CERN-TH.4707/87

A GROUP THEORETIC METHOD FOR STRING LOOP DIAGRAM

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A B S T R A C T

The new approach to arbitrary string scattering proposed by the authors is used to compute the planar tadpole operator, including its integrand, for the open bosonic string. The Virasoro gauge identities play a crucial role and are found to contain in general anomalous terms for open strings.

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In a recent paper¹⁾, hereafter referred to as I, a new method of computing the perturbative string S-matrix for arbitrary external states was given. In I, it was explicitly demonstrated how to recover the open bosonic trees. In a separate paper, we will discuss the general strategy for loop computations and explicitly derive the one loop diagrams for arbitrary excited states of the open bosonic string, as well as discuss their conformal properties. Here, we limit ourselves to show that the method works by computing the simplest example, namely the planar tadpole^{2),3)}. This result includes the correct determination of the integration measure over the Teichmüller parameter. It is clear from this example that the technique can be used to determine any loop diagram.

We will not repeat the discussion of the method, and we refer the reader to I. The tadpole diagram W is described by a bra-state $V(w)$, where w is a real parameter, and is of the form

$$W = \int dw V(w) \bar{f}(w) \quad (1)$$

Corresponding to going once around the loop, there is a cycling transformation T such that

$$V(w)T = V(w)$$

Also, for an operator R of conformal dimension d , the tadpole satisfies the overlap¹⁾

$$V(w) [R(z) - S(z)R(\tau z)] = 0 \quad (2)$$

up to a possible normal ordering constant. By a similar discussion to that used for the tree graphs, one can determine that

$$S(z) = S_{\tau}(z) = \left[\frac{z}{\tau(z)} \frac{d\tau(z)}{dz} \right]^d \quad (3)$$

In order to find an explicit form for T , we consider the distribution⁴⁾ of the charge and its images given in Fig. 1. We can thus consider the loop as a tree with an infinite set of external lines, and, by analogy with I, we take

$$T = \frac{z-1}{z(1-u)} \quad (4)$$

where u is the Chan variable between two adjacent charges, and is given by

$$1-u = \frac{w}{(1+w)^2} \quad (5)$$

Note that since u is a cross-ratio, it is the same no matter what two adjacent charges we choose.

Consequently, for $R = Q^\mu$, we have, inverting T ,

$$V(w) \left[Q^\mu(z) - Q^\mu \left(\frac{1}{1 - \frac{w}{(1+w)^2}} \right) \right] = 0 \quad (6)$$

While it would be correct to start from this relation, it is more convenient, for comparison with past literature, to carry out a gauge transformation of the form $G = \alpha^{L_0 - L_1 - 1}$ on $V(w)$, and so modify the cycling transformation. Since this gauge transformation leaves physical states inert, we are not affecting the final result. Using the relation

$$G R(z) G^{-1} = R(z') \left(\frac{z}{z'}, \frac{dz'}{dz} \right)^d \quad (7)$$

where

$$z' = \frac{\alpha z}{1 - (1-\alpha)z}$$

we find that after the gauge transformation, Eq. (2) becomes

$$V(w) \left[R(z) - \left(\frac{zw}{1+wz} \right)^d R(1+wz) \right] = 0 \quad (8)$$

In particular, we find the identity

$$V(w) \left[Q^\mu(z) - Q^\mu(1+wz) \right] = 0 \quad (9)$$

which in fact uniquely determines $V(w)$ to be

$$V(w) = \exp \left\{ -\frac{1}{2} (a | \Gamma(\mathcal{P}-1) | a) - \frac{1}{2 \ln w} [a | 1+w]^2 \right\} \quad (10)$$

where we use the notations of Ref. 3), with which this result agrees. On this result the reader may verify that of course the overlap identity for $R = P^\mu(z)$ is satisfied.

Hence, even though the oscillator form of the vertex is rather complicated, the overlap identities it obeys and which determine it are very simple.

