SEMI-CLASSICAL QUANTIZATION OF THE SUPERMEMBRANE

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ABSTRACT

We approach the quantization of the eleven-dimensional supermembrane from a semi-classical viewpoint. We quantize the fluctuations around a topologically-stabilized toroidal background, and find that the bosonic and fermionic contributions to the vacuum energy cancel. This result appears to depend crucially on the fermionic gauge symmetry and space-time supersymmetry.

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1. INTRODUCTION

In recent years there has been an upsurge of interest in theories of extended objects. While this activity has primarily been concerned with superstrings, interesting developments have also taken place in the theories of higher-dimensional extended objects. Hughes, Liu and Polchinski [1] have shown how to construct a Green-Schwarz type action for a 3-extended object propagating in flat six-dimensional spacetime. The crucial feature of this action is the existence of a Siegel-type symmetry [2], which eliminates half of the degrees of freedom of the fermionic fields. As emphasized by Schwarz [3], the existence of such a local symmetry is a highly non-trivial feature, which could be regarded as essential for the viability of a theory of extended objects. The attractiveness of two-dimensional extended objects (i.e. membranes) has been enhanced by the discovery [4] that a supersymmetric theory of membranes can be constructed and, furthermore, that it describes the propagation of the membrane in an eleven-dimensional supergravity background. We recall that eleven is the maximum dimension permitted by supersymmetry and yet the critical dimension for superstrings is only ten. The discovery of an eleven-dimensional supermembrane may thus resolve this puzzle.

The consistency of coupling to an eleven-dimensional supergravity background is suggestive of the analogous situation in ten-dimensional superstring theories, where the massless background fields are generated as excitations of the string itself. Thus this result suggests the presence of massless states in the supermembrane's spectrum too. Further support for this idea has been found in the existence of a consistent Kaluza-Klein truncation of the classical supermembrane theory to type IIA superstring theory coupled to $N = 2a$ supergravity [5].

In this paper, we shall be concerned only with extended objects of one time and two space dimensions, i.e. "2-branes". It would be interesting to know whether supersymmetry permits other extended objects. Possible "p-brane" theories exist whenever there is a closed $p + 2$ form in superspace [4]. Thus, in addition to superstrings coupled to ten-dimensional supergravity and the super 2-brane coupled to eleven-dimensional supergravity, one could envisage a super 5-brane coupled to the dual version of ten-dimensional supergravity but this theory has not yet been constructed. Starting from these fundamental p-branes in d dimensions, other $(p - k)$ -branes may be constructed in $(d - k)$ dimensions by the simultaneous dimensional reduction of the world-volume and the space-time as discussed in [5]. p-branes in $(d - k)$ dimensions may also be obtained by ordinary dimensional reduction of the space-time only. Note that the latter theories need not have a closed $p + 2$ form, since Chern-Simons terms will arise via the dimensional reduction.

The really interesting question is whether a consistent quantum theory of the supermembrane can be defined, and if so whether it contains massless states and preserves the classical structure described above. The quantization of membranes differs in many

respects from the quantization of strings, and these differences have sometimes been advanced to argue that only strings can provide a consistent quantum theory of extended objects. Progress in the understanding of superstring theories has depended upon a number of features apparently specific to the two-dimensional worldsheet. One specific feature is the Siegel symmetry. We have already discussed how this has been generalized to higher-dimensional objects. Another apparently specific feature of the superstring is the occurrence of the group $SO(8)$, with its attendant triality properties, as the invariance group of the space of physical variables. Thus the 8 bosonic variables corresponding to the transverse $10 - 2 = 8$ dimensions are matched by the $8 = 16 \div 2$ fermionic variables remaining after the Siegel symmetry has been taken into account. As we shall see, the same balance occurs for the supermembrane: here we have $11 - 3 = 8$ physical bosonic variables, and again $16 \div 2 = 8$ fermionic variables after taking account of the Siegel symmetry.

Clearly, any consistent theory must be free from ghosts. Both strings and membranes apparently suffer from the existence of an infinity of ghosts, owing to the negative sign of the kinetic term of the time coordinate X^0 . In the case of strings, this problem is known to be solved in the critical dimension. The proof [6] invokes the conformal symmetry, which in two dimensions is infinite dimensional. In more than two dimensions the conformal group is only finite dimensional, and is in any case unlikely to be relevant for membranes [7]. To address the ghost problem for membranes, one can proceed in one of two ways. In the first, covariant, approach one looks for an infinite dimensional symmetry that will substitute for the Virasoro algebra. Little has been achieved in this direction so far, although it is interesting to note that Hoppe [8] has demonstrated the existence of infinite-parameter symmetries of the membrane Hamiltonian. An alternative approach is to go to a physical gauge, where ghost-freedom is manifest but Lorentz invariance is not. The problem now is to demonstrate that Lorentz invariance is nevertheless preserved in the quantization. For strings, this is a problem owing to the existence of conformal anomalies which spoil the closure of the Lorentz algebra. Hence the need for a critical dimension, in which the conformal anomalies cancel. For membranes, it is difficult to see how a critical dimension could arise in this sense, when there is no obvious candidate for an anomalous symmetry at least in the case of the bosonic membrane. In particular, there is no classical conformal invariance. Moreover, there are no diffeomorphism anomalies in three dimensions. In fact it seems that many of the special features of string theory are needed in order to cure the problems of living in two dimensions.

An important question for any membrane theory is whether massless particles are present in the spectrum. In the case of the supermembrane, consistency presumably requires the occurrence of the graviton, gravitino and antisymmetric tensor of eleven-dimensional supergravity in the spectrum. For the purely bosonic membrane, Kikkawa and Yamasaki have concluded that the open membrane does not admit massless spin 1 or 2 particles in any integer dimension [9]. Their argument involved computing the shift of

the intercept of the classical mass/spin relation induced by 1-loop vacuum energy effects, by analogy with the string calculations of Brink and Nielsen [10].

The question naturally arises as to whether the inclusion of fermions would cure this problem. However, in order to follow the method of Ref.[10], it would be necessary to have a Neveu-Schwarz-Ramond (NSR) formalism for the supermembrane; i.e. a formalism which exhibits world-volume supersymmetry. The supermembrane theory of Ref.[4] is written in a Green-Schwarz formalism, in which spacetime supersymmetry is manifest. It is not clear at present whether an NSR formalism exists for the supermembrane, but we draw attention to the following problem. As discussed in Sec.3, the supermembrane, by contrast to the superstring, requires a cosmological term in the action. In three-dimensional supergravity, the cosmological term arises from eliminating an auxiliary field S which appears both linearly and quadratically in the action. However there is no super-invariant quadratic in S that does not also involve the Einstein-Hilbert gravitational action. This means that the Nambu-Goto action cannot be supersymmetrized. This problem was implicit in the original supermembrane paper of Howe and Tucker [11] (we are grateful to P. Howe for drawing this to our attention), and is further discussed in [12]. Although an NSR calculation of the shift in the intercept is thus not possible, it is gratifying to find, as we shall see in Sec. 4, that for the vacuum we are considering in this paper, the vacuum energy is zero in the Green-Schwarz formalism.

