The Background Field Method and the Non-Linear $\sigma$-Model

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ABSTRACT

We prove the renormalizability in the background field method of the two-dimensional bosonic non-linear $\sigma$-model with an arbitrary Riemannian manifold as target space. Particular attention is paid to the question of non-linear renormalization of the quantum fields and its effect on subdi-ergences. We give an algorithm that allows one to compute to arbitrary loop order.

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1. Introduction

The background field method is a useful computational tool in quantum field theories, particularly gauge theories, which allows one to compute radiative corrections while maintaining manifestly the symmetries of the theory under consideration [1]. For example, in non-Abelian gauge theories, the fact that the coupling constant renormalization is related to the background field wave function renormalization allows one to compute the $\beta$-function by calculating corrections to the propagator. It should be noted, however, that background gauge invariance is not a substitute for BRS invariance [2]; the latter has to be used for the quantum fields in order to establish multiplicative renormalizability of the non-Abelian gauge theory. It is this property, combined with a simple Ward Identity arising from the linear background-quantum split, that allows one to calculate quantities of interest by computing 1PT graphs with no external quantum lines without having to bother about renormalization of the quantum fields.

In many theories of interest, in particular two-dimensional non-linear $\sigma$-models and supersymmetric gauge theories, the background-quantum split is non-linear and the quantum field is not merely multiplicatively renormalized, due to the fact that it has canonical dimension zero. These circumstances require a modification of the background field method as they imply that renormalization of the quantum fields cannot be neglected in higher loop computations of the effective action with only external background lines. This is because the counterterms due to quantum field renormalizations do not cancel, nor can they simply be read off from lower loop calculations with no external quantum lines. In this paper, we study this question in the specific context of two-dimensional bosonic non-linear $\sigma$-models.

Before getting down to the real nitty-gritty, it may be useful to illustrate the issues involved in the context of four-dimensional $\phi^4$ theory. The action is

$$S = \int d^4x \left( \frac{1}{2} (\partial_{\mu} \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right)$$

(1.1)

and the generating functional for connected Green's functions $W[J]$ is defined by

$$e^{iW[J]} = \int \mathcal{D}\phi \exp i(I[\phi] + \int J \phi)$$

(1.2)
The generating functional for 1PI graphs is related to $W$ by a Legendre transformation

$$W[J] = \Gamma[\phi] + \int J\phi$$

(1.3)

where the argument of $\Gamma$ is the vacuum expectation value of the quantum field $\phi$ in the presence of the source $J$. We now split the total field $\phi$ into a background field $\varphi(x)$ and a quantum field $\pi(x)$

$$\phi(x) = \varphi(x) + \pi(x)$$

(1.4)

and define a new functional [3] $\bar{W}[\varphi, J]$ by

$$e^{i\bar{W}[\varphi, J]} = \int D\pi \exp i(S[\varphi + \pi] + \int J\pi).$$

(1.5)

Evidently

$$\bar{W}[\varphi, J] = W[J] - \int J\varphi.$$  

(1.6)

Further, we define a new $\Gamma$-functional by taking the Legendre transformation with respect to $J$ only,

$$\bar{\Gamma}[\varphi, \pi] = \bar{W}[\varphi, J] - \int J\pi.$$  

(1.7)

It is not difficult to show using the trivial shift symmetry $\delta\varphi(x) = \eta(x)$, $\delta\pi(x) = -\eta(x)$ that $\bar{\Gamma}$ depends only on $\varphi + \pi$, i.e.

$$\frac{\delta\bar{\Gamma}[\varphi, \pi]}{\delta\varphi(x)} = \frac{\delta\bar{\Gamma}[\varphi, \pi]}{\delta\pi(x)}$$

(1.8)

and that

$$\bar{\Gamma}[\varphi, 0] = \Gamma[\varphi].$$

(1.9)

(1.9) is the key equation; it states that the standard 1PI functional can be computed by calculating the 1PI background functional with no external quantum $\pi$ lines.

So far, all our considerations have been formal and we have not taken into account the effects of renormalization. Expanding out the interaction term in the action, one finds

$$\frac{\lambda}{4!}\phi^4 = \frac{\lambda}{4!} \left\{ \varphi^4 + 4\varphi^3\pi + 6\varphi^2\pi^2 + 4\varphi\pi^3 + \pi^4 \right\}$$

(1.10)

and in principle these various vertices could be renormalized differently. Now of course this doesn’t happen because of the linear-splitting Ward Identity (1.8); furthermore (1.8) also tells us that the wave-function renormalizations for $\varphi$ and $\pi$ are actually the same,

$$Z_\varphi = Z_\pi.$$

(1.11)
Thus, the Ward identity (1.8) has the consequence that the counterterms are functionals of the total field $\phi$, and may be deduced from graphs with no external quantum lines. In particular, the renormalization of the various vertices involving the quantum field is performed by renormalizing $\lambda$ as deduced from diagrams with only external background lines and then substituting the corresponding bare $\lambda$ into the right hand side of (1.10). The renormalization of the quantum field $\pi$ can also be deduced from graphs with only external background lines, as expressed in (1.11), but in fact these multiplicative renormalizations cancel out in graphs with no external quantum lines. This can clearly be seen diagramatically: the factors of $Z_\pi$ cancel between the propagators and vertices.

In the rest of this paper, we study the problem of background field method renormalization for two dimensional bosonic non-linear $\sigma$-models. In section 2, we review the $\sigma$-model and the background quantization method based on geodesics. In section 3, we study the generalization of the Ward identity (1.8). Although the shift symmetry is still Abelian, it is now non-linear, and this fact requires the use of BRS techniques. From the shift Ward identity, we find the forms of the counterterms: in addition to the expected metric counterterms which are functions of the total field, there are additional non-linear renormalizations of the quantum field. In section 4, we give an algorithm for computing these latter renormalizations. This involves the use of a modified source term which allows us to calculate all renormalizations from graphs without external quantum lines. The procedure is illustrated by a two-loop calculation.

2. The Non-Linear $\sigma$-Model

Let $\Sigma$ be two-dimensional spacetime and $M$ be a Riemannian manifold with metric $g$. Then the $\sigma$-model field is a map $\phi : \Sigma \to M$ represented in local coordinates by $\phi^i(x)$. The Lagrangian for the model is

$$L = \frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j.$$  \hspace{1cm} (2.1)

The quantization of general $\sigma$-models in two space-time dimensions based on the Lagrangian (2.1) has been discussed from a number of different points of view. Friedan proved the renormalizability of the $\sigma$-model in [4]. His approach considered fluctuations of the $\sigma$-model fields about a constant background. He introduced the notion of renormalizing the metric $g_{ij}$ and showed that the counterterms calculated in an expansion about a given...
constant background are geometrical expressions related to those for nearby constant backgrounds in a natural way. The other approaches to the quantization of $\sigma$-models have made use of various forms of the background field method [5,6]. This is the most convenient way to carry out renormalization calculations, and counterterms have been derived using the method in a number of theories at low loop orders. The great advantage of the background field method is that it allows one to restrict one's attention to a reduced subset of the set of all operators that might a priori be expected to mix in the renormalization. Of course, this requires a proof that renormalization can be carried out within the reduced subset of operators. In particular, the implications of the background field method at higher loop orders in $\sigma$-models have not so far been carefully investigated. This is the issue to which we now turn.