As discussed in I, the identities for $R(z) = \sum_{n=-\infty}^{+\infty} L_{-n} z^n$ can also be obtained by applying directly conformal transformations on $V(w)$. Unlike in the

tree level case, no power of the T's is the identity, but the strategy used in I to find vertices at different values of the Koba-Nielsen variables can be utilized here. The tadpole with a different value of w , say $\hat{w}$, will satisfy $V(\hat{w})T(\hat{w}) = V(\hat{w})$. There exists a transformation $H(w, \hat{w})$ which takes $V(w)$ to $V(\hat{w})$, namely

$$V(w) H(w, \hat{w}) = V(\hat{w}) \quad (11)$$

The action of cycling transformations on the two vertices implies

$$H(w, \hat{w}) T(\hat{w}) = T(w) H(w, \hat{w}) \quad (12)$$

The T's are conformal transformations of the variable z , and as such, H is also realized as a conformal transformation on z . If $\hat{w} = w + \delta w$, where δw is small, we can represent H by

$$H: z \rightarrow z + g(z) \delta w \quad (13)$$

Eq. (12) now becomes

$$z + g(1+wz) = w g(z) \quad (14)$$

Examining the above argument for a transformation H which does not change w , we find the condition on g to be instead:

$$g(1+wz) = w g(z) \quad (15)$$

We now must solve these equations, beginning with Eq. (14). Defining $t = \ln(z-1/1-w)$, and

$$g(t) = e^t j(t) - \frac{1}{(1-w)^2} \quad (16)$$

we find that Eq. (14) is equivalent to

$$\frac{1}{w} + j(t + \ln w) = j(t) \quad (17)$$

However, we must also demand that the solution be single valued in z , in which case we also have

$$j(t + 2\pi) = j(t) \quad (18)$$

Of course, such functions are well known. One example of interest to us which has only one simple pole in its fundamental domain is constructed from the Weierstrass ζ -function⁵⁾:

$$j(t) = \frac{1}{\omega} \bar{\zeta}(t - r) + \text{const.} \quad (19)$$

where

$$\bar{\zeta}(t) = \zeta(t) - \frac{t}{2\pi} \eta \quad (20)$$

r is a constant, and $\eta = \zeta(i\pi)$. The "periods" of ζ are $2i\pi$ and $\ln\omega$.

Equation (15) is solved by taking

$$g(t) = e^t \bar{j}(t) \quad (21)$$

provided that

$$\bar{j}(t + \ln\omega) = \bar{j}(t), \quad \bar{j}(t + 2i\pi) = \bar{j}(t) \quad (22)$$

That is, $\bar{j}$ is an automorphic (in this case, elliptic) function, and so can be constructed also out of combinations of ζ with appropriate arguments.

Taking $\bar{j}(t) = 1$ in Eq. (22), we find the resulting identity:

$$V \left(L_0 - \frac{L-1}{1-\omega} \right) = 0 \quad (23)$$

Using $j(t)$ in Eq. (19), we find the identity

$$\frac{\partial V}{\partial \omega} = V \left\{ \oint \frac{dz}{z} \left[\left(z - \frac{1}{1-\omega} \right) \frac{1}{\omega} \bar{\zeta}(u) - \frac{1}{(1-\omega)^2} \right] \frac{L(z)}{z} - c \right\} \quad (24)$$

where

$$u = \ln [1 - (1-\omega)z] \quad (25)$$

corresponds to the choice of r required to place the pole of $\bar{\zeta}$ at $z = 0$ and c is the normal ordering constant. Using Eq. (20), we find that the identity becomes

$$\frac{\partial V}{\partial w} = V \left[\frac{L_2}{w(1-w)^2} - \frac{3L_1}{2w(1-w)} - \frac{L_1}{(1-w)^2} + 6 \left(\frac{1}{2w} - \frac{1}{12w} - \frac{1}{12w} \right) - c \right] \quad (26)$$

+ terms involving $L_{n \geq 1}$

The constant c is determined by putting the vacuum with zero momentum on each side of the equation, and using the form of V , Eq. (10), from the Q^μ -overlap. One finds

$$c = - \frac{D}{w} \sum_{n=1}^{\infty} \frac{w^n}{(1-w^n)^2} + \frac{D}{2w \ln w} \quad (27)$$

This corresponds to $V(w)$ having a weight factor in its conformal properties.