Perhaps the most important question about the membrane theory, but about which little can be said at present, relates to the issue of renormalizability. In contrast to the two-dimensional string σ -model, the three-dimensional membrane σ -model is non-renormalizable by power-counting. One possibility, which has been discussed by Weinberg [13], is that the theory has only a finite number of relevant parameters, despite the non-renormalizability, by virtue of being asymptotically safe.

In this paper we take some first steps towards quantizing the membrane, by applying the methods of semiclassical quantization. The difficulties in defining a proper quantum theory of membranes stem from the inherent non-linearity of the field equations, as well as the related non-renormalizability in three dimensions. Semiclassical methods focus on humbler questions such as the perturbative structure of the spectrum and the issue of the vacuum energies.

We shall begin in Secs. 2 and 3 by discussing some exact features of the classical theory of membranes and supermembranes. In this we make use of the work of Hoppe [8] in choosing a gauge which closely parallels the light-cone gauge in string theory. In Sec. 4 we proceed to the semiclassical quantization. Stable classical solutions of the membrane equations are hard to find. In order to work with the simplest possible stable classical solution we adopt the device of having the membrane propagate not in eleven-dimensional Minkowski spacetime but rather in a spacetime in which two of the spatial dimensions have

been compactified to $S^1 \times S^1$. Then the membrane can be stabilized by being stretched over the 2-torus. We then quantize the linearized fluctuations around this background. Owing to the existence of the Siegel symmetry and world-volume general coordinate invariance, there is a match between the numbers of bosonic ($11 - 3 = 8$) and fermionic ($16 \div 2 = 8$) degrees of freedom. We find that the vacuum energies of the bosonic and fermionic modes cancel, as in the case of the Green-Schwarz superstring.

In Sec. 5 we discuss the issue of the membrane spectrum, deriving the constraints that must be satisfied by the physical states, analogous to the equality of left and right Hamiltonians in string theory. We discuss the supermultiplet structure of the spectrum, and the limits in which the supermembrane reduces to the superstring. Certain useful formulae and conventions are relegated to the appendix.

2. THE CLASSICAL BOSONIC MEMBRANE

By analogy with the Nambu-Goto action for the string, the action for the bosonic membrane is given by the world-volume swept out by the membrane,

$$I = \frac{-1}{4\pi^2\Xi} \int d^3\xi \sqrt{-\det(\partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu})}, \quad (2.1)$$

where Ξ^{-1} is the membrane tension, $\xi^i = (\tau, \sigma, \rho)$ ($i = 0, 1, 2$) are the three world-volume coordinates, and $X^\mu(\xi)$ gives the embedding of the world-volume in a d -dimensional Minkowski spacetime. It is easy to see that this action is classically equivalent to

$$I = \frac{-1}{8\pi^2\Xi} \int d^3\xi \sqrt{-g} [g^{ij} \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} - 1], \quad (2.2)$$

where g_{ij} is an auxiliary world-volume metric whose field equation is

$$g_{ij} = \partial_i X^\mu \partial_j X_\mu. \quad (2.3)$$

Note that the algebraic field equation determines g_{ij} entirely, since, unlike the case of the string, the action (2.2) is not conformally invariant.

The action (2.2) is invariant under world-volume general coordinate transformations, and under Poincaré transformations in the d -dimensional Minkowski spacetime. Varying (2.2) with respect to X^μ yields the field equation

$$\partial_i (\sqrt{-g} g^{ij} \partial_j X^\mu) = 0. \quad (2.4)$$

In this paper, we shall be considering only *closed* orientable membranes, so there are no boundary conditions to be imposed. Owing to the general coordinate invariance on the

world-volume, the system of equations (2.3) and (2.4) for the d fields $X^\mu(\xi)$ is underdetermined, i.e. there are 3 degrees of freedom that may be specified arbitrarily.

The independent degrees of freedom of the theory can be analyzed in the Hamiltonian formalism [14]. We begin by making an ADM parametrization of the world-volume metric g_{ij} [see, for example, 15], in terms of a shift-vector N^a ($a = 1, 2$), a lapse function N and a 2-metric γ_{ab} , all of which depend on τ, σ and ρ :

$$g_{00} = -N^2 + \gamma_{ab}N^aN^b, \quad g_{0a} = g_{a0} = \gamma_{ab}N^b, \quad g_{ab} = \gamma_{ab}. \quad (2.5)$$

$$g^{00} = -N^{-2}, \quad g^{0a} = g^{a0} = N^aN^{-2}, \quad g^{ab} = \gamma^{ab} - N^aN^bN^{-2}, \quad (2.6)$$

where γ^{ab} is the inverse of γ_{ab} , i.e. $\gamma^{ab}\gamma_{bc} = \delta_c^a$. It follows from (2.5) that

$$\sqrt{-g} = N\sqrt{\gamma}. \quad (2.7)$$

In terms of these variables, the action (2.2) takes the form

$$I = \frac{-1}{8\pi^2} \int d^3\xi N\sqrt{\gamma}[-N^{-2}(\dot{X} - N^a\partial_a X)^2 + \gamma^{ab}\partial_a X^\mu\partial_b X_\mu - 1], \quad (2.8)$$

where we have set $\Xi = 1$. The canonical momentum is given by

$$K_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = -\frac{1}{4\pi^2}\sqrt{-g}g^{\mu\alpha}\partial_\alpha X_\mu \quad (2.9)$$

$$= (4\pi^2 N)^{-1}\sqrt{\gamma}(\dot{X}_\mu - N^a\partial_a X_\mu). \quad (2.10)$$

The equations of motion of N and N^a are the gauge constraints for the theory, $\mathcal{H} \approx 0$, $\mathcal{H}_a \approx 0$, where

$$\mathcal{H} = \frac{1}{2}[\gamma^{-1/2}K^\mu K_\mu + \gamma^{1/2}(\gamma^{ab}\partial_a X^\mu\partial_b X_\mu - 1)], \quad (2.11)$$

$$\mathcal{H}_a = K^\mu\partial_a X_\mu. \quad (2.12)$$

As a consequence of these constraints, the Hamiltonian density H is weakly-vanishing,

$$4\pi^2 H \equiv K^\mu \dot{X}_\mu - L = N\mathcal{H} + N^a\mathcal{H}_a \approx 0. \quad (2.13)$$

As in general relativity, to determine the evolution of a system with a weakly-vanishing Hamiltonian one needs to fix gauges. Corresponding to (2.11) and (2.12) we can impose 3 gauge conditions. Together with the constraints, these eliminate 3 coordinates and their conjugate momenta. The remaining non-vanishing components γ_{ab} of the world-volume metric g_{ij} do not constitute independent degrees of freedom, since they satisfy the algebraic equation of motion

$$\gamma_{ab} = \partial_a X^\mu\partial_b X_\mu. \quad (2.14)$$

Thus the world-volume metric is eliminated entirely, but at the price of non-linearities in the equations of motion for the independent physical degrees of freedom.

