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LOW ENERGY LIGHT SCATTERING:
A MULTIPLE SCATTERING DESCRIPTION

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ABSTRACT

A multiple scattering development is used to compute extinction, absorption and scattering cross-sections for arbitrarily shaped particles whose dimensions are comparable to the incident radiation wavelength. Various numerical applications to spheres and spheroids exhibit a remarkable agreement with exact solutions.

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INTRODUCTION

The electromagnetic scattering theory is a research field of crucial importance in today's physics, both from the theoretical and experimental points of view. Unfortunately, very little is known as long as the scattering object does not have a simple shape : exact theories only exist in the cases of perfect spheres (Mie theory [2]), spheroidal particles [3] and infinitely long circular cylinders [4]. In general, it is not possible to get such an exact solution, since it involves the knowledge of the boundary conditions. Many models have been proposed to describe the light scattering, valid in some particular ranges of the ratio a/λ (λ being the wavelength of the incident radiation and a a characteristic length of the scatterer) [5]; the aim of this letter is to present a model, valid for $a \lesssim \lambda$ and to exhibit its remarkable agreement with the exact theories, in the cases where such theories do exist. In this model, following an idea of ref. [6] [7], the scatterer is modelled by an array of polarisable elements located on the sites of a cubic lattice; the cross sections are deduced from the total dipole moment of the scatterer. The letter is organized as follows : in section I, we briefly describe the model; in section II, we discuss the numerical results, and section III is devoted to some conclusive remarks and perspectives.

I. A BRIEF DESCRIPTION OF THE MODEL

We consider the scattering of light by a dielectric particle, considered as an aggregate of polarisable elements located on a cubic lattice. The complex electric polarisability α_e is related to the index of refraction $m = n + ik$ by the Clausius-Mossoti relation (see Van de Hulst [1] for instance) :

$$\alpha_e = \frac{3}{4\pi\rho} \frac{m^2 - 1}{m^2 + 2} \quad (1)$$

ρ being the number of polarisable elements by volume unit ($\rho = d_0^{-3}$, d_0 being the spacing of the lattice). The scattering object is acted on by an incident electromagnetic field, linearly polarized :

$$\vec{E}_{\text{ext}}(\vec{r}, t) = \vec{E}_0 \exp \{ i (\vec{k} \cdot \vec{r} - \omega t) \} \quad (2)$$

Following Jackson [1], the dipole moment $\vec{d}_i$ of the i -th element is proportional to the complex vector amplitude of the electric field at the site i

$$\vec{d}_i(t) = \alpha_e \vec{E}(\vec{r}_i, t) \quad (3)$$

$\vec{r}_i$ being the position of the site i .

Moreover, the electric field $\vec{E}(\vec{r}_i, t)$ is the sum of the external field (2) and the fields radiated by all the other dipoles, given for instance by Jackson [1] :

$$\vec{E}_{\text{rad}}(\vec{r}_i, t) = \sum_{\substack{j=1 \\ j \neq i}}^N \frac{e^{i k r_{ij}}}{r_{ij}} \left[((\vec{r}_{ij} \wedge \vec{d}_j) \wedge \vec{r}_{ij}) k^2 + \frac{1-i k r_{ij}}{2} (3 (\vec{d}_j \cdot \vec{r}_{ij}) \vec{r}_{ij} - r_{ij}^2 \vec{d}_j) \right] \quad (4)$$

where $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ and N is the total dipole number. It is convenient to introduce the 3×3 matrix :

$$\Pi(\vec{r}) = \frac{e^{i k r}}{r} \left[k^2 r^2 \left(1 - \frac{\vec{r} \otimes \vec{r}}{r^2} \right) + (1 - i k r) \left(3 \frac{\vec{r} \otimes \vec{r}}{r^2} - \mathbf{1} \right) \right] \quad (5)$$

$\mathbf{1}$ being the unit matrix; this allows a compact notation for $\vec{E}_{\text{rad}}$:

$$\vec{E}_{\text{rad}}(\vec{r}_i, t) = \sum_{\substack{j=1 \\ j \neq i}}^N \Pi(\vec{r}_{ij}) \cdot \vec{d}_j(t). \quad (6)$$

To compute the complex vector amplitudes $\vec{d}_i$ we will not solve the system of linear equations they satisfy as in ref. [6] but we will approximate $\vec{d}_i$ by a multiple scattering development [7] :

$$\begin{aligned} \vec{d}_i(t) = & \alpha_e \vec{E}_0 e^{i(\vec{k} \cdot \vec{r}_i - \omega t)} + \alpha_e^2 \sum_{\substack{j=1 \\ j \neq i}}^N \Pi(\vec{r}_{ij}) \cdot \vec{E}_0 e^{i(\vec{k} \cdot \vec{r}_j - \omega t)} \\ & + \alpha_e^3 \sum_{\substack{j=1 \\ j \neq i}}^N \sum_{\substack{l=1 \\ l \neq j}}^N \Pi(\vec{r}_{ij}) \Pi(\vec{r}_{jl}) \cdot \vec{E}_0 e^{i(\vec{k} \cdot \vec{r}_l - \omega t)} + \dots \end{aligned} \quad (7)$$

which can be interpreted as a development on the different Born approximations.