To find the integration measure, we place the one external line on shell and demand that W of Eq. (1) vanish when the line is a spurious state. Taking $L_{-1} | \Psi \rangle$ trivially gives zero as a result of Eq. (23), and so the first Virasoro gauge gives no condition on the integration measure. However, taking $L_{-2} | \Psi \rangle$ leads to the on-shell result^{*)}

$$\frac{\partial V}{\partial w} = V \left[\frac{L_2}{w(1-w)^2} + \frac{1}{w} + (D-2) \frac{\partial}{\partial w} \ln f(w) + \frac{D}{2w \ln w} + \frac{1}{1-w} \right] \quad (28)$$

Here, we have used the result

$$\eta = -i\pi \left[2w \frac{\partial}{\partial w} \ln f(w) + \frac{1}{2} \right] \quad (29)$$

where

$$f(w) = \prod_{n=1}^{\infty} (1-w^n) \quad (30)$$

Consequently, we find that the second Virasoro gauge condition determines the integration measure to be

$$\bar{f}(w) = w^{-2} f^{-D+2}(w) (\ln w)^{-\frac{1}{2}D} (1-w)^{-1} \quad (31)$$

We have evaluated the integrand for the general one loop graph. The method is identical, and the result will be published elsewhere.

^{*)} This is a heuristic derivation, since there is no state at zero momentum with $L_0 = -1$. It nevertheless gives the correct answer, Eq. (31).

It is interesting to note that no mention has been made of ghosts, but still the result has the correct reduction in the number of factors of the partition function. When a multiperipheral tree is attached to this tadpole, it is straightforward to verify that the standard answer is recovered for the one-loop planar diagram, in all details; in particular, the $(1-w)^{-1}$ factor of Eq. (31) is reabsorbed in a change of variables²⁾. The actual value of D is not determined by the plasma graph. Rather, $D = 26$ comes from demanding that physical closed string poles occur in the orientable non-planar diagram⁶⁾. This minimal dependence on $D = 26$ of the present method contrasts sharply with that found in string field theory and may bode well for applications to string theories outside 10 and 26-dimensions.

The reader will notice that we have neglected one important term in the derivation of the measure given in Eq. (31), namely, when we integrate by parts, there is the boundary term. Due to the fact that the tadpole has only one leg, these terms are difficult to discuss in its case. As such, we will give these boundary effects for an arbitrary one-loop graph, the complete results being given later.

Following the above steps for a planar one-loop graph with N external legs, one finds the following expression

$$b g^N \int \prod_{i=1}^N dx_i V w^{-2} f^{-24}(w) \left(\frac{\ln w}{-2\pi} \right)^{-13} \prod_{i=1}^N \frac{1}{1-x_i}$$

and V is the vertex. After performing the Jacobi transformation

$$q = \exp \frac{2\pi^2}{\ln w}, \quad v_i = \frac{\ln x_i}{\ln w} \quad (32)$$

the amplitude is

$$W = b \frac{g^N}{\pi} \int \prod_{i=1}^{N-1} dv_i \theta(v_i, v_i) \int_0^1 \frac{dq}{q^3} \left(-\frac{2\pi^2}{\ln q} \right)^N f^{-24}(q^2) \prod_{i=1}^N \frac{1}{1-x_i} V \quad (33)$$

b being a constant determined by unitarity which includes the Chan-Paton group factor $\text{tr} \lambda^1 \lambda^2 \dots \lambda^N$, and V having the form

$$V = \prod_{i,j} \left[\psi_{ij} \frac{1}{\sqrt{(1-x_i)(1-x_j)}} \right]^{P_i \cdot P_j} M \quad (34)$$

where

$$\begin{aligned} \psi_{ij} &= -\frac{2\pi}{\ln q} \sin \pi(v_j - v_i) \prod_{n=1}^{\infty} \frac{1 - 2q^{2n} \cos 2\pi(v_j - v_i) + q^{4n}}{(1 - q^{2n})^2} \\ &= -\frac{2\pi}{\ln q} f_{ij}(q^2) \end{aligned} \quad (35)$$