In the case of the string, by making a judicious gauge choice one can turn the ostensibly non-linear system of equations into a linear system. The non-linearities of the membrane equations are much more severe. Nonetheless, significant simplification can be achieved by making a string-inspired light-cone gauge choice. Thus we begin by using the three general coordinate invariances to set

$$X^+ = p^+ \tau, \quad (2.15)$$

$$g_{0a} = 0 \quad , \quad (a = 1, 2) \quad (2.16)$$

where $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{10})$.

Although these conditions fix the gauge-freedoms that are functions of all three coordinates (τ, σ, ρ) , there remains a residual τ -independent reparametrization invariance of (2.15) and (2.16),

$$\delta \xi^0 = 0, \quad \delta \xi^a = f^a(\sigma, \rho). \quad (2.17)$$

Consequently, at a fixed time τ^* , one may impose further gauge conditions. Following Hoppe [8], we use one of the two gauge freedoms (2.17) to impose

$$g_{00} = -\gamma \quad (2.18)$$

at $\tau = \tau^*$, where $\gamma \equiv \det \gamma_{ab}$. Eq.(2.18) is in fact maintained for all τ , by virtue of the $\mu = +$ component of the field equation (2.4). This may be seen by substituting (2.15) and (2.16) into (2.4), to obtain

$$\frac{\partial}{\partial \tau} \left(\frac{\gamma}{g_{00}} \right)^{\frac{1}{2}} = 0. \quad (2.19)$$

There still remains one reparametrization freedom in (2.17) that leaves the Hoppe gauge condition (2.18) invariant, namely diffeomorphisms of the form (2.17) for which $\partial_a(\delta \xi^a) = 0$, i.e. [8]

$$\delta \xi^0 = 0, \quad \delta \xi^a = \varepsilon^{ab} \partial_b f(\sigma, \rho), \quad (2.20)$$

where $\varepsilon^{12} = -\varepsilon^{21} = 1$. The Jacobian of this transformation is unity. It is the generalization for membranes of the residual freedom in the case of the string to perform a shift of σ by a constant. For the time being, we shall leave this residual gauge freedom unfixed.

In order to count the number of physical degrees of freedom of the membrane, we must now take into account the constraints associated with the gauge freedoms discussed above. The coordinate X^+ is eliminated as an independent degree of freedom by (2.15). The algebraic equation (2.3) for the world-volume metric g_{ij} may be combined with the gauge choices (2.15), (2.16) and (2.18) to eliminate the coordinate X^- . Thus we have

$$\partial_a X^- = \frac{1}{p^+} \partial_a \underline{X} \cdot \dot{\underline{X}}, \quad (2.21)$$

$$\dot{X}^- = \frac{1}{2p^+} (\dot{\underline{X}}^2 + \gamma), \quad (2.22)$$

where $X^\mu = (X^+, X^-, \underline{X})$, $\dot{\underline{X}} \equiv \frac{\partial}{\partial \tau} \underline{X}$, and $\gamma = \det(\partial_a \underline{X} \cdot \partial_b \underline{X})$. These equations allow one to solve for X^- , but this does not exhaust the content of (2.21). Taking the curl of (2.21), we obtain the X^- -independent condition

$$\partial_a \underline{X} \cdot \partial_b \dot{\underline{X}} - \partial_b \underline{X} \cdot \partial_a \dot{\underline{X}} = 0. \quad (2.23)$$

Note that since X^- is a globally defined function on the membrane, Eq.(2.21) implies the constraint that

$$\oint \partial_a \underline{X} \cdot \dot{\underline{X}} d\ell^a = 0, \quad (2.24)$$

where the integral is performed around any closed curve on the membrane. For integrals around contractible loops, (2.24) contains the same information as the local constraint (2.23). However, if the membrane is not simply-connected, (2.24) contains additional global constraints in the case of non-contractible loops. In these cases, we obtain the membrane analogues of the equality of left and right Hamiltonians in closed string theory. It is straightforward to show that the left-hand side of the constraint equation (2.24) is independent of τ and continuous deformations of the path, by virtue of the field equations and the local constraint (2.23). The τ independence just expresses the usual preservation of the constraints by the field equations. The path independence implies that the left-hand side of (2.24) is a constant for any given non-contractible loop. The constraint is that this constant is zero.

In the Hamiltonian formalism the left-hand side of (2.23) is in fact the generator of the residual diffeomorphism (2.20). One may impose a gauge choice to fix the residual diffeomorphism freedom (2.20) at a particular time τ^* . Eq.(2.22) then determines the first time derivative of a variable to be eliminated, and the elimination is preserved for all τ by (2.4). This leaves $(d-3)$ independent physical degrees of freedom. In practice, we find it more convenient to leave the gauge freedom (2.20) unfixed for the time being.

After imposing the gauge conditions (2.15), (2.16) and (2.18), the equation of motion (2.4) for the transverse coordinates $\underline{X}$ becomes

$$\ddot{\underline{X}} = \partial_a (\gamma \gamma^{ab} \partial_b \underline{X}), \quad (2.25)$$

where $\gamma \gamma^{ab}$ is the matrix of minors of the metric

$$\gamma_{ab} = \partial_a \underline{X} \cdot \partial_b \underline{X}. \quad (2.26)$$

Thus (2.24) may be written as

$$\begin{aligned} \dot{\underline{X}} = & \partial_\sigma [(\partial_\rho \underline{X} \cdot \partial_\rho \underline{X}) \partial_\sigma \underline{X} - (\partial_\sigma \underline{X} \cdot \partial_\rho \underline{X}) \partial_\rho \underline{X}] \\ & + \partial_\rho [(\partial_\sigma \underline{X} \cdot \partial_\sigma \underline{X}) \partial_\rho \underline{X} - (\partial_\sigma \underline{X} \cdot \partial_\rho \underline{X}) \partial_\sigma \underline{X}], \end{aligned} \quad (2.27)$$

where $\partial_\sigma = \frac{\partial}{\partial \sigma}$, $\partial_\rho = \frac{\partial}{\partial \rho}$. The inherent non-linearity of (2.27) shows clearly the greater complexity of the membrane as compared with the string.

The action (2.2) is invariant under rigid Poincaré transformations in the target space. Correspondingly, we have conserved momentum and angular momentum charges

$$P^\mu = \int d\sigma d\rho K^\mu, \quad (2.28)$$

$$M^{\mu\nu} = \int d\sigma d\rho (X^\mu K^\nu - X^\nu K^\mu), \quad (2.29)$$

where K^μ is the canonical momentum given by (2.9). Note that (2.28) and (2.29) are invariant under world-volume general coordinate transformations, as they must be since the Poincaré transformations commute with reparametrizations on the world-volume. The freedom to pick an arbitrary foliation of the world-volume into a family of spacelike hypersurfaces parametrized by some “time” coordinate is a direct analogue of the “many-fingered time” of general relativity [15].

