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ON THE EQUILIBRIUM OF A HEAVY STRING
WITH A GAS OF LIGHT STRINGS

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Abstract

The thermodynamics of a gas of light strings in equilibrium with a heavy string, all contained in a box and at a temperature above the Hagedorn temperature, is described. Lower and upper bounds for the mass of the heavy string are derived. Validity limits of the thermodynamic approach are discussed. As an illustration the general formulae are applied to a particular case - the heterotic string.

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There is a variety of string models. It is believed that in some of them the string gas at sufficiently high energy density has an unusual energy distribution: one string has most of the energy, while the others form a gas of light strings in equilibrium with the heavy one [1], [2], [3]. Further increasing the energy of the gas at constant volume increases the mass of the heavy string without much affecting the light strings. Moreover, the gas cannot be in equilibrium with a heat bath, whatever the temperature of the bath [4]. In the present paper we give a thermodynamic description of the phase with the heavy string (H-phase). The two main points are

- The heavy string is an analogue of the heat bath.
- Each subset of strings with a given mass is discussed separately.

This eliminates the convergence difficulties encountered, when trying to describe the string gas at high energy density as a whole.

For high mass strings the degeneracy of the N -th string mass level is given by the formula

$$P(N) = C N^{-(a+1)/2} e^{b\sqrt{N}}, \quad (1)$$

where a , b , C are model-dependent positive constants. The mass levels corresponding to these degeneracies are

$$M = c \sqrt{N/\alpha'} , \quad (2)$$

where c is another model-dependent constant and α' is the string parameter believed to be roughly of order unity, when the mass scale is chosen so that the Planck mass $m_{Pl} = 1$. From (1) and (2) one obtains for the density of mass levels

$$\rho(m) = 2C(c^2/\alpha')^{(a-1)/2} m^{-a} e^{m/T_H} . \quad (3)$$

The constant $T_H = c/(b\sqrt{\alpha'})$ is known as the Hagedorn temperature¹. This density of states increases so fast, that in spite of the canonical damping factor $\exp(-m/T)$ the sum over masses in the partition function diverges for $T > T_H$. Because for $T < T_H$ the H-phase is not stable, the application of statistical thermodynamics to the description of the H-phase requires some care. Since the pioneering papers on the statistical bootstrap, where formula (3) also holds and where the H-phase has been analyzed, [5], [6] the usual procedure has been to use the microcanonical ensemble and to work with approximations to phase space integrals [1], [3]. For a recent attempt to use thermodynamics in a simplified model cf. [2].

Consider a gas of strings in equilibrium in the H-phase. The gas is contained in a $D+1$ dimensional spherical vessel of volume V and has fixed energy E . The degeneracy (number of quantum states) of the gas is some function $\Gamma(E,V)$. Entropy can be calculated from Boltzmann's formula $S = \ln \Gamma$ and temperature can be defined by the identity $dE = TdS$ (cf.e.g. [4])

¹ We use a temperature scale, where the Boltzmann constant $k = 1$.

where the heat transfer, to which dE and dS correspond must be reversible. The temperature cannot depend on whether the energy supplied is used to increase the mass of the heavy string or to increase the kinetic energy of the strings. Otherwise, the entropy of the gas could be increased by converting mass into kinetic energy or vice versa, contrary to the assumption that the system is in equilibrium i.e. has maximum entropy possible at energy E . Thus the temperature can be calculated from small changes of the mass of the heavy string. Using formulae (1) and (2), one finds

$$T = T_H [1 - (a+1)T_H/M]^{-1} \approx T_H, \quad (4)$$

where M is the mass of the heavy string. The temperature is slightly higher than T_H , but when the heavy string is very heavy, one can use the approximation $T \approx T_H$.

Consider now the gas of massless strings², which is a subsystem of the gas in the box. For this system the grand partition function is of course well defined also for $T > T_H$. The standard argument gives

$$\ln Z_0 = V(2\pi)^{-D-1} \sum \epsilon_i \int d^{D+1}p \ln(1 + \epsilon_i e^{-p/T}). \quad (5)$$

The summation is over the quantum states of a massless string at a given momentum and $\epsilon_i = +1$ for fermionic states and -1 for bosonic states. The integrals are well known and one finds the free energy

² Negative energy states, if any, are excluded from our analysis.