The action $I = \int d^2x \; L$ is not invariant under any symmetries for a general target manifold $M$, but it is reparameterization invariant in the sense that $L$ has the same form in any coordinate chart. This can be rephrased slightly: let $f$ be a diffeomorphism of $M$ onto itself; then it induces a new map, $\phi' = f \circ \phi$, and we have

$$I[f \ast g, \phi'] = I[g, \phi]$$

(2.2)

or, infinitesimally,

$$I[g - \mathcal{L}_v g, \phi + v] = I[g, \phi]$$

(2.3)

where $v$ is the vector field generating the diffeomorphism and $\mathcal{L}_v$ denotes the Lie derivative. Evidently, a diffeomorphism only induces a symmetry of $I$, $I[\phi'] = I[\phi]$, if it is an isometry, $\mathcal{L}_v g = 0$. The standard generating functional of connected Green's functions is defined as before

$$e^{iW[I]} = \int D\phi \ exp i \left\{ I[\phi] + \int d^2x \ J_i(x) \ \phi^i(x) \right\}$$

(2.4)

but the source term clearly spoils reparameterization invariance.

Formally, the effect of a diffeomorphism is given by

$$e^{iW_2-\mathcal{L}_v g[I]} = \int D\phi \ exp i \left\{ I[g, \phi] + \int J \cdot (\phi + v) \right\} .$$

(2.5)

Since coupling the source to a function of $\phi$ leads to the same $S$-matrix, one concludes that the latter is reparameterization invariant and that models with metrics which are related by a diffeomorphism are physically equivalent.
It is possible to avoid spoiling reparameterization invariance in the Green's functions by the source term if one uses an unconventional source term \([4] \int d^2x \ h(x; \phi(x))\). The functional

\[
\delta W[h] = \int \mathcal{D}\phi \ exp \ i \{I[g, \phi] + \int h(x; \phi)\}
\]

(2.6)

will be reparameterization invariant provided that \(h(x; \phi(x))\) is defined to transform as a scalar. Note that the explicit functional dependence of \(h\) on \(x^\mu\) and \(\phi^i(x^\mu)\) means that \(\int h(x; \phi)\) is equivalent to a sum of an infinite number of source terms coupling to all powers of the quantum field. Diffeomorphic metrics yield completely equivalent functionals of the form (2.6) since

\[
W_{g-\mathcal{L}_v}[h - \mathcal{L}_v h] = W_g[h].
\]

(2.7)

Thus, the non-linear sigma model is really a theory of an equivalence class of models defined by diffeomorphic metrics. In practice, renormalization calculations will be performed for the conventional functional (2.4) with a single source coupled to the quantum field, but the counterterms can then easily be transformed into forms appropriate to (2.6). For the time being, we will concentrate on renormalizing the functional (2.4), but will return to (2.6) later on.

If \(L\) is regarded as the tree-level Lagrangian, then the fully renormalized theory will be calculated from \(L + \Delta L\), where \(\Delta L\) is the counter Lagrangian which serves to remove the divergences from the theory. On general grounds, one expects

\[
L + \Delta L = \frac{1}{2} g^{0}_{ij}(\phi^0) \partial_\mu \phi^0_i \partial_\mu \phi^0_j
\]

(2.8)

where the bare metric \(g^0\) differs from \(g\) by covariant metric counterterms \(T_{ij}\)

\[
\mu^{-\epsilon} g^0_{ij} = g_{ij} + T_{ij},
\]

(2.9)

where \(\epsilon\) is the dimensional regularization parameter and \(\mu\) is the renormalization scale. The bare field \(\phi^0\) is related to \(\phi\) by a non-linear renormalization, the corresponding counterterms being in general non-covariant. Since the arbitrary scale \(\mu\) is introduced during renormalization, the renormalized metric \(g_{ij}\) depends on \(\mu\) and its evolution with changing \(\mu\) is governed by the \(T_{ij}\)'s. One expects that these \(T_{ij}\)'s will not be arbitrary tensors on \(M\), but will be local tensorial functionals of the metric, i.e. they must satisfy

\[
f_* T(g) = T(f_* g)
\]

(2.10)
where \( f \) is a diffeomorphism from one open submanifold of \( M \) onto another open submanifold. The \( T_{ij} \)'s will therefore be local functionals constructed from the metric, the curvature tensor and covariant derivatives of finite order. While the foregoing is all very plausible it is not so easy to demonstrate in ordinary perturbation theory, which would involve splitting up the metric into a constant metric plus a deviation. Nevertheless, the results of [4] show that this can be done. On the other hand, the background field method provides a way to keep manifest the reparameterization invariance of these models, and is therefore the preferred method for explicit calculations.

However, a straightforward linear background-quantum split, \( \phi^i = \varphi^i + \pi^i \), does not lead to a manifestly covariant formalism, since \( \pi^i \) cannot be interpreted as a vector. In order to achieve manifest covariance, it is therefore necessary to use a non-linear split and this can be based on geodesics [5]. Let \( \Phi^i(s) \) be an interpolating field with

\[
\Phi^i(0) = \varphi^i \quad ; \quad \left. \frac{d\Phi^i}{ds} \right|_{s=0} = \xi^i \quad \text{and} \quad \Phi^i(1) = \phi^i
\]  

(2.11)

that satisfies the geodesic equation

\[
\frac{d^2\Phi^i}{ds^2} + \Gamma^i_{jk} \left( \frac{d\Phi^j}{ds} \right) \left( \frac{d\Phi^k}{ds} \right) = 0.
\]  

(2.12)

Then we can solve (2.12) with the initial conditions of (2.11) to get

\[
\phi^i = \varphi^i + \pi^i \quad ; \quad \pi^i = \xi^i + \chi^i(\varphi, \xi)
\]  

(2.13)

where

\[
\chi^i = -\sum_{n=2}^{\infty} \frac{1}{n!} \Gamma^i_{j_1 \ldots j_n} \xi^{j_1} \ldots \xi^{j_n}
\]  

(2.14)

and

\[
\Gamma^i_{j_1 \ldots j_n} \equiv \tilde{\nabla}_{(j_1 \ldots j_{n-2}} \Gamma^i_{j_n-1 j_{n-2})}(\varphi)
\]  

(2.15)

In (2.15) \( \tilde{\nabla} \) indicates that the covariant derivative is to be taken with respect to the lower indices only. In this way of splitting, the quantum field is taken to be \( \xi^i \), the tangent vector to the geodesic \( \Phi^i(s) \) at \( s = 0 \). Now, \( \xi \) has a geometrical interpretation: it is a cross-section of the bundle over \( \Xi \) obtained by pulling back the tangent bundle of \( M \) with
the background field \( \phi, \xi \in \Gamma[\varphi^*(T\mathcal{M})] \). This fact ensures the covariance of the expansion. To expand the Lagrangian one sets \(^7\)

\[
L(s) = \frac{1}{2} g_{ij}(\Phi(s)) \partial_{\mu} \Phi^i(s) \partial^\mu \Phi^j(s)
\]

so that

\[
L(\phi) = L(1) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{d^n L(s)}{ds^n} \right)_{s=0}
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} (\nabla_s)^n L(s)_{s=0}
\]