3. THE CLASSICAL SUPERMEMBRANE

The action for the supermembrane in flat superspace is [4]

$$I = \frac{-1}{4\pi^2} \int d^3\xi \left[\frac{1}{2} \sqrt{-g} g^{ij} \Pi_i^\mu \Pi_{j\mu} - \frac{1}{2} \sqrt{-g} + \epsilon^{ijk} \Pi_i^A \Pi_j^B \Pi_k^C B_{CBA} \right], \quad (3.1)$$

where $\Pi_i^A = (\Pi_i^\mu, \Pi_i^\alpha)$ with

$$\Pi_i^\mu = \partial_i X^\mu - i\bar{\psi} \Gamma^\mu \partial_i \psi, \quad (3.2)$$

$$\Pi_i^\alpha = \partial_i \psi^\alpha, \quad (3.3)$$

ψ^α is a 32-component Majorana spinor, $\mu = 0, 1, \dots, 10$, and we have set the membrane tension Ξ equal to unity. The super 3-form B_{CBA} is such that $dB = H$, with all components of H vanishing except $H_{\mu\nu\alpha\beta} = -\frac{1}{3}(C\Gamma_{\mu\nu})_{\alpha\beta}$. Our conventions are that $\epsilon^{012} = -1$, $\{\Gamma^\mu, \Gamma^\nu\} = -2\eta^{\mu\nu}$ with $\eta^{\mu\nu} = \text{diag}(-1, 1, \dots, 1)$, and $\bar{\psi}_\alpha = -\psi^\beta C_{\beta\alpha}$. The charge conjugation matrix $C_{\alpha\beta}$ is antisymmetric.

Solving for B_{CBA} , the action may be written as

$$\begin{aligned} I = \frac{-1}{8\pi^2} \int d^3\xi \left[\sqrt{-g} g^{ij} \Pi_i^\mu \Pi_{j\mu} - \sqrt{-g} + \epsilon^{ijk} \Pi_i^\mu \Pi_j^\nu \bar{\psi} \Gamma_{\mu\nu} \partial_k \psi \right. \\ \left. - i\epsilon^{ijk} \Pi_i^\mu \bar{\psi} \Gamma_{\mu\nu} \partial_j \psi \bar{\psi} \Gamma^\nu \partial_k \psi \right. \\ \left. - \frac{1}{3} \epsilon^{ijk} \bar{\psi} \Gamma_{\mu\nu} \partial_i \psi \bar{\psi} \Gamma^\mu \partial_j \psi \bar{\psi} \Gamma^\nu \partial_k \psi \right]. \quad (3.4) \end{aligned}$$

In addition to world-volume general coordinate invariance and Poincaré invariance, the action is also invariant under rigid spacetime supersymmetry transformations

$$\delta\psi = \varepsilon, \quad (3.5)$$

$$\delta X^\mu = i \bar{\varepsilon} \Gamma^\mu \psi, \quad (3.6)$$

and local Siegel transformations [4,2,1]

$$\delta\psi = (1 + \Gamma)\kappa, \quad (3.7)$$

$$\delta X^\mu = i \bar{\psi} \Gamma^\mu \delta\psi, \quad (3.8)$$

where κ is a world-volume scalar, spacetime Majorana spinor function of τ, σ and ρ , and

$$\Gamma = \frac{i}{6\sqrt{-g}} \varepsilon^{ijk} \Pi_i^\mu \Pi_j^\nu \Pi_k^\rho \Gamma_{\mu\nu\rho}. \quad (3.9)$$

For the action (3.4), Siegel invariance requires that g_{ij} also transform. The variation of g_{ij} is complicated, and may be found in Ref.[4]. In practice, it is more convenient to exhibit the Siegel invariance by going on “half-shell”, i.e. substituting the algebraic equation of motion for g_{ij} ,

$$g_{ij} = \Pi_i^\mu \Pi_{j\mu}, \quad (3.10)$$

into the action (3.4), and working in 1.5 order formalism.

Varying (3.4) with respect to X^μ and ψ yields

$$\partial_i(\sqrt{-g} g^{ij} \Pi_j^\mu) + \varepsilon^{ijk} \Pi_i^\nu \partial_j \bar{\psi} \Gamma^\mu{}_\nu \partial_k \psi = 0, \quad (3.11)$$

$$(1 - \Gamma) g^{ij} \Pi_i^\mu \Gamma_\mu \partial_j \psi = 0, \quad (3.12)$$

with Γ given by (3.9). In deriving (3.12) we have used the identity

$$\Gamma \Pi_k^\mu \Gamma_\mu = \Pi_k^\mu \Gamma_\mu \Gamma = \frac{-i}{2\sqrt{-g}} g_{k\ell} \varepsilon^{ij\ell} \Pi_i^\mu \Pi_j^\nu \Gamma_{\mu\nu}. \quad (3.13)$$

Note that after using the half-shell condition (3.10), we have $\Gamma^2 = 1$, and hence $\frac{1}{2}(1 \pm \Gamma)$ act as projection operators. Therefore (3.12) gives equations of motion for only 16 of the 32 components of ψ . This is a consequence of the Siegel invariance of the action; equation (3.7) may be used to gauge away the other 16 unphysical components of ψ . A convenient gauge to choose is

$$\Gamma\psi = -\psi. \quad (3.14)$$

The canonical momentum $K^\mu = \frac{\partial L}{\partial \dot{X}^\mu}$ is given by

$$K^\mu = -\frac{1}{4\pi^2} \sqrt{-g} g^{\alpha i} \Pi_i^\mu - \frac{1}{4\pi^2} \varepsilon^{0ij} (\Pi_i^\nu + \frac{i}{2} \bar{\psi} \Gamma^\nu \partial_i \psi) \bar{\psi} \Gamma^\mu{}_\nu \partial_j \psi. \quad (3.15)$$

Substituting this into (2.28) yields the conserved total momentum. One can easily verify that P^μ is invariant under supersymmetry transformations as the supersymmetry algebra requires it to be. The formula for the total angular momentum $M^{\mu\nu}$ is now more complicated than (2.29), and we find

$$M^{\mu\nu} = \int d\sigma d\rho [X^\mu K^\nu - X^\nu K^\mu - \frac{i}{8\pi^2} \sqrt{-g} g^{\alpha i} \Pi_i^\rho \bar{\psi} \Gamma_\rho \Gamma^{\mu\nu} \psi + \frac{1}{16\pi^2} \epsilon^{0ij} \Pi_i^\rho \Pi_j^\sigma \bar{\psi} \Gamma_{\rho\sigma} \Gamma_{\mu\nu} \psi + \frac{i}{16\pi^2} \epsilon^{0ij} (\Pi_i^\rho + \frac{i}{3} \bar{\psi} \Gamma^\rho \partial_i \psi) (\bar{\psi} \Gamma^\sigma \Gamma^{\mu\nu} \psi \bar{\psi} \Gamma_{\rho\sigma} \partial_j \psi + \bar{\psi} \Gamma_{\rho\sigma} \Gamma^{\mu\nu} \psi \bar{\psi} \Gamma^\sigma \partial_j \psi)]. \quad (3.16)$$

The remaining generator in the eleven-dimensional super-Poincaré algebra is the supercharge Q^α , which we find to be

$$Q = \frac{1}{4\pi^2} \int d\sigma d\rho [-2i \sqrt{-g} g^{\alpha i} \Pi_i^\mu \Gamma_\mu \psi + \epsilon^{0ij} \Pi_i^\mu \Pi_j^\nu \Gamma_{\mu\nu} \psi + \frac{4i}{3} \epsilon^{0ij} (\Pi_i^\mu + \frac{2i}{5} \bar{\psi} \Gamma^\mu \partial_i \psi) (\Gamma_{\mu\nu} \psi \bar{\psi} \Gamma^\nu \partial_j \psi + \Gamma^\nu \psi \bar{\psi} \Gamma_{\mu\nu} \partial_j \psi)]. \quad (3.17)$$

4. SEMICLASSICAL QUANTIZATION

Owing to the intrinsically non-linear structure of the membrane theory, we are forced to adopt a semiclassical approach to quantization. This involves choosing a stable classical solution and quantizing the fluctuations about it. From this procedure we may investigate the relative contributions of the bosons and fermions to the vacuum energy and the structure of the spectrum of states, in the sector of the theory defined by the classical solution.

