \quad (\text{A5}) \end{aligned}$$

Now recalling that θ is complex and σ Hermitian one can choose

$$\theta_e = i \delta_{em} \quad \text{and} \quad (\theta \sigma) = i \delta_{em} \sigma_m = i \sigma_e$$

$$(\theta \sigma)^2 = (-1) \sum_e \sigma_e^2 = -3 \cdot \mathbb{1}$$

Rewriting (A5) with matrix indices gives:

$$\begin{aligned} 0 = & (-3) \langle \text{Tr } \phi_1 \dots \mathbb{1} \phi_i \dots \phi_n \rangle \\ & + \frac{\beta'}{4} \langle \text{Tr } (\phi_1 \dots i \sigma_{ab}^e \phi_i \dots \phi_n) \sum_{\pi} [i \sigma_{cd}^e (\phi_i \phi_{\pi})_{dc} - i (\phi_{\pi}^+ \phi_i^+)_{dc} \sigma_{cd}^e] \rangle \quad (\text{A6}) \end{aligned}$$

Now we use the identity for σ matrices:

$$\sum_e \sigma_{ab}^e \sigma_{cd}^e = 2 \delta_{ad} \delta_{bc} - \delta_{ab} \delta_{cd} \quad (\text{A7})$$

$$\begin{aligned} 0 = & 3 \langle \text{Tr } \phi_1 \dots \phi_i \dots \phi_n \rangle \\ & + \frac{\beta'}{2} \langle \sum_{\pi} (\phi_i \dots \phi_n \phi_1 \dots)_{ba} \\ & \quad \{ (\phi_i \phi_{\pi})_{ab} - (\phi_{\pi}^+ \phi_i^+)_{ab} \} \rangle \quad (\text{A8}) \end{aligned}$$

since $\text{Tr} \{ (\phi_1 \phi_{\pi}) - (\phi_{\pi}^+ \phi_1^+) \} = 0$ in $\text{SU}(2)$.

The last equation has, for one plaquette $\phi_4\phi_3\phi_2\phi_1$ containing ϕ_1 , the form:

$$\begin{aligned} & \frac{\beta'}{2} \left\langle \sum_{\tilde{\square}} \text{Tr} (\phi_1^+ \phi_7^+ \phi_6^+ \phi_5^+ \phi_4 \phi_3 \phi_2 \phi_1 - \phi_5 \phi_6 \phi_7 \phi_1 \phi_4 \phi_3 \phi_2 \phi_1) \right\rangle \\ &= 3 \langle \text{Tr} \phi_4 \phi_3 \phi_2 \phi_1 \rangle. \end{aligned} \quad (\text{A.9})$$

where the summation goes over all plaquettes $\tilde{\square}$ with links $\phi_1, \phi_5, \phi_6, \phi_7$ containing ϕ_1 . Graphically, Eq. (A.9) can be represented as

$$\frac{\beta'}{2} \sum_{\tilde{\square}} \left(\text{Tr} \begin{array}{c} \begin{array}{|c|c|c|c|} \hline & 3 & & \\ \hline 4 \downarrow & 1 & \uparrow 2 & \\ \hline 5^+ \downarrow & 1^+ & \uparrow 7^+ & \\ \hline & 6^+ & & \\ \hline \end{array} \\ - \text{Tr} \begin{array}{|c|c|c|c|} \hline & 3 & & \\ \hline 4 \downarrow & & \uparrow 2 & \\ \hline 5 \downarrow & 1 & \uparrow 7 & \\ \hline & 6 & & \\ \hline \end{array} \end{array} \right) = 3 \cdot \text{Tr} \begin{array}{|c|c|c|c|} \hline & 3 & & \\ \hline 4 \downarrow & & \uparrow 2 & \\ \hline & 1 & & \\ \hline \end{array} \quad (\text{A.10})$$

$\beta(a)$	0.01	1.0	1.5	2.0	2.2	2.55
$\beta'(2a)$	0.4(3)	3.7(2)	5.6(2)	7.08(5)	7.35(5)	7.54(6)
$m^2(2a)$	13.5(2)	5.6(2)	1.4(1)	-4.7(2)	-10.9(4)	-36.0(5)
$\lambda(2a)$	6.6(6)	7.3(4)	8.1(1)	11.9(2)	17.1(3)	35.8(3)
$\beta'(4a)$	109(281)	-10(45)	17(37)	25(7)	19(1)	15.0(4)
$m^2(4a)$	128(3)	70(3)	41(2)	10(1)	-5(1)	-66(3)
$\lambda(4a)$	3(90)	13(33)	44(15)	31(3)	33(2)	88(3)

TABLE 1 - Results for the effective couplings after blocking transformations.

β	0.01	1.0	1.5	2.0	2.2	2.55
$\epsilon = \langle \chi^4 \rangle (2a)$	0.0227(3)	0.0585(7)	0.1050(5)	0.219(1)	0.312(2)	0.462(2)
$\epsilon = \langle \chi^4 \rangle (4a)$	$3.7(2)10^{-4}$	$1.15(5)10^{-3}$	$2.77(8)10^{-3}$	0.0156(6)	0.052(2)	0.230(4)
$\beta_{PL}(2a)$	0.01(1)	0.21(1)	0.56(3)	1.10(6)	1.4(1)	2.2(1)
$\beta_A(2a)$	$\pm 0.2 \cdot 10^{-3}$	± 0.03	0.13(2)	0.34(3)	0.33(3)	0.3(1)
$\beta_{PL}(4a)$	± 0.10	± 0.05	± 0.11	0.4(2)	0.8(1)	2.4(4)
$\beta_A(4a)$	± 0.2	$\pm 0.6 \cdot 10^{-2}$	± 0.05	0.07(5)	0.19(3)	0.30(3)

TABLE 2 - Values of the dielectric constant as $\epsilon = \langle \chi^4 \rangle$ and effective coupling constants β_{PL} , β_A as a function of β .

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FIGURE CAPTIONS

Fig. 1 Evolution of the expectation value $\langle \frac{1}{2} \text{Tr} U \rangle$ inside the hypercubes (which are eliminated) as a function of the iterations.

Fig. 2 Schematic picture of the block transformation.

Fig. 3 Dielectric constant $\varepsilon = \langle \chi^4 \rangle$ as a function of β .

Fig. 4 Shape of the effective action $S(\chi)$ obtained from the probability distribution $P(\chi)$, $P(\chi) = e^{-S(\chi)} \chi^3$.

Fig. 5 Shape of the effective χ action $\mathcal{V}(\chi)$

Fig. 6 Values of β_{PL} and β_{A} after one and two blockings.

- Figure 1 -

- Figure 2 -

- Figure 3 -

- Figure 4 -

- Figure 5 -

- Figure 6 -