(2.17)

where \( \nabla_s \) is the covariant derivative along the curve \( \Phi^i(s) \). The series (2.17) is easily evaluated using the formulae

\[
\nabla_s \partial_\mu \Phi^i = \nabla_\mu \frac{d\Phi^i}{ds} = \partial_\mu \frac{d\Phi^i}{ds} + \partial_\mu \Phi^k \Gamma^i_{kj}(\Phi) \frac{d\Phi^j}{ds}, \quad \nabla_s g_{ij} = 0,
\]

\[
\nabla_s \frac{d\Phi^i}{ds} = 0, \quad [\nabla_s, \nabla_\mu] X^k = \frac{d\Phi^i}{ds} \partial_\mu \Phi^j R_{ij}^k X^l,
\]

(2.18)

where in the last equation \( X^i \) is an arbitrary vector. Hence, all the vertices derived in this expansion involve tensorial functionals of the background metric \( g_{ij}(\phi) \). If we introduce the split (2.13) into \( W[J] \) and drop the term \( \int J \cdot \varphi \), we obtain

\[
e^{iW[\varphi, J]} = \int D\pi \exp i[I[\phi] + \int J_i \pi^i]
\]

\[
= \int D\xi \exp i \left( I[\phi] + \int J_i (\xi^i + \chi^i) \right).
\]

(2.19)

Now (2.19) is not quite what we want since the source is still coupled to the (non-covariant) function \( \chi^i \), so we define a new functional

\[
e^{iW[\varphi, J]} = \int D\xi \exp i \left( I[\phi] + \int J_i \xi^i \right).
\]

(2.20)

This functional is now manifestly covariant under reparameterization of the background field \( \phi \). In perturbation theory, however, we must still handle another potential source of non-covariance due to the necessity of separating the covariant kinetic term into a free part for the calculation of the propagator plus non-covariant interaction terms involving the connection. Following reference \(^6\), one can introduce orthonormal frames \( e^a_\alpha \) on the
manifold \( (e_i^a e_j^a = g_{ij}) \), in which case the propagator is taken from the term \( \frac{1}{2} \partial_\mu \xi^a \partial^\mu \xi^a \), where \( \xi^a = e_i^a \xi^i \). This leaves the non-covariant two-point vertices \( \partial_\mu \xi^a \partial^\mu \varphi^i \omega_i^{ab} (\varphi) \xi^b \) and \( \frac{1}{2} \partial_\mu \varphi^i \partial^\mu \varphi^j \omega_i^{ab} (\varphi) \omega_j^{ac} (\varphi) \xi^b \xi^c \), where \( \omega_i^{ab} \) is the spin connection for the orthonormal frame \( e_i^a \). All the vertices aside from these two involve the background via tensors.

Despite the apparent non-covariance introduced by these two vertices, the generating functionals calculated in perturbation theory will be covariant under background reparameterizations provided one uses an invariant regularization scheme to handle the divergences. In particular, the divergences themselves will be covariant. To see this, we note that for an infinitesimal reparameterization with parameter \( \nu^i (\varphi) \), we have

\[
\delta \varphi^i = \nu^i (\varphi) \quad \delta g = -L_{\nu} g \quad \delta \xi^i = -\partial_j \nu^i \xi^j
\] (2.21)

and the fact that this is a linear transformation for the quantum field \( \xi^i \) allows us to immediately establish the following generalized Ward identity for the invariantly regulated effective action \( \Gamma [g, \varphi, \xi] \) (which is the Legendre transform of \( W \) with respect to \( J_i \) in (2.20)):

\[
L_{\nu} g \frac{\partial \Gamma}{\partial g} - \nu^i \frac{\delta \Gamma}{\delta \varphi^i} - \partial_j \nu^i \xi^j \frac{\delta \Gamma}{\delta \xi^i} = 0 . \] (2.22)

This is a generalized Ward identity because of the first term, which involves varying the parameters of the model. Since (2.22) is linear in \( \Gamma \), the same identity holds for the divergent part of \( \Gamma \). Equivalently, it is clear by inspection of any Feynman graph that the operation \( L_{\nu} g \frac{\partial}{\partial g} \) cannot render a finite diagram divergent. Furthermore, the two-point vertices involving \( \omega_i^{ab} \) explicitly cannot contribute to divergent graphs. Thus, the background-quantum split (2.13) together with an invariant regulator yields a perturbation theory that is manifestly invariant under reparameterizations of the background field. However, as we shall shortly see, it is not enough to compute \( 1PI \) graphs with only external background lines if one is to determine the counterterms necessary for higher loop calculations from lower loop graphs.

3. The Non-Linear Splitting Ward Identity

The action

\[
I[\phi] \equiv I[\varphi, \xi] = I[\varphi + \pi]
\] (3.1)
is invariant under the obvious symmetry
\[ \delta \varphi^i = \eta^i(x) \]
\[ \delta \pi^i = -\eta^i(x) \] (3.2)
and this leads to a simple Ward Identity for the linear splitting \( \tilde{\Gamma}[\varphi, \pi] \) functional:
\[ \frac{\delta \tilde{\Gamma}}{\delta \varphi^i} = \frac{\delta \tilde{\Gamma}}{\delta \pi^i} \] (3.3)
with the obvious solution
\[ \tilde{\Gamma}[\varphi, \pi] = \tilde{\Gamma}[\varphi + \pi]. \] (3.4)
Now, we can reformulate (3.2) in terms of \# \( \varphi \) and \( \xi \):
\[ \delta_{\eta} \varphi^i = \eta^i \]
\[ \delta_{\eta} \xi^i = F^i_j(\varphi, \xi) \eta^j \] (3.5)
where the functional \( F^i_j \) is determined by the requirement that \( \phi^i \) be invariant, i.e.
\[ \delta_{\eta} \phi^i = \eta^i \delta_{\pi} \phi^i + \eta^j \phi^i F^i_k \eta^k = 0 \] (3.6)
where
\[ \delta_i \equiv \frac{\delta}{\delta \phi^i}, \quad \delta_i' \equiv \frac{\delta}{\delta \xi^i} \] (3.7)
(note that since the \( \phi \leftrightarrow (\varphi, \xi) \) relation is local in \( x^\mu \), we could have used ordinary partial derivatives and ordinary summation here). It is easy to derive the identity
\[ \delta_{[\eta} F^i_{k]} + F^i_{[j} \xi^i \xi^j F^i_{k]} = 0 \] (3.8)
with the aid of which it can be verified that the transformations (3.5) are Abelian. However, they are nevertheless non-linear and this fact requires that they be handled with care at the quantum level, since the transformations themselves will require renormalization. To study the associated Ward Identities it is therefore convenient to consider the associated B.R.S. transformations. The discussion that we shall give is similar in certain respects to