The factor M is of the form

$$\langle 0 | \exp \sum_{i,j} (a^{(i)})^\dagger (1-x_i) H^{ij} (1-x_j) a^{(j)} \rangle \quad (36)$$

and H^{ij} are essentially normalized derivatives²⁾ of ϕ_{ij} with respect to the x 's, there being $n + m$ such derivatives in H_{nm}^{ij} , $n + m > 0$.

The L_{-2} identity of the above graph with all lines on shell reads

$$\begin{aligned} V_{L_{-2}}^{(1)} = & \omega (1-x_1)^2 \left\{ \frac{x_N}{\omega} \frac{\partial V}{\partial x_N} - \frac{\partial}{\partial \omega} \ln \left[\omega \left(-\frac{\ln \omega}{2\pi^2} \right)^{13} f^{24}(\omega) \right] \right. \\ & \left. - \sqrt{\frac{x_N}{\omega}} \frac{\partial}{\partial x_N} \ln(1-x_N) \right\} \quad (37) \end{aligned}$$

The expression of Eq. (33) has divergences whose physical origin is by now well understood. The one at $w = 0$ comes from the tachyon propagating in the loop and is due to an inappropriate integral representation of the open string propagator. It can be removed by analytic continuation, which is achieved by a suitable integration by parts. The other divergences, at $q = 0$, can be understood as coming from the planar loop containing the possibility of a tachyon or an on shell zero momentum closed string dilaton disappearing into the vacuum. The tachyon divergence can also be removed by analytic continuation^{7),8)} leaving the only one of interest, namely the one due to the dilaton. Performing this latter analytic continuation in q through partial integration gives W of the form

$$\begin{aligned} W = & \frac{g^N}{\pi} \left\{ -\frac{1}{2} \int_0^1 d\nu_i \theta(\nu_{i+1} - \nu_i) \frac{1}{q^2} \left(-\frac{2\pi^2}{\ln q} \right)^N f^{24}(q^2) \frac{\sqrt{V}}{\prod_{i=1}^N (1-x_i)} \right\}_{q=1} \\ & + \frac{1}{2} \int_0^1 \frac{dq}{q^2} \int_0^1 d\nu_i \theta(\nu_{i+1} - \nu_i) \frac{\partial}{\partial q} \left[\left(-\frac{2\pi^2}{\ln q} \right)^N \frac{1}{f^{24}(q^2)} \frac{\sqrt{V}}{\prod_{i=1}^N (1-x_i)} \right] \quad (38) \end{aligned}$$

Using Eq. (37) after a Jacobi transformation, we find the action of L_{-2} , on line 1 say, with all external lines on shell and physical to reduce to

$$\begin{aligned} W_{L_{-2}}^{(1)} = & \frac{g^N}{\pi} \left(\frac{1}{2} q \frac{\partial}{\partial q} - 1 \right) \int_0^1 d\nu_i \theta(\nu_{i+1} - \nu_i) \frac{1}{q^2} \left(\frac{\ln q}{-2\pi^2} \right)^{-N+2+\frac{1}{2}\sum p_i^2} \\ & f^{24}(q^2) \prod_{i=1}^N \left(\frac{1}{1-x_i} \right)^{-\frac{1}{2}P_i^2+1} (1-x_i)^2 \prod_{i,j} f_{ij}^{P_i \cdot P_j}(q^2) M \Big|_{q=0} \quad (39) \end{aligned}$$