In order to simplify the problem as much as possible, we consider a membrane propagating in a spacetime with topology $R^9 \times S^1 \times S^1$. With such a topology there is a classical solution with a toroidal membrane stretched around the $S^1 \times S^1$. Choosing the $S^1 \times S^1$ to be in the X^1 and X^2 directions, the classical solution takes the form of a purely bosonic background, with

$$X^1 = \ell_1 R_1 \sigma, \quad X^2 = \ell_2 R_2 \rho, \quad X^I = 0, \quad I = 3, \dots, 9, \quad (4.1)$$

$$\psi = 0, \quad (4.2)$$

where $0 \leq \sigma \leq 2\pi$, $0 \leq \rho \leq 2\pi$, R_1 and R_2 are the radii of the two circles, and ℓ_1 and ℓ_2 are integers characterizing the winding numbers of the membrane around the two circles. The most general flat 2-torus is characterized by 3 parameters, but for simplicity we take the two circles to be orthogonal. We are working in the gauge discussed in Sec. 2, with $X^+ = p^+ \tau$. The world-volume metric in this background is flat,

$$g_{ij} = \text{diag}(-(\ell_1 \ell_2 R_1 R_2)^2, (\ell_1 R_1)^2, (\ell_2 R_2)^2), \quad (4.3)$$

and solving for X^- we find

$$X^- = \frac{1}{2p^+} (\ell_2 \ell_2 R_1 R_2)^2 \tau. \quad (4.4)$$

The classical $(\text{mass})^2$ of the solution is given by $-P^\mu P_\mu$, where P^μ is given by (2.28), and is

$$(\text{mass})^2 = (\ell_1 \ell_2 R_1 R_2)^2. \quad (4.5)$$

For simplicity we shall set $\ell_1 = \ell_2 = R_1 = R_2 = 1$ for the rest of the calculations in this section, restoring them in the final results. Thus $X^+ = p^+ \tau$, $X^- = \frac{1}{2p^+} \tau$, $X^1 = \sigma$, $X^2 = \rho$ and $g_{ij} = \text{diag}(-1, 1, 1)$ in the background.

We now consider fluctuations $\underline{Z}$ of the transverse coordinate around the classical solution, and write $\underline{X} = \underline{X}_{\text{classical}} + \underline{Z}$. Thus

$$\begin{aligned} X^1 &= \sigma + Z^1, \\ X^2 &= \rho + Z^2, \\ X^I &= Z^I. \end{aligned} \quad (4.6)$$

The fermions, being zero in the background, are pure fluctuations ψ . Substituting (4.6) into (3.11) and (3.12), and keeping only the terms of linear order in $\underline{Z}$ and ψ , we find, in the gauges discussed in Secs. 2 and 3 ($X^+ = p^+ \tau$, $g_{\alpha\alpha} = 0$, $g_{\alpha\sigma} = -\gamma$, $\psi = -\Gamma\psi$),

$$\ddot{Z}^1 = \partial_\sigma \partial_\sigma Z^1 + \partial_\sigma \partial_\rho Z^2, \quad (4.7)$$

$$\ddot{Z}^2 = \partial_\rho \partial_\rho Z^2 + \partial_\sigma \partial_\rho Z^1, \quad (4.8)$$

$$\ddot{Z}^I = \partial_\sigma \partial_\sigma Z^I + \partial_\rho \partial_\rho Z^I, \quad (4.9)$$

$$\dot{\psi} = i\Gamma_2 \partial_\sigma \psi - i\Gamma_1 \partial_\rho \psi. \quad (4.10)$$

At this stage, we must fix the remaining gauge invariance of Eq.(2.20). The gauge choice $g_{\alpha\alpha} = 0$ yields an equation generalizing (2.21) for the supermembrane. This equation can be solved for $\partial_\alpha X^-$, leaving a residual constraint that generalizes (2.23). Upon linearization about our background, this constraint gives

$$\partial_\rho \dot{Z}^1 = \partial_\sigma \dot{Z}^2. \quad (4.11)$$

This can be integrated to give $\partial_\rho Z^1 = \partial_\sigma Z^2 + h(\sigma, \rho)$, which leads us to exploit the residual gauge symmetry (2.20) to set the undetermined function $h(\sigma, \rho)$ to zero, thus obtaining for all τ

$$\partial_\rho Z^1 = \partial_\sigma Z^2. \quad (4.12)$$

This allows us to cast (4.7) and (4.8) into the form of standard wave equations,

$$\ddot{Z}^1 = \partial_\sigma \partial_\sigma Z^1 + \partial_\rho \partial_\rho Z^1, \quad (4.13)$$

$$\ddot{Z}^2 = \partial_\sigma \partial_\sigma Z^2 + \partial_\rho \partial_\rho Z^2. \quad (4.14)$$

The relation (4.12) reduces the number of independent functions of τ, σ and ρ by one, yielding 8 bosonic degrees of freedom. Since the fermion equation of motion (4.10) is of first order, the 16 Siegel–gauge–fixed components of ψ satisfying (3.14) also give rise to 8 fermionic degrees of freedom.

Using the representation for the eleven–dimensional Γ –matrices given in the appendix, we can express ψ in the Siegel gauge (3.14) as

$$\psi = (16\sqrt{2} p^+)^{-\frac{1}{2}} \begin{pmatrix} \chi \\ -i\chi^* \\ -\sqrt{2}p^+\chi \\ -i\sqrt{2}p^+\chi^* \end{pmatrix} \quad (4.15)$$

where χ is a complex 8–component spinor of $SO(7) \times U(1) \subset SO(9)$. Substituting (4.15) into (4.10) casts the fermion equation into the form

$$\dot{\chi} = \partial_\sigma \chi^* - i\partial_\rho \chi^*. \quad (4.16)$$

Substituting (4.6) and (4.15) into the gauge conditions $g_{oo} = -\gamma$ and $g_{oa} = 0$, and keeping terms to quadratic order in fluctuations,

$$\begin{aligned} \dot{X}^- = \frac{1}{2p^+} & \left[1 + \dot{\underline{Z}}^2 + (\partial_\sigma \underline{Z})^2 + (\partial_\rho \underline{Z})^2 + G(\underline{Z}), \right. \\ & \left. + \frac{i}{2} [(\chi^\dagger \partial_\sigma \chi^* + \chi^T \partial_\sigma \chi) - i(\chi^\dagger \partial_\rho \chi^* - \chi^T \partial_\rho \chi)] \right], \end{aligned} \quad (4.17)$$