\# From now on, we use DeWitt notation, so that an index \( \xi^\nu \) does double duty as a tensor index and a spacetime point \( x^\mu \). Summation over indices includes integration over spacetime.
the proof in [4] of the renormalizability of the \( \sigma \)-model about a constant background. We introduce a ghost field \( c^i(x) \) and define

\[
\begin{align*}
\sigma \varphi^i &= c^i \\
\sigma \xi^i &= F^i_j c^j \\
sc^i &= 0.
\end{align*}
\] (3.9)

Then \( s^2 = 0 \) by virtue of (3.8). To obtain the Ward Identity it is necessary to modify the action by including \( s \xi^i \) coupled to an anticommuting source \( L_i \). However, since \( F^i_j c^j \) has power counting weight zero it will mix with all other possible dimension zero operators. To allow for this operator mixing we therefore define

\[
\Sigma = I + L_{\alpha i} N^i_{\alpha}
\] (3.10)

where

\[
N^i_{\alpha} = s \Lambda^i_{\alpha} \quad \alpha = 0, 1, \ldots \infty \\
\Lambda^i_0 \equiv s \xi^i \quad L_{0 i} \equiv L_i.
\] (3.11)

The set \( \{ \Lambda^i_{\alpha} \} \) are all possible dimension zero vectorial functions of \( \varphi \) and \( \xi \), this set being sufficient as we shall see. If we assign dimension zero and ghost number \( n_g = 1 \) to \( c \) then the \( L \)'s have dimension 2 and ghost number \( -1 \). It is clear that

\[
s \Sigma = 0,
\] (3.12)

since \( s \phi = 0 \) and \( s^2 = 0 \). This can be rewritten as

\[
s \Sigma = c^i \frac{\delta \Sigma}{\delta \varphi^i} + \frac{\delta \Sigma}{\delta L_i} \frac{\delta \Sigma}{\delta \xi^i} = 0
\] (3.13)

since

\[
\delta \xi^i = \frac{\delta \Sigma}{\delta L_i}.
\]

For future reference, we define the operators

\[
A = c^i \frac{\delta}{\delta \varphi^i}, \quad B_X = \delta'_i X \delta_i + (-1)^x \delta_i X \delta'_i.
\] (3.14)
where
\[ \delta_i = \frac{\delta}{\delta L_i}, \quad \delta_i' = \frac{\delta}{\delta \xi_i} \]  
(3.15)

and \( x \) is the Grassmann parity of the functional \( X \). Then (3.13) can be written
\[ s \Sigma = A \Sigma + \frac{1}{2} B_\Sigma \Sigma = 0. \]  
(3.16)

We have
\[ B_X Y = (-1)^{x+y+z} B_Y X \]  
(3.17)

and the generalized Jacobi Identity
\[ B_X B_Y Z + (-1)^{x+y+z(x+y)} B_Z B_X Y + (-1)^{y+z+x(y+z)} B_Y B_Z X = 0, \]  
(3.18)

which can be proved by straightforward computation. If \( X, Y \) and \( Z \) are even, (3.17) and (3.18) imply
\[ B_X B_X X = 0. \]  
(3.19)

If, in addition,
\[ A X + \frac{1}{2} B_X X = 0, \]  
(3.20)

then the operator \( D_X \) defined by
\[ D_X = A + B_X \]  
(3.21)

is nilpotent, i.e.
\[ D_X D_X Y = 0 \quad \forall Y. \]  
(3.22)

Clearly \( \Sigma \) satisfies (3.20), so \( D_\Sigma \) is nilpotent.

We now pass to the quantum theory, assuming the existence of an invariant regularization scheme (dimensional regularization is such a scheme for the bosonic case). The Ward Identity for the 1PI functional \( \Gamma[\varphi, \xi, L, c] \) is
\[ e^i \frac{\delta \Gamma}{\delta \varphi^i} + \frac{\delta \Gamma}{\delta L_i} \frac{\delta \Gamma}{\delta \xi_i} = 0 \]  
(3.23)

# Two dimensional massless theories also require infrared regularization. This can be done in a reparameterization and shift (3.5) invariant way by including a potential \( m^2 V(\phi) \), where \( V \) is a scalar function of the total field. The function \( V(\phi) \) will also have to be renormalized, e.g. at the one-loop order by a term proportional to \( D^i D_j V \). Since these renormalizations are proportional to \( m^2 \), they are clearly distinguishable from the other ultraviolet divergences with which we are chiefly concerned.
and we wish to subtract the divergences so that the renormalized $\Gamma$, $\Gamma'$, continues to satisfy (3.23). As in the Yang-Mills case, we proceed order by order in $\hbar$. For example, the one-loop divergences satisfy
\begin{equation}
eq e^{\delta \Gamma_D^{(1)} / \delta \phi^i} + \delta \Gamma_D^{(1)} / \delta L_i \frac{\delta \Sigma}{\delta \xi^i} + \frac{\delta \Sigma}{\delta L_i} \frac{\delta \Gamma_D^{(1)}}{\delta \xi^i} = 0 \tag{3.24}
\end{equation}
or
\begin{equation}
D_\Sigma \Gamma_D^{(1)} = 0 \tag{3.25}
\end{equation}
The solution to (3.25) is given by
\begin{equation}
\Gamma_D^{(1)} = G[\phi] + D_\Sigma X[\phi, \xi, L, c] \tag{3.26}
\end{equation}
where $G[\phi]$ is a reparameterization invariant functional of the total field $\phi$, and $X$ is an arbitrary functional with ghost number $-1$ and dimension zero (remembering that it is an integrated functional). That (3.26) solves (3.25) follows from the fact that $sG[\phi] = 0$ and from (3.22); that it is the most general solution can be proved using similar arguments to the Yang-Mills case [8]. $X$ has the general form
\begin{equation}
X = Z_{\alpha \beta} L_{\alpha i} \Lambda_\beta^i \tag{3.27}
\end{equation}
for some (infinite) constants $Z_{\alpha \beta}$. Expanding out (3.26) yields
\begin{equation}
\Gamma_D^{(1)} = G[\phi] - Z_{\alpha \beta} L_{\alpha i} N_\beta^i + Z_{0 \alpha} \Lambda_\alpha^i \delta \Sigma / \delta \xi^i \tag{3.28}
\end{equation}
Since all the vertices coming from $I$ are covariant, it follows that $\Gamma_D^{(1)}[\phi, \xi, c = L = 0]$ is covariant, so that the $\Lambda_\alpha^i$ are indeed vectors as claimed, and in addition $G[\phi]$ must be of the form $-\frac{1}{2} T_{ij} \partial_\mu \phi^i \partial_\mu \phi^j$. To summarize, the one loop divergences comprise metric divergences (from $G$), non-linear $\xi$ renormalizations ($Z_{0 \alpha} \Lambda_\alpha^i \delta \Sigma / \delta \xi^i$) and multiplicative renormalizations of the infinite set of sources $\{L_{\alpha i}\}$. The sources $\{L_{\alpha i}\}$ are needed only in the proof of renormalizability and will ultimately be set to zero. Thus, the important renormalizations are those of the metric and of the quantum field $\xi$.