$$F_0 = -T \ln Z_0 = -a_0 V T^{D+2}, \quad (6)$$

where

$$a_0 = [n_B + (1-2^{-D-1})n_F] D! \zeta(D+2) / [2^D \pi^{(D+1)/2} \Gamma(D/2+1/2)], \quad (7)$$

n_B is the number of bosonic states and n_F the number of fermionic states, both for a massless string at given momentum. Knowing the contribution of the gas of massless strings to the free energy, one can calculate from standard thermodynamic identities the corresponding contribution to any thermodynamic function. For instance, the contribution to the energy of the gas is

$$E_0 = F_0 - T \partial F_0 / \partial T = -(D+1) F_0 \quad (8)$$

and the contribution to the heat capacity at constant volume

$$C_{V0} = -T \partial^2 F_0 / \partial T^2 = -(D+2)(D+1) F_0 / T. \quad (9)$$

Our standard example will be the heterotic string [7], which is reputed to be phenomenologically the most promising one. For the heterotic string $D=8$, $n_B = n_F = 4032$ and $T_H^{-1} = \pi(2 + \sqrt{2})\sqrt{\alpha'}$. Thus $a_0 = (31/15)\pi^5$ and

$$F_0 \approx 3.14 \cdot 10^{-8} V_\alpha (T/T_H)^{D+2} (\alpha')^{-1/2}, \quad (10)$$

where $V_\alpha = V \alpha'^{-(D+1)/2}$.

At fixed temperature the energy of the gas of massless strings fluctuates. The average is given by formula (8) and the mean square fluctuation is $T^2 C_{V0}$, which is small compared to E^2 if

$$V \gg (D+2)/[a_0(D+1)T_H^{D+1}] . \quad (11)$$

This is the necessary condition for the applicability to the gas of massless strings of the standard thermodynamics, where both temperature and energy are well defined. Condition (11) is not necessary to derive averages like (8). For the heterotic string this condition reduces to $V_\alpha \gg 3.3 \cdot 10^6$. Another limitation is that we have used the quasiclassical approximation to count the translational states of the strings. Thus the radius of the box must be much larger (in units where $\hbar = c = 1$) than the inverse of the temperature

$$R \gg T_H^{-1} . \quad (12)$$

For the heterotic string this condition reads $R \gg 10\alpha'^{-1/2}$ and since $V=3.30R^9$, one finds $V_\alpha \gg 6.2 \cdot 10^9$ - a condition stronger than that following from (11). Formally it is easy to remove the quasiclassical approximation. Then, however, the results become strongly dependent on the shape of the box and on the boundary conditions, which makes the general considerations not very useful.

For massive strings we will use the non-relativistic approximation. The relativistic calculation is straight-forward, but it leads to more complicated formulae without giving more physical insight. For strings of mass m corresponding to the n -th mass level of the string one finds

$$\ln Z_n = C_m CV(2\pi)^{-D-1} n^{-(a+1)/2} e^{-m/T_H} \int d^{D+1} p e^{-m/T_e p^2/(2mT)} . \quad (13)$$

Here C_m is the ratio of the degeneracy of the n -th level (corresponding to mass m) to its asymptotic approximation (1). For not too high values of n this may be a significant correction. The dependence of C_m on n for $n \leq 100$ is shown in Fig 1. The calculation of C_m uses $C = 2^{-11/4}$, which has been obtained taking the coefficient C for the bosonic string given by Huang and Weinberg [8] and an identity for the Jacobi function $\Theta_4(0, z)$ given e.g. in ref. [2]. For the contribution to the free energy one finds

$$F_n = -a_n VT^{(D+3)/2} n^{-(a/2-D/4+1/4)} (\alpha')^{-D/4-1/4} e^{m(1/T_H-1/T)} , \quad (14)$$

where

$$a_n = CC_m [c/(2\pi)]^{(D+1)/2} . \quad (15)$$

Since $T \approx T_H$, the exponential factor is unimportant in the low mass region unless differentiation with respect to temperature is considered. Without this factor the sum of contributions (14) for $n = 1, 2, \dots$ converges if

$$a > D/2 + 3/2 . \quad (16)$$

This is the condition for the existence of the H-phase originally derived by Frautschi [5] for $D = 2$ and more recently generalized to more dimensions [1], [2], [3]. When this condition is satisfied, and the mass M of the heavy string is sufficiently large, the gas of low mass strings may be described using formula (6) and the approximations to F_n :