$$\partial_\sigma X^- = \frac{1}{p^+} \left[\dot{Z}^1 + \partial_\sigma \underline{Z} \cdot \dot{\underline{Z}} + \frac{i}{8} [(\chi^T \dot{\chi} + \chi^\dagger \dot{\chi}^*) + (\chi^\dagger \partial_\sigma \chi + \chi^T \partial_\sigma \chi^*)] \right], \quad (4.18)$$

$$\partial_\rho X^- = \frac{1}{p^+} \left[\dot{Z}^2 + \partial_\rho \underline{Z} \cdot \dot{\underline{Z}} + \frac{i}{8} [i(\chi^T \dot{\chi} - \chi^\dagger \dot{\chi}^*) + (\chi^\dagger \partial_\rho \chi^* + \chi^T \partial_\rho \chi^*)] \right], \quad (4.19)$$

where

$$G(\underline{Z}) = 2(\partial_\sigma Z^1 + \partial_\rho Z^2) + 4(\partial_\sigma Z^1 \partial_\rho Z^2 - \partial_\rho Z^1 \partial_\sigma Z^2). \quad (4.20)$$

When we come to construct the mass formula, $G(\underline{Z})$ will drop out in the $\int d\sigma d\rho$ integral.

As a preliminary to quantization, we write the general solutions of (4.13), (4.14), (4.9) and (4.16) as

$$\underline{Z} = \underline{z}_o + \underline{p}\tau + \frac{1}{\sqrt{2}} \sum_{m^2+n^2 \neq 0} \frac{1}{\omega_{mn}} e^{i(m\sigma+n\rho)} [\underline{\alpha}_{mn}^\dagger e^{i\omega_{mn}\tau} + \underline{\alpha}_{-m-n} e^{-i\omega_{mn}\tau}], \quad (4.21)$$

$$\chi = \sqrt{2} S_{oo} + \sum_{m^2+n^2 \neq 0} e^{i(m\sigma+n\rho)} \left[\frac{m-in}{\omega_{mn}} S_{mn}^\dagger e^{i\omega_{mn}\tau} + S_{-m-n} e^{-i\omega_{mn}\tau} \right], \quad (4.22)$$

and

$$\omega_{mn} = (m^2 + n^2)^{\frac{1}{2}}, \quad (4.23)$$

where, in (4.17), the Fourier coefficients have been chosen so that $\underline{Z}$ is real.

Proceeding to the semiclassical quantization, we must distinguish between the unconstrained variables Z^I ($I = 3, \dots, 9$) and the two variables Z^1 and Z^2 which satisfy the constraint (4.11) and the gauge condition (4.12). For the unconstrained variables we have the canonical commutation relation $[K^I, Z^J] = -i\delta^{IJ}\delta(\sigma - \sigma')\delta(\rho - \rho')$ which implies, from (2.9), that

$$[\dot{Z}^I, Z^J] = -(2\pi)^2 i \delta^{IJ} \delta(\sigma - \sigma') \delta(\rho - \rho'). \quad (4.24)$$

For the constrained variables Z^1 and Z^2 , we must proceed differently, owing to the need to incorporate (4.11) and (4.12). For this, we adopt Dirac's procedure for quantizing constrained variables [16]. Eqs.(4.11) and (4.12) constitute a non-commuting (i.e. second class) system of constraints. The canonical momenta conjugate to Z^1 and Z^2 are $k^1 = (2\pi)^{-2} \dot{Z}^1$, $k^2 = (2\pi)^{-2} \dot{Z}^2$. Denoting the constraints (4.11) and (4.12) by $\phi_s = 0$, $s = 1, 2$, with

$$\begin{aligned} \phi_1 &= \partial_\rho k^1 - \partial_\sigma k^2, \\ \phi_2 &= \partial_\rho Z^1 - \partial_\sigma Z^2, \end{aligned} \quad (4.25)$$

we have the Poisson bracket algebra

$$[\phi_1(\sigma, \rho), \phi_2(\sigma', \rho')]_P = \nabla^2 \delta(\sigma - \sigma') \delta(\rho - \rho') \equiv C_{12}, \quad (4.26)$$

where $\nabla^2 = \partial_\sigma^2 + \partial_\rho^2$.

The Dirac brackets are now defined by

$$[A, B]_D = [A, B]_P - [A, \phi_s]_P (C^{-1})^{st} [\phi_t, B]_P, \quad (4.27)$$

for arbitrary operators A and B. The Dirac brackets have the property that ϕ_1 and ϕ_2 now commute with each other and with the Hamiltonian. The Dirac brackets of the conjugate pairs (Z^1, k^1) and (Z^2, k^2) are:

$$\begin{aligned} [k^1(\sigma, \rho), Z^1(\sigma', \rho')]_D &= - \left(1 - \frac{\partial_\rho^2}{\nabla^2} \right) \delta(\sigma - \sigma') \delta(\rho - \rho'), \\ [k^2(\sigma, \rho), Z^2(\sigma', \rho')]_D &= - \left(1 - \frac{\partial_\sigma^2}{\nabla^2} \right) \delta(\sigma - \sigma') \delta(\rho - \rho'). \end{aligned} \quad (4.28)$$

We now pass from the Dirac brackets to the quantum commutators

$$\begin{aligned} [\dot{Z}^1(\sigma, \rho), Z^1(\sigma', \rho')] &= -(2\pi)^2 i \left(1 - \frac{\partial_\rho^2}{\nabla^2} \right) \delta(\sigma - \sigma') \delta(\rho - \rho'), \\ [\dot{Z}^2(\sigma, \rho), Z^2(\sigma', \rho')] &= -(2\pi)^2 i \left(1 - \frac{\partial_\sigma^2}{\nabla^2} \right) \delta(\sigma - \sigma') \delta(\rho - \rho'). \end{aligned} \quad (4.29)$$

For the fermions, one could proceed as above using the Dirac formalism for the eleven-dimensional spinor satisfying the Majorana and Siegel conditions. In practice, we find it more convenient to work with the unconstrained spinor χ . Substituting (4.15) into the action (3.1), one can quantize χ canonically, yielding

$$\{\chi^{*A}, \chi^B\} = 2(2\pi)^2 \delta^{AB} \delta(\sigma - \sigma') \delta(\rho - \rho'), \quad (4.30)$$

where $A, B = 1, \dots, 8$ are $\text{SO}(7)$ spinor indices.