Now, as all states are on shell, $N-1$ of them satisfy $L_0 = 1$, but leg one on which L_{-2} was applied consists of a state which has $L_0 = +1$. A particular case of Eq. (39) is obtained by taking $L_{-2} |0\rangle$ on line 1 and $|0\rangle$ on all other lines. Then, $M = 1$, and $\sum_1 p_i^2/2 = N-2$, and it reduces to

$$W L_{-2}^{(1)} |0\rangle = \frac{\theta q^N}{\pi} \int dv_i \delta(v_{i+1} - v_i) \frac{\partial}{\partial q^2} \tilde{F}(q^2) \Big|_{q=0} \quad (40)$$

which, as $\tilde{F}$ is a meromorphic function of q^2 , is a finite result.

Consider now arbitrary on shell physical external legs, which have a state of level N_k on leg k . The oscillator algebra brings down, according to Eq. (36), an additional factor $[(1-x_k) \ln q]^{N_k}$ times a meromorphic function of q^2 . The $(\ln q)^{N_k}$ follows from the fact that the matrix H_{nm}^{ij} is a derivative with respect to the x_i 's of ϕ_{ij} . Consequently, the non-meromorphic factors disappear for the reason discussed above. Thus, even for excited states, the result can be put in the form of Eq. (40) for a suitable meromorphic F_{ij} .

The computation for the non-orientable graphs is much the same, except for a different choice of Θ -function, and slightly different factors in the measure. However, the final result, Eq. (39) is of the same form on shell. The orientable non-planar graphs are finite by themselves and do not give rise to any anomaly. The planar and non-orientable graphs taken together contain a factor

$$[2^{-13} (\kappa + 1)] \kappa \lambda_1 \lambda_2 \dots \lambda_N$$

where $\lambda = 1, 0, -1$ for symplectic, unitary and orthogonal groups respectively^{9), 10)}. Consequently, this anomalous term vanishes only for $SO(2^{13})$. The final steps of the above calculation are similar to those that occur in the cancellation of divergences at one loop^{11), 8)}. Of course, as in usual quantum field theory, it is not necessary to cancel anomalies between graphs; one may also try to remove them by counterterms. It has been noted in Ref. 12) that the planar one-loop infinities can be removed by a finite counterterm obtained by going to a de Sitter space of small curvature, and it is possible that such a mechanism also removes the above anomalous terms.

In our approach, it is clear that at one loop the anomalous term can come only from a boundary of the integration region over the modulus, the measure being determined from a differential equation.

Almost by definition, an anomaly can only arise as a result of a need to regulate infinities, and it has been noticed by utilizing the Polyakov approach that there is a direct connection between divergences and BRST-anomalies¹³⁾, both being associated with the boundary of moduli space.

In this paper we have demonstrated that the group theoretic method proposed in I can compute loop graphs for arbitrary external states, including the integration measure. The method can also derive properties of S-matrix elements without having to know their detailed structure. In particular, one finds that in general the graphs of the open bosonic string have anomalous terms in their conformal properties. In accord with general considerations, these are closely related to the divergences, and cancel only for $SO(2^{13})$, as do the divergences.

We hope it is clear to the reader that the general technique used here can be applied to any loop graph. For each loop there is one cycling relation. In this sense, each loop is treated separately. The presence of the other loops is felt only through extra periodicity requirements, beyond those of Eqs. (17) and (18) which the function $g(z)$ involved in the conformal properties of the graph must satisfy. The reader will also have realized that we have obtained an explicit expression for the measure without any discussion of ghosts.

Note added

After the completion of this work, we saw a paper by C. Callan, C. Lovelace, C. Nappi and S.A. Yost, "Adding holes and crosscaps to the superstring", Princeton preprint PUPT-1045 (1987) which also discussed from a very different point of view the BRST-anomalies in open string theory.

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FIGURE