Figure 1: Ratio of the degeneracy of the n -th level of a heterotic string to the asymptotic estimate (1) of this degeneracy. The parameters are $C = 2^{-11/4}$, $a = 10$, $b = 2\pi(2 + \sqrt{2})$.

$$F'_n = -a_n V_\alpha (T\sqrt{\alpha'})^{(D+3)/2} n^{-a/2+D/4-1/4} (\alpha')^{-1/2} . \quad (17)$$

For the heterotic string $C = 2^{-11/4}$, $c = 2$ and $a = 10$, so that formula (17) reduces to

$$F'_n = -2.62 \cdot 10^{-9} C_m V_\alpha n^{-11/4} (T/T_H)^{11/2} (\alpha')^{-1/2}. \quad (18)$$

Summing over n one finds

$$F = \Sigma F_n = -10^{-10} [314(T/T_H)^{10} + 1.6(T/T_H)^{11/2}] V_\alpha (\alpha')^{-1/2}. \quad (19)$$

The second term in the square bracket is the contribution of the massive states. It is about 0.5 per cent. The corresponding contributions to the energy and to the heat capacity are about 2 per cent and about 6 per cent respectively. The number of strings in the n -th mass state can be obtained as the ratio of the total energy of these strings to the average energy of one such string. The result is

$$P_n = -F'_n / T_H, \quad (20)$$

which for the heterotic string gives $2.8 \cdot 10^{-8} C_m V_\alpha n^{-13/4}$. This supports the picture of the H-phase [2] as composed mainly of the gas of massless strings in equilibrium with the heavy string.

Formula (20) is applicable only for $m \ll M$. An extension to higher masses can be found as follows. The probability, assumed very small, that the heavy string decays into two strings is equal to the ratio of the degeneracy of the two-string final state to the degeneracy of the parent string.

Thus using formula (1) for the heavy string and the quasiclassical approximation to count the states of the relative motion of the two strings, the decay probability is

$$P(M, m) = C_m (c/\sqrt{\alpha'})^{a+1} m^{-(a+1)} V (2\pi)^{-D-1} \int C_{M-m-p^2/2\mu} [M/(M-m-p^2/2\mu)]^{a+1} e^{-p^2/(2\mu T_H)} d^D p. \quad (21)$$

Here p is the momentum of the relative motion of the two final strings and μ is their reduced mass. It is enough to consider $m < M/2$. Then p^2 in the square bracket and in the subscript of C can be neglected and

$$P(M, n) = C_m C_{M-m} (c/\sqrt{\alpha'})^{a+1} (2\pi)^{-(D+1)/2} m^{-a-1} \mu^{(D+1)/2} T_H^{(D+1)/2} V [M/(M-m)]^{a+1}. \quad (22)$$

For $m \ll M$ this reduces to (20) with F'_n taken from (17).

The H-phase corresponds to a maximum, possibly local, of entropy only if, for a small transfer of energy from the heavy string to the gas of light strings, the temperature of the gas increases at least as much as that of the string. Calculating the heat capacity of the heavy string from the formula $1/C_V = -T^2 \partial^2 S / \partial M^2$ one finds the stability condition

$$\Sigma C_{Vn} \leq M^2 T_H^{-2} / (a+1). \quad (23)$$

In particular for the heterotic string this condition reduces to

$$V_\alpha \leq 3.25 \cdot 10^5 M^2 \alpha'. \quad (24)$$

Thus in the present approximation for every volume V there is a minimum mass M of the heavy string, which makes the H-phase stable. Using formula (4) this minimum mass can be converted into a maximum temperature. Combining conditions (24) and (12) one finds $N \gg 5000$. Therefore, the excitation of the heavy string must be very high in order to justify the quasiclassical description. On the other hand, conditions (24) and (12) give $N \gg 3$. Therefore, the condition for the applicability of standard thermodynamics is not very restrictive.

Let us now comment on the validity limits of the present analysis. When at fixed volume the energy of the gas increases, gravitational collapse must happen at some point³. Thus there is an upper limit on M . A rigorous calculation of this upper limit would be difficult. A rough estimate from above can be obtained as follows. We assume that all the mass of the string gas, which is approximately M , is concentrated at the centre of the box and that the box must have collapsed, when the radius of the horizon created by mass M is larger than the original radius of the box. From a formula given by Acetta and Gleiser [9] for the radius of the horizon of a spherically symmetrical black hole of mass M in $D+1$ space dimensions, one finds

$$M < D\pi^{(D-1)/2} [8\Gamma(D/2+1/2)]^{-1} R^{D-1} m_{Pl}^D. \quad (25)$$

³ The resulting black hole cannot be considered as a way of describing the H-phase of the string gas, because at large mass the entropy $S \sim M^2$ for the black hole and $S \sim M$ for the H-phase of the string gas.