Substituting (4.21) and (4.22) into (4.24), (4.29) and (4.30), we find the following commutation relations for the α and S oscillators:

$$[\alpha_{mn}^1, \alpha_{m'n'}^{1\dagger}] = \frac{m^2}{\omega_{mn}} \delta_{mm'} \delta_{nn'}, \quad (4.31)$$

$$[\alpha_{mn}^2, \alpha_{m'n'}^{2\dagger}] = \frac{n^2}{\omega_{mn}} \delta_{mm'} \delta_{nn'}, \quad (4.32)$$

$$[\alpha_{mn}^I, \alpha_{m'n'}^{J\dagger}] = \omega_{mn} \delta^{IJ} \delta_{mm'} \delta_{nn'}, \quad (4.33)$$

$$\{S_{mn}^A, S_{m'n'}^{B\dagger}\} = \delta^{AB} \delta_{mm'} \delta_{nn'}, \quad (4.34)$$

$$[p^1, z_0^1] = [p^2, z_0^2] = -i, \quad [p^I, z_0^J] = -i\delta^{IJ}, \quad (4.35)$$

$$\{S_{00}^A, S_{00}^{B\dagger}\} = \delta^{AB}, \quad (4.36)$$

with all other independent commutators vanishing. Note that the constraint (4.12) implies that

$$n\alpha_{mn}^1 = m\alpha_{mn}^2 \quad (4.37)$$

and that this is consistent with (4.31) and (4.32). It also implies that $[\alpha_{mn}^1, \alpha_{m'n'}^{2\dagger}] = \frac{mn}{\omega_{mn}} \delta_{mm'} \delta_{nn'}$, etc.

Substituting (4.21) and (4.22) into (4.17), and then the result into (3.15) and (2.28), we find

$$\begin{aligned} P^- = \frac{1}{2p^+} [1 + \underline{p}^2 + \sum_{m^2+n^2 \neq 0} (\underline{\alpha}_{mn} \cdot \underline{\alpha}_{mn}^\dagger + \underline{\alpha}_{mn}^\dagger \cdot \underline{\alpha}_{mn}) \\ + \sum_{m^2+n^2 \neq 0} \omega_{mn} (-S_{mn}^A S_{mn}^{A\dagger} + S_{mn}^{A\dagger} S_{mn}^A)]. \end{aligned} \quad (4.38)$$

Using the fact that $P^+ = p^+$ and $\underline{P} = \underline{p}$, we therefore have the eleven-dimensional mass formula

$$(\text{mass})^2 = (\ell_1 \ell_2 R_1 R_2)^2 + H, \quad (4.39)$$

where

$$H = 2 \sum_{m^2+n^2 \neq 0} (\underline{\alpha}_{mn}^\dagger \cdot \underline{\alpha}_{mn} + \omega_{mn} S_{mn}^{A\dagger} S_{mn}^A). \quad (4.40)$$

In obtaining the result (4.39) from (4.38), we have restored the winding numbers ℓ_1 and ℓ_2 , and radii R_1 and R_2 of the two circles, which we had set equal to unity earlier; now

$$\omega_{mn} = [(m\ell_2 R_2)^2 + (n\ell_1 R_1)^2]^{\frac{1}{2}}. \quad (4.41)$$

A key result of this paper is that the mass formula (4.39, 4.40) is correct as it stands, without a vacuum energy term. This is because the bosonic contribution, which takes the form of an Epstein zeta function,

$$\Delta_B(\text{mass})^2 = \sum_{m^2+n^2 \neq 0} [(m\ell_2 R_2)^2 + (n\ell_1 R_1)^2]^{\frac{1}{2}}, \quad (4.42)$$

coming from normal-ordering (4.38), is cancelled mode-by-mode by an equal but opposite fermionic contribution. Note that $\alpha_{mn}^\dagger$ and $S_{mn}^\dagger$ are creation operators for all m and n , while α_{mn} and S_{mn} are annihilation operators. The difference between this convention and the usual string convention stems from the form of our oscillator expansions (4.21) and (4.22).

The cancellation of the vacuum energy contributions (4.41) between bosons and fermions depends crucially on the Siegel symmetry (3.7, 3.8), which caused the number of fermionic physical degrees of freedom to be halved, (Eq.(4.15)). This cancellation indicates a striking difference in the quantum behaviour of the supermembrane as compared to the bosonic membrane. In the case of the bosonic membrane, the non-integral value of the vacuum energy sum has been advanced as an argument for the absence of massless states [9]. By contrast, to the semiclassical order studied in the present paper, the supermembrane's vacuum energy cancellation has taken place in a fashion analogous to that of the superstring in the Green-Schwarz formalism. We shall discuss the implications of this result in more detail in Sec. 6.

5. ON THE SPECTRUM OF STATES

In closed string theories, the structure of the spectrum is determined by the constraint that relates the left and right Hamiltonians, and by the quantum numbers of the vacuum. For the closed bosonic membrane, the constraints analogous to the equality of left and right Hamiltonians are provided by the loop integral in Eq.(2.24). For the supermembrane, these constraints acquire fermionic terms, and to the order to which we are working, they are given by integrals around non-contractible loops of (4.18) and (4.19),

$$\oint \partial_a X^- d\ell^a = 0. \quad (5.1)$$

At first sight this constraint seems to contain far more information for any given loop than in the string case, since it involves an integral over only one of the two spatial membrane

coordinates. However, as already explained in Sec. 2, the field equations and the local constraint (2.23) for the bosonic membrane make the left-hand side of (5.1) depend only on the homotopy class of the closed curve around which the integration is performed. The same comment applies to the supermembrane, using the field equations and the constraints (4.18) and (4.19). Thus there is only one constraint for each homotopically inequivalent loop.

Substituting (4.21) and (4.22) into (4.18) and (4.19), and then using (5.1), we obtain two constraints for our toroidal membrane:

$$\ell_1 R_1 p^1 + N_B^{(1)} + N_F^{(1)} = 0, \quad (5.2)$$

$$\ell_2 R_2 p^2 + N_B^{(2)} + N_F^{(2)} = 0, \quad (5.3)$$

where the “number operators” are

$$N_B^{(1)} = \sum_{m^2+n^2 \neq 0} \frac{m}{\omega_{mn}} \underline{\alpha}_{mn}^\dagger \cdot \underline{\alpha}_{mn}, \quad N_B^{(2)} = \sum_{m^2+n^2 \neq 0} \frac{n}{\omega_{mn}} \underline{\alpha}_{mn}^\dagger \cdot \underline{\alpha}_{mn}, \quad (5.4)$$

$$N_F^{(1)} = \sum_{m^2+n^2 \neq 0} m S_{mn}^{A\dagger} S_{mn}^A, \quad N_F^{(2)} = \sum_{m^2+n^2 \neq 0} n S_{mn}^{A\dagger} S_{mn}^A. \quad (5.5)$$

The commutation relations for these operators follow from (4.31)–(4.37), and are

$$\begin{aligned} [N_B^{(1)}, \underline{\alpha}_{mn}^\dagger] &= m \underline{\alpha}_{mn}^\dagger, & [N_B^{(2)}, \underline{\alpha}_{mn}^\dagger] &= n \underline{\alpha}_{mn}^\dagger, \\ [N_B^{(1)}, \underline{\alpha}_{mn}] &= -m \underline{\alpha}_{mn}, & [N_B^{(2)}, \underline{\alpha}_{mn}] &= -n \underline{\alpha}_{mn}, \\ [N_F^{(1)}, S_{mn}^{A\dagger}] &= m S_{mn}^{A\dagger}, & [N_F^{(2)}, S_{mn}^{A\dagger}] &= n S_{mn}^{A\dagger}, \\ [N_F^{(1)}, S_{mn}^A] &= -m S_{mn}^A, & [N_F^{(2)}, S_{mn}^A] &= -n S_{mn}^A. \end{aligned} \quad (5.6)$$

We recall that all the daggered operators are creation operators, and all the operators without daggers are annihilation operators.