To prove that this procedure can be carried through loop by loop, one first subtracts the 1-loop counterterms from $\Sigma$
\begin{equation}
\Sigma \rightarrow \Sigma' = \Sigma - \Gamma_D^{(1)} \tag{3.29}
\end{equation}
but this new $\Sigma'$ does not satisfy (3.23) so we have to add divergent $\hbar^2$ terms $Q$ such that

$$\tilde{\Sigma} = \Sigma - \Gamma^D_{(1)} + Q$$

(3.30)

does satisfy (3.23), up to order $\hbar^2$. In order for (3.23) to be satisfied to this order, we require that

$$D_\Sigma Q = \frac{1}{2} B_{\Gamma^D_{(1)}} \Gamma^D_{(1)}.$$  \hfill (3.31)

The integrability condition for (3.31) is

$$D_\Sigma B_{\Gamma^D_{(1)}} \Gamma^D_{(1)} = 0$$

(3.32)

and this is readily seen to be satisfied using (3.25) and

$$ABX X + 2BXAX = 0,$$  \hfill (3.33)

valid for any even $X$. The new quantum 1PI functional, $\tilde{\Gamma}$, constructed from $\tilde{\Sigma}$ is finite at one loop and satisfies (3.23), from which it follows that the two-loop divergences satisfy (3.25) and so we may continue to all orders. Note that in the $\ell$ loop renormalizations, $Q^\ell$ is subtracted out and a new $Q^{\ell+1}$ is needed in order that (3.23) be maintained at order $(\hbar)^{\ell+1}$. $Q^{\ell+1}$ is itself subtracted out at the $\ell + 1$ loop order. Thus, the renormalized action $\Sigma^{(r)}$ will also satisfy (3.23) but its explicit form is not so obvious due to the need to introduce and then later subtract the $Q$'s.

Anticipating the need for quantum field renormalizations, we may nonetheless write an expression for the renormalized action that satisfies (3.23):

$$\Sigma^{(r)}[\varphi, \xi, L, c] = I^0[\varphi, \xi^0] + L^0_{\alpha i} N^i_{\alpha}(\varphi, \xi^0)$$

(3.34)

where $I^0$ includes the metric counterterms, i.e. $g_{ij} \rightarrow g^0_{ij} = g_{ij} + \Sigma T_{ij}$ and

$$L^0_{\alpha i} = L_{\beta i} Z^\beta_{\alpha}$$

$$Z_{\alpha i}(\varphi, \xi^0) = \xi^i.$$  \hfill (3.35)

The proof that (3.34) satisfies

$$\frac{c^i \delta \Sigma^{(r)}}{\delta \varphi^i} + \frac{\delta \Sigma^{(r)}}{\delta L_i} \frac{\delta \Sigma^{(r)}}{\delta \xi^i} = 0$$

(3.36)
follows from the observation that
\[ s^0 \Sigma^{(r)} = 0 \]  
(3.37)

where
\[ s^0 \varphi^i = c^i \]
\[ s^0 \xi^0_0 = F^i_j (\varphi, \xi^0) c^j = \frac{\delta \Sigma^{(r)}}{\delta L^0_i} \]
\[ s^0 c^i = 0 \]  
(3.38)

because
\[ N^i_0 (\varphi, \xi) = s \Lambda^i_0 (\varphi, \xi) \Rightarrow N^i_0 (\varphi, \xi^0) = s^0 \Lambda^i_0 (\varphi, \xi^0) \]  
(3.39)

Using (3.38), (3.37) is
\[ e^i \frac{\delta \Sigma^{(r)}}{\delta \varphi^i} + \frac{\delta \Sigma^{(r)}}{\delta L^0_i} \frac{\delta \Sigma^{(r)}}{\delta \xi^0_i} = 0 \]  
(3.40)

If one then changes variables from \((\varphi, \xi^0)\) to \((\varphi, \xi)\) and uses (3.35), one readily sees that (3.40) implies (3.36). Strictly speaking, we should show that the solution (3.34) is unique; this can be done along the same lines as the Yang-Mills case as discussed, for example, in Ref. [8].

To summarize then, the consequences of the non-linear splitting Ward Identity are given in equations (3.34) and (3.35); in addition to the metric renormalizations, there are non-linear renormalizations of the quantum field \(\xi^i\) that are not derivable from expanding out the metric counterterms. Furthermore, they do not cancel out in higher loop graphs and therefore must be taken into account. In the next section we give an algorithm for computing them.

4. Computational Algorithm

In this section we show how to renormalize the functional
\[ e^{i W[\varphi, J]} = \int D\xi e^{i (I[\varphi, \xi] + \int J\xi)} \]  
(4.1)

From the preceding section we know that the renormalized functional is given by
\[ e^{i W[\varphi, J]} = \int D\xi e^{i (I^0[\varphi, \xi^0] + \int J\xi)} \]  
(4.2)
which follows from (3.34) and (3.35) upon setting the \( L_{ai} \) to zero. The task is to compute
the metric contributions and the (non-linear) quantum wave function renormalizations.
This could, of course, be done straightforwardly by computing all \( 1PI \) graphs with both
external quantum and background lines. To do this would be to violate the spirit of the
background field method, however, in which the effective action \( \Gamma[\varphi] \) is computed from
graphs with no external quantum lines. Nonetheless, as we have shown, the non-linear
renormalizations of the quantum field must be taken into account in order to correctly
subtract the theory. Moreover, these renormalizations cannot be deduced from the divergent parts of \( \Gamma[\varphi] \) alone.

In order to deduce the necessary renormalizations of \( g_{ij} \) and \( \xi^i \) without computing
graphs with external \( \xi \) lines, we consider instead of (4.1) the generalized functional (2.6),

\[
e^{iW[h]} = \int D\xi e^{i\{I[\varphi,\xi] + \int h(\xi,\phi)\}}
\]

(4.3)
The functional (4.3) can be renormalized in the same way as (4.1) since the modified
source is a functional of the total field, but the wave-function renormalizations cancel out
since \( W[h] \) is essentially the vacuum functional for the modified action \( I + \int h \). So (4.3) is
renormalized by

\[
\begin{align*}
  g_{ij} &\rightarrow g_{ij}^0 = \mu^\epsilon \left(g_{ij} + T_{ij}(g)\right) \\
  h &\rightarrow h^0 = \mu^\epsilon \left(h + H(g, h)\right)
\end{align*}
\]

(4.4)
where the source counterterms \( H \) are linear in \( h \) because \( h \) has dimension two. As with
the metric counterterms, the \( h \) counterterms can be classified according to their conformal
weights under the scalings \( g_{ij} \rightarrow \lambda^{-1} g_{ij} \), \( h \rightarrow \lambda^{-1} h \). For example, the one-loop counterterms have conformal weight zero, so there are only two possibilities, \( \nabla^i \nabla_i h \) and \( Rh \). The latter does not occur since the Feynman rules involve only derivatives of \( h \).

In terms of the non-linear \( \varphi - \xi \) split, \( h \) has the expansion

\[
h(x, \phi(x)) = h(x, \varphi(x)) + \sum_{n=1}^\infty \frac{1}{n!} \xi^{i_1} \ldots \xi^{i_n} (\nabla_{i_1} \ldots \nabla_{i_n} h)_{\phi=\varphi}
\]

(4.5)
If we choose the condition

\[
\nabla_{i_1} \ldots \nabla_{i_n} h \bigg|_{\phi=\varphi} = 0 \quad \text{for} \quad n \neq 1
\]

(4.6)
then the source term \( \int h \) reduces to a conventional source term as in (4.1). When this is done after renormalization, it is necessary to perform a quantum field redefinition in order to recast the renormalized integrand into the form of (4.2).