Given the dipole moments of all the polarisable elements, one can now derive the expression of the integrated cross sections .

From the expression of the mean energy absorbed by a dipole per unit time (see e.g. Jackson [1]) one gets the absorption cross-section :

$$\sigma_{\text{abs}} = 4 \pi k \frac{\text{Im}(\alpha_e)}{|\alpha_e|^2} \sum_{i=1}^N \frac{|\vec{d}_i|^2}{|\vec{E}_0|^2} \quad (8)$$

From the optical theorem [1], the extinction cross section reads :

$$\sigma_{\text{ext}} = -4 \pi k \sum_{i=1}^N \text{Im}(e^{i\vec{k} \cdot \vec{r}_i} \vec{d}_i^* \cdot \vec{E}_0) / |\vec{E}_0|^2 \quad (9)$$

And finally the scattering cross section reads :

$$\sigma_{\text{scat}} = \sigma_{\text{ext}} - \sigma_{\text{abs}} \quad (10)$$

We will also be interested in the efficiency factor :

$$Q = \sigma / G \quad (11)$$

where G is the area of the geometrical cross section of the scattering particle (for spheres $G : \pi \Omega_e^2$, Ω_e being the "equivalent radius"). To compute the differential cross section let us first remark that asymptotically eq (5) reads :

$$\Pi(\vec{r}) \underset{r \rightarrow \infty}{\cong} k^2 \frac{e^{ikr}}{r} (1 - \vec{n} \otimes \vec{n}) \quad (12)$$

where $\vec{n} = \vec{r}/r$ represents the observation direction. One can easily deduce the field created by the scatterer :

$$\vec{E}(\vec{r}, t) \cong k^2 \frac{e^{ikr}}{r} \sum_{i=1}^N e^{-i\vec{k} \cdot \vec{r}_i} (\mathbf{1} - \vec{n} \otimes \vec{n}) \cdot \vec{d}_i(t) \quad (13)$$

The differential cross section reads :

$$\frac{d\sigma}{d\Omega} = \frac{(\vec{I}_s \cdot \vec{n})^2}{I_0} r \quad (14)$$

where $\vec{I}_s$ (resp. I_0) is the time-averaged Poynting vector of the scattered wave (resp. incident wave). We can then deduce the intensity functions $i_1(\theta)$ (perpendicular to the scattering plane) and $i_2(\theta)$ (parallel to the scattering plane) as :

$$i_\epsilon(\theta) = \lim_{r \rightarrow \infty} \frac{k^6}{\pi} \int_0^{2\pi} d\phi \left[\sum_{i=1}^N e^{-i\vec{k} \cdot \vec{r}_i} \vec{d}_i \cdot \vec{\epsilon} \right] \left[\sum_{j=1}^N e^{-i\vec{k} \cdot \vec{r}_j} \vec{d}_j \cdot \vec{\epsilon} \right]^* \quad (15)$$

where (θ, ϕ) are the spherical coordinates of $\vec{n}$. $i_1(\theta)$ (resp. $i_2(\theta)$) corresponds to $\vec{e} = \vec{e}_\phi$ (resp. $\vec{e} = \vec{e}_\theta$).

II. NUMERICAL RESULTS

In order to test the validity of this multiple scattering model, we have performed some numerical computations to compare it with exact theories [2], [3].

From now on, we use an incident field of the form

$$\vec{E}_{inc}(\vec{r}, t) = E_0 \vec{e}_y \exp \{ i k (x \sin \zeta + z \cos \zeta) - i \omega t \} \quad (16)$$

where ζ is the incident angle.

To compare with the Mie theory we have first computed the light scattering by spheres at $\zeta = 0^\circ$. In table 1, we have compared the exact values for the extinction, absorption and scattering efficiency factors to the numerical results of our model using 305 polarizable elements for several values of the ratio $x = 2\pi a/\lambda$ (a being the radius of the sphere) and an index of refraction $m = 1.7 + .1i$. We are in close agreement up to $x = 1.9$. The number of iterations needed to get a stable result is of the order of 20 for $x < 1.5$. For greater x values we need a bigger number of iterations (~ 50 for $x = 1.9$). Our results are slightly better than those obtained in ref. [6]. The plots of the functions $i_1(\theta)$ and $i_2(\theta)$ (fig. 1a) and of the polarization $p(\theta)$ (fig. 1b) exhibit a very close agreement with the Mie theory for $x = 1.5$ and $m = 1.33 + .1i$. Our results are also in good agreement with the exact theory for $x = 1$ and $m = 1.33 + .1i$ or $1.7 + .1i$. Unfortunately the agreement is no longer so good when the real part of m and x increase as shown in figs. 2a and 2b. We can propose two complementary explanations. The first one deals with a roughness effect : the scatterer being composed by an aggregate of polarizable entities, it cannot be considered as a smooth grain. Experimental results for large particles ($ka \gg 1$) [8] exhibit an enhancement of the backscattered intensity in the rough case, like in fig. 2a, and a small negative polarization (absent in our case). Secondly, we have not taken into account the magnetic multipoles, whose contribution would be a modification of the complex index of refraction [9] when x increases at fixed number of polarizable elements, and for high values of the index of refraction.