Combining the inequalities (25) and (23) one gets an upper limit on R if $D < 3$ and a lower limit if $D > 3$. For the heterotic string one finds $M < 4.7(m_{Pl}R)^7 m_{Pl}$ and $R > 0.05(m_{Pl}\sqrt{\alpha'})^{-11/5} m_{Pl}^{-1}$.

An alternative approach to collapse would be to compare the entropies of a black hole and of the string gas each at energy E . The system with the higher entropy should be stable. This has been done in an approximate way in ref [9] and the conclusion is that the strings are stable up to very high masses. This is much higher than what we get from our estimate. Since our estimate is very conservative, this suggests that the neglect of interactions, which is implicit in the calculation of the entropy of the strings, becomes unjustified for strings in a box at energies much below the transition range found in ref [9].

Another upper limit on M comes from the obvious requirement that the size of the heavy string should not exceed the size of the box [4]. For the closed bosonic string, Mitchel and Turok [3] found a root mean square radius

$$r = [(D+1)2\pi\alpha']^{1/2}(n/6D)^{1/4} . \quad (26)$$

Applying this as an order of magnitude estimate to the heterotic string one finds

$$M \ll 0.25 R^2(\alpha')^{-3/2} , \quad (27)$$

which is a very strong limitation. When combined with (24) it yields $V_\alpha \ll 3 \cdot 10^9$ and would exclude the applicability of the quasiclassical approximation. The difficulty of squeezing a heavy string into a small box casts some doubt on the physical relevance of the H-phase for heterotic strings [4], but in order to make a firm statement, one would have to know the analogue of formula (26) for the heterotic string.

All the bounds derived in the present paper for the H-phase of heterotic strings are summarized in Fig. 2. A reduction of the mean string radius by a factor of about two in comparison with estimate (26) would be necessary to produce an allowed region. If an allowed region exists, it is limited by conditions (12), (23) and the analogue of (26). Note that except for condition (23), these are conditions for the applicability of the model of quasiclassical non-interacting strings in a classical box and not on the existence of a long string in equilibrium with short ones.

Like its predecessors, the present analysis has important limitations. Let us concentrate on one of them - the handling of the conservation laws. The heavy string is so big that it can easily compensate the momentum and the shifts of the centre of mass of the gas of light strings. Because of the necessity of this compensation, however, we never integrated over the phase space of the heavy string. On the other hand, defining the entropy of the heavy string as the logarithm of its degeneracy, one implies summation over all angular momenta, electric charges, colours etc. Conservation of these quantum numbers would thus have to be secured by emission of massless and very low-energy strings, which somehow should not contribute to the calculated probabilities. Otherwise one should impose constraints on

Figure 2: Regions in the V - M plane forbidden for the H-phase of heterotic strings: C - stability with respect to gravitational collapse, S - stability with respect to internal energy transfer, ΔE - smallness of energy fluctuations, Q - applicability of the quasiclassical approximation, $?$ - condition that the string is smaller than the box. Units are chosen so that $\alpha' = 1$. Curve C is calculated with the additional assumption that $m_{Pl} = 1$. Otherwise it should be shifted to the right by $8 \ln m_{Pl}$.

the quantum numbers of the strings involved. Frautschi [5] showed that constraints on scalar additive quantum numbers like baryon number, electric charge and strangeness affect the spectrum of fireballs by increasing the parameter a without changing the parameter b . On the other hand according to [10], when for strings angular momenta are constrained to be less than some arbitrary but fixed J_0 , not only a strongly increases, but also the exponent bm changes to $2\sqrt{bm}$. Thus constraints strongly reduce the decay probability. Usually closed strings satisfy (16) and open strings do not (cf. e.g. [2]). The foregoing comment suggests that it would be necessary to take into account more of the dynamics, before concluding with any certainty that open strings in a box are unstable with respect to decays. Also the discussion of the transition between a string and a Schwarzschild black hole [11] would be changed if constraints were imposed on the electric charge and angular momentum of the string. Whether this is necessary is a dynamical question, which cannot be settled by thermodynamics alone.

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