To see how this works, consider the bare source \( h^0 \) which occurs in the renormalized functional

\[
e^{iW^{(\gamma)}[h]} = \int \mathcal{D}\xi \exp i \left( I^0[\phi, \xi] + \int h^0(x; \phi) \right);
\]

one finds

\[
\mu^{-\epsilon}h^0 = h + \frac{1}{4\pi \epsilon} \nabla^i \nabla_i h + \ell \geq 2 \text{ loop terms}.
\]  

(4.8)

Expanding now \( h^0 \) using (4.5) and (4.6), one finds

\[
\mu^{-\epsilon}h^0 = h_i \left( \xi^i + \sum_{\ell=1}^{\infty} X_{(\ell)}^i (\phi, \xi) \right), \quad h_i = \nabla_i h \bigg|_{\phi=\phi},
\]

(4.9)

where the \( X_{(\ell)}^i \) are covariant expressions corresponding to \( \ell \) loops involving all powers in the quantum field \( \xi \). These arise from the renormalization of \( h \) upon the imposition of (4.6). Note that \( X_{(\ell)}^i \) is determined entirely by the \( \ell \)-loop contribution to \( h^0 \) in (4.8); \( X_{(\ell)}^i \) contains terms of arbitrary order in the quantum field \( \xi \) and is a power series in the regulator \( \epsilon^{-1} \) up to order \( \epsilon^{-\ell} \). For example, from the one-loop contribution to \( h^0 \) given in (4.8) one obtains

\[
X_{(1)}^i = -\frac{1}{4\pi \epsilon} \left( \frac{2}{3} R^i_j \xi^j + \frac{1}{2} \nabla_j R^i_k \xi^j \xi^k - \frac{1}{12} \nabla^i R_{jk} \xi^j \xi^k + O(\xi^3) \right).
\]  

(4.10)

We can regain the form (4.2) by changing variables in the functional integral so that the source \( h_i \) couples to \( \xi^i \), with the resulting bare quantum field \( \xi^{0i} \) in the action given by

\[
\xi^{0i} = \xi^i - X^i_{(1)}(\phi, \xi) - X^i_{(2)}(\phi, \xi) + X^i_{(1)} \frac{\delta}{\delta \xi^j} X^j_{(1)} + \ell \geq 2 \text{ loop terms}.
\]

(4.11)

Again, we emphasize that the expression for the bare quantum field \( \xi^{0i} \) at a given loop order \( \ell \) is derived from the renormalization of \( h \) at loop orders up to \( \ell \) given in (4.8).

In order to compute the renormalization of \( I \) and \( h \) in (4.7), it is convenient in practice to consider a more general functional

\[
e^{iW_{[\phi, h; J]}[\xi]} = \int \mathcal{D}\xi \exp i \left( I[\phi, \xi] + \int h + \int J_i \xi^i \right).
\]

(4.12)
This functional obviously reduces on the one hand to (4.3) for \( J_i = 0 \) and on the other hand to (4.1) for \( h = 0 \). In order to renormalize (4.12), we require metric, \( h \) and \( \xi \) renormalizations. Since \( h \) has dimension two, its presence does not affect the renormalization of \( \xi^i \), which has dimension zero. Thus, by our previous discussion, the renormalization of \( \xi^i \) is given in terms of the renormalization of \( h \) by (4.11). We also know from the results of section 3 that the renormalization of \( h \) in (4.12) is given by functionals of the total field \( \phi(\varphi, \xi) \), so it may be deduced from diagrams with no external \( \xi \) lines. Hence, by calculating with the general functional (4.12), we may deduce the renormalization of the quantum field \( \xi^i \) via the renormalization of \( h \) from diagrams with no external quantum lines, and then set \( h = 0 \) to obtain the renormalized (4.2).

To illustrate the general scheme for renormalization of non-linear \( \sigma \)-models, we present the calculation of metric and quantum field renormalizations through the two-loop order.

For calculations through two loops, one needs the Lagrangian expanded out to order \( \xi^4 \). Using (2.17, 2.18), one finds

\[
\frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j = \frac{1}{2} g_{ij}(\varphi) \partial_\mu \varphi^i \partial^\mu \varphi^j + g_{ij} \nabla_\mu \xi^i \nabla^\mu \xi^j + \frac{1}{2} R_{ikl} \xi^k \xi^l \partial_\mu \varphi^i \partial^\mu \varphi^j \\
+ \frac{1}{2} R_{ikl} \xi^k \nabla_\mu \xi^i \nabla^\mu \xi^j + \frac{1}{6} R_{ikl} \xi^k \nabla_\mu \xi^i \nabla^\mu \xi^j + \frac{1}{6} R_{ikl} \xi^k \nabla_\mu \xi^i \nabla^\mu \xi^j + \frac{1}{4} R_{ikl} \xi^k \nabla_\mu \xi^i \nabla^\mu \xi^j + \ldots .
\]

(4.13)

The expansion of \( h(x; \phi(x)) \) is given in (4.5).

The propagator is taken from the term \(-\frac{1}{2} g_{ij}(\varphi) \xi^i \xi^j \partial_\mu \varphi^i \partial^\mu \varphi^j - \frac{m^2}{2} g_{ij}(\varphi) \xi^i \xi^j \), where we have introduced a mass term for \( \xi \) as an infrared regulator. The remaining quadratic terms in \( \xi \) will be considered as two-point interaction vertices. Quadratic vertices which involve \( \Gamma_\mu \xi^i = \partial_\mu \varphi^k \Gamma_{kj} \xi^i \) cannot contribute to the divergences, since reparameterization covariance would require \( \Gamma_\mu \xi^i \) to appear through the curvature \( R_{\mu \nu} \xi^i \), and there is no dimension two parity even invariant that can be constructed from \( R_{\mu \nu} \xi^i \).
The divergent one-loop graphs are then

Figure 1

where the fat lines denote expressions in background fields and the thin lines are the quantum propagators. From graph (a) we obtain

\[
\frac{1}{8\pi} \left\{ \frac{2}{\epsilon} + \gamma_E + \ell n \left( \frac{m^2}{4\pi\mu^2} \right) + O(\epsilon) \right\} R_{ij}(\varphi) \partial_{\mu}\varphi^i \partial_{\nu}\varphi^j
\]

(4.14)

while from graph (b) we obtain

\[
-\frac{1}{8\pi} \left\{ \frac{2}{\epsilon} + \gamma_E + \ell n \left( \frac{m^2}{4\pi\mu^2} \right) + O(\epsilon) \right\} \nabla_i \nabla^i h(\varphi)
\]

(4.15)

where \( \gamma_E \) is the Euler-Mascheroni constant.