Henceforth, it should be interesting to reconsider this example with a more important number of dipoles using a CRAY computer instead of a IBM one [10].

The second set of numerical results is concerned with light scattering by spheroidal particles, and should be compared with the results of ref. [3]. Once more, the agreement with the exact theory is very impressive, since the model used seems "a priori" very coarse. We have computed light scattering by prolate spheroids (of equation $(x^2 + y^2)/b^2 + (z^2/a^2) = 1$) and oblate spheroids ($(x^2/b^2) + (y^2 + z^2)/a^2 = 1$) for many values of the number $c = k \sqrt{a^2 - b^2}$, like in ref. [3]. In fig. 3, we have plotted the scattering for spheroids of index of refraction $m = 1.33$ efficiency factor Q_{sca} , as a function of c for three values of the ratio a/b in the prolate case and several incident angles ζ and for $a/b = 2$ and $\zeta = 0$ in the oblate case. Finally fig. 4 shows the functions $i_1(\theta)$ and $i_2(\theta)$ at $\zeta = 0^\circ$ for a prolate spheroid with $a/b = 2$ and $c = 1, 2, 3, 4, 5$. "built" with 507 polarizable elements. In both cases the curves exhibit a remarkable agreement with Asano-Yamamoto's results. A good agreement is also obtained for absorbing spheroids ($m = 1.5 + .1i$) of ratio $a/b = 2$ up to $c' \equiv ka = 5$.

III. CONCLUSIONS AND PERSPECTIVES

In this letter we have presented a model able to describe accurately the light scattering by an arbitrarily shaped object, in the range of validity $a \lesssim \lambda$. We have just exhibited some numerical results showing very nice agreement with the results of exact theories [2], [3]. Let us stress the simplicity of the model leading to numerical computations faster than those involved by Asano-Yamamoto theory for spheroids. Moreover it provides a very nice tool to investigate light scattering by objects different than spheres or spheroids and to study for instance the effects of roughness or irregularities (a more detailed study will be found in ref. [10]). Lets us point out that this work is relevant in astrophysics to compute light scattering by interplanetary and interstellar dust.

Finally, one could also ask oneself why such a rough model fits so well with exact theories. This is probably related to the quantum "low energy scattering theorems"

[11]; in the quantum scattering theory (see Newton [1] for a description, similar to the electromagnetic scattering theory), such theorems describe some independence of the scattering function on the scattering potential, valid at low energy ($a/\lambda < 1$). An application of such theorems, in connection with our multiple scattering model, is currently under study [12].

Table 1

x	Q_{ext}	Q_{abs}	Q_{sca}
. 5	. 14 (.13)	. 10 (.10)	. 03 (.03)
1 .	. 71 (.72)	. 29 (.30)	. 42 (.42)
1 . 5	1 . 88 (1.93)	. 56 (.60)	1 . 32 (1.33)
1 . 9	2 . 81 (2.78)	. 69 (.69)	2 . 12 (2.09)

The efficiency factors Q_{ext} , Q_{abs} , Q_{sca} for spheres of index of refraction $m = 1.7 + .1i$ as a function of x . The values obtained from the Mie theory are quoted between commas.

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FIGURE CAPTIONS

Fig. 1a : Angular distribution of the intensity functions $i_1(\theta)$ and $i_2(\theta)$ for spheres of size $x = 1.5$ and index of refraction $m = 1.33 + .1i$. Solid curve : Mie result. Dashed curve : our result.

Fig. 1b : Polarization $p(\theta)$ for spheres of size $x = 1.5$ and index of refraction $m = 1.33 + .1i$. Solid curve : Mie result. Dashed curve : our result.

Fig. 2 : The same as fig. 1 for spheres of size $x = 1.5$ and index of refraction $m = 1.7 + .1i$.

Fig. 3 : Scattering efficiency factor Q_{sca} as a function of the size parameter c for spheroids of index of refraction $m = 1.33$. Solid curve : prolate spheroids of $a/b = 10$. Dashed curve : prolate spheroids of $a/b = 5$. Small dashed curve : prolate spheroids of $a/b = 2$ for several incident angles ζ . Dot dashed curve : oblate spheroids of $a/b = 2$ at $\zeta = 0^\circ$.

Fig. 4 : Angular distribution of the intensity functions $i_1(\theta)$ (full curve) and $i_2(\theta)$ (dashed curve) for prolate spheroids of $a/b = 2$ and index of refraction $m = 1.33$ at $\zeta = 0^\circ$.

Fig. 4a : $c = 1$. Fig. 4b : $c = 2$. Fig. 4c : $c = 3$. Fig. 4d : $c = 4$.
Fig. 4e : $c = 5$.

Fig. 1 a

Fig. 1 b

Fig. 2 a

Fig. 2 b

Fig. 3

Fig. 4 a

Fig. 4 b

Fig. 4 c

Fig. 4 d

Fig. 4 e