In order to calculate the two-loop divergences, we will need to subtract the one-loop subdivergences and for this we need the expansions of the counter-Lagrangians corresponding to the pole terms in (4.14) and (4.15) as well as the renormalization of the quantum field \( \xi^i \). The latter has already been given in (4.10); it is derived from the pole part of (4.15) as previously explained. In order to calculate the two-loop graphs, we need the counter-Lagrangians for (4.14, 4.15) to order \( \xi^2 \). Substituting the total field \( \phi \) for \( \varphi \) in the pole part of (4.14) and expanding by the differentiation technique explained earlier, we obtain

\[
-\frac{1}{2\pi\epsilon} \left\{ \frac{1}{2} R_{ij}(\varphi) \partial_{\mu}\varphi^i \partial_{\nu}\varphi^j + R_{ij} \nabla_\mu \xi^i \partial_{\nu}\varphi^j + \frac{1}{2}\xi^k \nabla_k R_{ij} \partial_{\mu}\varphi^i \partial_{\nu}\varphi^j + \frac{1}{4}\xi^k \xi^l \nabla_k R_{ij} \nabla_\ell \partial_{\mu}\varphi^i \partial_{\nu}\varphi^j + \frac{1}{2} R_{ij} R^i_{\ell m} \xi^k \xi^l \partial_{\mu}\varphi^m \partial_{\nu}\varphi^j \right\}
\]

(4.16)
Similarly, from the pole part of \((4.15)\) we obtain
\[
\frac{1}{4\pi\varepsilon} \left\{ \nabla^i \nabla_j h(\varphi) + \xi^j \nabla_j \nabla^i \nabla_i h + \frac{1}{2} \xi^j \xi^k \nabla_j \nabla_k \nabla^i \nabla_i h \right\}.
\] (4.17)

From the renormalization of the quantum field \(\xi\), we obtain the corresponding counter-Lagrangian by substituting \((4.11)\) into the original action and the \(\int h\) source term (keeping \(\xi^2\) terms only):
\[
-X^{(1)}(I + \int h) = \frac{1}{4\pi\varepsilon} \left\{ \left( \frac{1}{2} \nabla_j R^i_k - \frac{1}{12} \nabla^i R_{jk} \right) \frac{\delta I}{\delta \varphi^i} \xi^j \xi^k - \frac{2}{3} R^i_j \xi^i \nabla_\mu \nabla^\mu \xi_i 
+ \frac{2}{3} R^i_k R_{lijm} \partial_\mu \varphi \partial_\nu \varphi \xi^i \xi^j \xi^k \nabla^l \nabla_j h
+ \left( \frac{1}{2} \nabla_j R^i_k - \frac{1}{12} \nabla^i R_{jk} \right) \xi^j \xi^k \nabla_i h - \frac{2}{3} R^i_k \xi^i \xi^j \nabla_i \nabla_j h \right\}.
\] (4.18)

The basic two loop graphs are

**Figure 2**

![Graphs](image_url)

- a) \(R^2 + \nabla \nabla R\)
- b) \(R\)
- c) \(R\partial \Phi\)
- d) \(\nabla^2 h\)
- e) \(R\)

19
and the counterterm graphs from (4.16) and (4.17) are

**Figure 3**

![Counterterm graphs](image)

There are also counterterm graphs coming from the non-linear renormalizations of the $\xi$ field, as given in (4.18):

**Figure 4**

![Counterterm graphs](image)

There are actually additional graphs coming from the linear term in $X^{i}_1$, corresponding to a linear renormalization of $\xi$ (linear in $\xi$ but dependent on $\varphi$). However, these graphs completely cancel. This cancellation is analogous to the cancellation of constant multiplicative renormalizations discussed in the introduction. In the present case, however, the
presence of non-linear renormalizations of $\xi$ limits such cancellations of the linear $\xi$ renormalizations to graphs with one loop more than those in which the linear renormalizations first occurred.

The two loop divergences arising from the sum of the contributions from the graphs in Figures 2, 3 and 4 are $\Gamma_{(2,1)}^D + \Gamma_{(2,2)}^D + \Gamma_{(2,2)}^{\prime D}$, where

$$\Gamma_{(2,1)}^D = \frac{1}{32\pi^2\epsilon^2}R_ik_{\ell\mu}R_{jk\ell\mu}\partial_\mu\varphi^i\partial_\mu\varphi^j$$

$$\Gamma_{(2,2)}^D = \frac{1}{32\pi^2\epsilon^2}\Delta_L R_{ij}\partial_\mu\varphi^i\partial_\mu\varphi^j - \frac{1}{16\pi^2\epsilon^2} \left( R_{ij}\nabla^i\nabla^j h + \frac{1}{2}\nabla^i\nabla_i\nabla^j\nabla_j h \right)$$

$$\Gamma_{(2,2)}^{\prime D} = -\frac{1}{96\pi^2\epsilon^2}\nabla^i R \left[ \frac{\delta}{\delta\xi^i}(I + f h) \right]_{\xi = 0}$$

and where

$$\nabla_L R^{ij} = \nabla^k\nabla_k R^{ij} + [\nabla^i, \nabla_k]R^{kj} + [\nabla^j, \nabla_k]R^{ki}$$

is the Lichnerowitz Laplacian acting on the Ricci tensor. The $\frac{1}{\epsilon}$ term $\Gamma_{(2,1)}^D$ gives rise to the two-loop $\beta$-function [4]; this is the new divergence at the two-loop order that is not derivable from the one-loop results. The $\frac{1}{\epsilon^2}$ terms of $\Gamma_{(2,2)}^D$ are the ones that one expects at two loops from the standard renormalization group pole equation of Ref. [6].

$\Gamma_{(2,2)}^{\prime D}$ are extra $\frac{1}{\epsilon}$ terms that arise as a consequence of the non-linear renormalizations of $\xi$ at the one-loop order, which are unrelated to the metric renormalization. If one sets $h = 0$, then $(-\Gamma_{(2,2)}^{\prime D})$ is the $\xi$-independent part of the counter-Lagrangian that extends (4.18) to the two-loop order, which will be necessary for renormalization at higher orders.

As we have explained above, the full set of non-linear $\xi$ renormalizations may actually be derived from the $h$-renormalizations, given in this case in (4.20). Setting $J_i = 0$ means setting $\frac{\delta}{\delta\xi^i}\Gamma(g, h, \varphi, \xi) = 0$, thus setting (4.21) to zero. On the other hand, renormalizing $h$ by the $\frac{1}{\epsilon}$ terms in (4.20) and then imposing anew the condition (4.6), we find new two-loop, $\frac{1}{\epsilon^2}$ order terms that extend (4.11). Constructing the corresponding counter-Lagrangian terms to extend (4.18) to the two-loop order, and setting $\xi = 0$ we again reobtain precisely (4.21). This provides a non-trivial check on the consistency of the renormalization procedure.

The non-linear renormalizations of $\xi$ at the one-loop order give rise to the two-loop counterterm graphs of Figure 4. If we had not performed these renormalizations, the
divergences of the Figure 4 graphs would have remained in the theory. These divergences are local and contain parts that are simultaneously infrared and ultraviolet divergent as well as pure ultraviolet divergent $\frac{1}{\epsilon}$ and $\frac{1}{\epsilon^2}$ terms. The divergences from the Figure 4 graphs are

$$-\frac{1}{96\pi^2\epsilon} \left\{ 2 + \tilde{\gamma}_E \right\} \frac{\delta}{\delta \phi^i} \left[ S + \int h \right] \nabla^i R \quad (4.23)$$

where

$$\tilde{\gamma}_E = \gamma_E + \ell n \left( \frac{m^2}{4\pi\mu^2} \right). \quad (4.24)$$

Thus, the non-linear $\xi$ renormalizations are essential if one is to correctly calculate the leading $\frac{1}{\epsilon}$ divergences in graphs with only external background $\varphi$-lines and also to ensure the cancellation of $\frac{1}{\epsilon} \ell n \left( \frac{m^2}{\mu^2} \right)$ terms that are off-shell infrared and ultraviolet divergent. Although the divergences in (4.23) vanish to this order subject to the classical field equations for the background $\varphi$ in the presence of the source $h$, this feature will not be maintained at higher orders, where second and higher variations of $I - \int h$ will appear.

5. Conclusion

In this paper, we have shown how to renormalize the non-linear $\sigma$-model model using the background field method. In addition to the expected counterterms which are functionals of the total field $\phi^i$, there are additional non-linear field renormalizations of the quantum field $\xi^i$ which must be performed even if one wishes to calculate only diagrams without external quantum lines. These non-linear renormalizations of $\xi^i$ can be calculated from diagrams without external $\xi$-lines if one couples to a generalized source $h(x; \phi^i(x))$ and then renormalizes this source as if it were a potential for the $\sigma$-model.

The generating functional $W[g, h]$ given in (4.7) is a convenient functional to use in deriving the renormalization group equation [4], since the renormalized $W^r$ is related to the bare $W^0$ by

$$W^r[g^r, h^r] = W^0[g^0, h^0] \quad (5.1)$$

From this, it immediately follows, since the bare quantities are independent of the renormalization scale $\mu$, that

$$\mu \frac{\partial W^r}{\partial \mu} + \int \beta_{ij} \frac{\delta W}{\delta g_{ij}} + \int \gamma \frac{\delta W}{\delta h} = 0 \quad (5.2)$$
Here, \( \beta_{ij}(\phi) = \mu \frac{\partial}{\partial \phi} f_{ij}^r(\phi) \) is the metric \( \beta \)-function and \( \gamma(g(\varphi), h(x; \phi)) = \mu \frac{\partial}{\partial \mu} h^r(x; \phi) \) is the anomalous dimension function, which is linear in \( h \). Note that for the standard generating functional \( W[g, J] \) of eqn.(4.1), there is no such simple relation as (5.1) or renormalization group equation such as (5.2), precisely because there are non-linear renormalizations of the quantum field \( \xi \).

The \( \beta \)- and \( \gamma \)-functions given in (5.2) are not uniquely determined because of the reparameterization invariance of \( W \):

\[
\int \mathcal{L}_v g_{ij} \frac{\delta W}{\delta g_{ij}} + \int \mathcal{L}_v h \frac{\delta W}{\delta h} = 0 \quad ,
\]

where \( v^i(\phi) \) is a vector field generating a diffeomorphism of the target manifold \( M \). Equation (5.3) implies that the renormalization group equation (5.2) is unchanged if one shifts \( \beta_{ij} \) and \( \gamma \) simultaneously by [4]

\[
\beta_{ij} \rightarrow \beta_{ij} + 2 \nabla_i (v_j) \\
\gamma \rightarrow \gamma + v^i \nabla_i h \quad .
\]

From the definitions of \( \beta_{ij} \) and \( \gamma \), one can derive the renormalization group pole equations [6]. There is a slight subtlety here in that some counterterms may be ambiguous, in the sense that they can be interpreted either as field renormalizations or as metric (and \( h \)) renormalizations. The extra divergences \( \Gamma^{PD}_{(2,2)} \) are of this type. If one considers them as metric (and \( h \)) renormalizations, then the standard pole equations of Ref. [6] are apparently no longer true. The resolution of this point is that the change in the counterterms by terms of this type amounts to field redefinitions which cancel out in \( W[g, h, J = 0] \) as expressed in (5.1). For example, ignoring the \( \Gamma^{PD}_{(2,2)} \) terms gives

\[
g_{ij}^0(\phi) \partial_\mu \phi^i \partial^\mu \phi^j = \mu^x [g_{ij}(\phi) + T_{ij}(\phi)] \partial_\mu \phi^i \partial^\mu \phi^j \quad .
\]

The derivative terms can be crossed off from both sides of (5.5), leading to the standard renormalization group pole equations. On the other hand, to linear order in the vector field \( v^i = \nabla^i R \), (5.5) can be rewritten as

\[
g_{ij}^0(\phi + v) \partial_\mu (\phi^i + v^i) \partial^\mu (\phi^j + v^j) = \mu^x [g_{ij}(\phi) + T_{ij}(\phi) + \mathcal{L}_v g_{ij}(\phi)] \partial_\mu \phi^i \partial^\mu \phi^j 
\]

(5.6)
thus recovering the $\Gamma^{(2,2)}_{\nu\nu}$ counterterms in the form $L_{\nu\mu}g\partial\phi\partial\phi$. The point is that in (5.6), we cannot simply cross off the derivative terms from both sides of the equation as one could in (5.5). Thus, the standard renormalization group pole equations do not hold for the $\Gamma^{(2,2)}_{\nu\nu}$ terms.

There is therefore a preferred renormalization of the metric (and $h$) in which the higher order pole terms are in accord with the standard pole equations. In this scheme, the divergences in 1PI graphs with no external $\xi$ lines at the $\ell$th loop order will divide into three categories: the "new" $1/\xi$ divergences, the higher order pole terms up to $1/\xi^2$ that are determined by the pole equations, and some additional terms proportional to the classical field equations (including contributions from $h$). As we have seen, these divergences may be ignored in the renormalization group pole equations for $W[g, h, J = 0]$. The remaining $g$ and $h$ renormalizations do not involve terms that can be removed by field redefinitions. The specific forms of this last class of divergences may be recovered from the $h$ renormalizations at the same $\ell$th loop order as explained in Section 4.

We note that, since there is no two-loop $1/\xi$ divergence in the $h$-sector, the wave function renormalizations necessary for 2 loop and 3 loop calculations are determined entirely by the graph of Fig. 1a. Although it is straightforward in principle to calculate the wave function renormalizations necessary for a 3 loop calculation, in practice this is a somewhat lengthy procedure. For practical calculations it may therefore be more convenient to employ BPHZ style techniques, as advocated by the authors of Ref. [9], especially if one is interested in a divergence involving a subset of graphs.

To conclude, we remark that in this paper we have been concerned with the renormalization of the bosonic $\sigma$-model without torsion, and this allowed the use of the symmetry-preserving dimensional regularization scheme. For bosonic $\sigma$-models with torsion, or if there are chiral fermions present, this type of approach is no longer feasible, and in these situations there may be obstructions to implementing the shift symmetry in the renormalized theory. As in Yang-Mills theory, this essentially boils down to a cohomology problem for the operator $D_U$ of section three.
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