The vacuum in the solitonic sector that we are considering will be chosen to satisfy

$$\alpha_{mn}|vac\rangle = 0, \quad S_{mn}^A|vac\rangle = 0, \quad m^2 + n^2 \neq 0. \quad (5.7)$$

These conditions are all that are necessary in order to determine the various mass levels for the spectrum. These are determined by the commutation relations of the operator H given in (4.40), which are

$$[H, \underline{\alpha}_{mn}^\dagger] = 2\omega_{mn} \underline{\alpha}_{mn}^\dagger, \quad [H, \underline{\alpha}_{mn}] = -2\omega_{mn} \underline{\alpha}_{mn}, \quad (5.8)$$

$$[H, S_{mn}^{A\dagger}] = 2\omega_{mn} S_{mn}^{A\dagger}, \quad [H, S_{mn}^A] = -2\omega_{mn} S_{mn}^A. \quad (5.9)$$

Thus the mass of a state obtained by acting on the vacuum with creation operators

$$\alpha_{m_1 n_1}^\dagger \cdots \alpha_{m_i n_i}^\dagger S_{p_1 q_1}^\dagger \cdots S_{p_j q_j}^\dagger |vac\rangle \quad (5.10)$$

will be

$$(\text{mass})^2 = (\ell_1 \ell_2 R_1 R_2)^2 + \omega_{m_1 n_1} + \cdots + \omega_{m_i n_i} + \omega_{p_1 q_1} + \cdots + \omega_{p_j q_j}. \quad (5.11)$$

The constraints (5.2) and (5.3) determine which combinations of creation operators in (5.10) are allowed. For example, if the quantized momenta p^1 and p^2 in the toroidal directions are both zero, the constraints imply that

$$m_1 + \cdots + m_i + p_1 + \cdots + p_j = 0, \quad n_1 + \cdots + n_i + q_1 + \cdots + q_j = 0. \quad (5.12)$$

The conditions (5.7) are not sufficient to determine the full structure of the vacuum. As in the Green-Schwarz formulation of the superstring, the vacuum actually is a supermultiplet. The structure of this supermultiplet may be determined by examining the consistency of the various possible assignments of quantum numbers using the explicit forms for the Lorentz generators and the supercharge [see, for example, 17].

We have not carried out this analysis in detail. Nevertheless, from the mass formula (4.39), we see that in the solitonic sector under consideration the vacuum will correspond to a set of states with non-vanishing eleven-dimensional mass. Even though nine-dimensional states are classified by $SO(8) \times U(1) \times U(1)$, the eleven dimensional mass is still relevant to the classification of states because it determines the structure of the vacuum supermultiplet. Although the quantized momenta p^1 and p^2 correspond to central charges in the $N = 2$ nine-dimensional supersymmetry that will classify the fluctuations about our classical solution, the fact that the $(\text{mass})^2$ (4.39) is non-vanishing means that no multiplet-shortening will take place and so the vacuum must correspond to a long massive $N = 2$, $d = 9$ multiplet with $(\text{mass})^2 = (\ell_1 \ell_2 R_1 R_2)^2$. The simplest such multiplet has 2^{15} boson plus 2^{15} fermion on-shell states. The dimensional reduction of this multiplet to four dimensions is an $N = 8$, $d = 4$ multiplet with maximal spin 4, and its off-shell form has been given explicitly in Ref.[18].

It is instructive to establish a correspondence between the above semiclassical results and the known structure of superstring theories. If we let R_2 tend stepwise to zero while at the same time increasing ℓ_2 such that the product $\ell_2 R_2$ is kept fixed, say $\ell_2 R_2 = 1$, then the ρ -dimension is shrunk while keeping the energy of the classical solution constant. Essentially, one is rolling up the membrane, keeping its area constant, to produce a closed string. In this limit ρ -dependence and non-zero momentum p^2 are excluded and one obtains from (4.39) the mass formula for a ten-dimensional superstring compactified on a circle of radius R_1 . The nine-dimensional mass is given by

$$(\text{mass}_9)^2 = \ell_1^2 R_1^2 + \frac{m_1^2}{R_1^2} + 2 \sum_{m \neq 0} [\alpha_{m0}^\dagger \cdot \alpha_{m0} + \omega_{m0} S_{m0}^{A\dagger} S_{m0}^A] \quad (5.13)$$

where $\frac{m_1}{R_1}$ is the quantized momentum p^1 on the circle, and

$$\omega_{m0} = |m|. \quad (5.14)$$

In this limit, the constraint (5.3) becomes empty, while (5.2) yields

$$\ell_1 m_1 + \sum_{m \neq 0} \frac{m}{\omega_{m0}} \alpha_{m0}^\dagger \cdot \alpha_{m0} + \sum_{m \neq 0} m S_{m0}^{A\dagger} S_{m0}^A = 0. \quad (5.15)$$

The bosonic parts of the above results agree with those obtained in [19], where the compactification of the bosonic string on a circle was discussed. Note that with our definitions of α and S oscillators in (4.21) and (4.22), $\alpha_{m0}^\dagger$ and $S_{m0}^\dagger$ create left-moving states when $m > 0$ and right-moving states when $m < 0$, while α_{m0} and S_{m0} annihilate left-moving states when $m > 0$ and right-moving states when $m < 0$. Thus (5.15) is equivalent to the usual constraint relating left and right Hamiltonians for the string.

6. CONCLUSIONS

In this paper we have developed the formalism for semiclassical quantization of the supermembrane about a topologically stabilized classical solution. In the semiclassical approximation, the supermembrane's quantum behaviour has proved to be very similar to that of the superstring in the Green-Schwarz formalism. In particular, the vacuum energies of the boson and fermion modes have cancelled. This is encouraging for the prospect of the theory's spectrum containing massless states.

In fact, the calculation we have performed conveys some direct information on massless states in the spectrum. If we set $\ell_2 = 0$, the stable classical solution will be a collapsed string-like membrane wound around only one of the compact directions in the target space. This is the same configuration as that studied in the Kaluza-Klein discussion of Ref. 5. For this classical configuration, the mass formula (4.39) indicates the presence of massless states. Although the world-volume metric for this classical solution is degenerate, the field equations are nevertheless non-singular. It is not clear whether the semiclassical approximation can be trusted for such a configuration. In particular, (4.41) shows that all values of m give the same frequency ω_{mn} in the semiclassical approximation. This infinite degeneracy will presumably be lifted when the higher order terms in the Hamiltonian are included. On the other hand, in the $\ell_1 \neq 0, \ell_2 \neq 0$ case that we have been primarily concerned with in this paper, there are some grounds for confidence in the semiclassical results. Semiclassical calculations in the case of a massive bosonic string [20] give results in close accord with the exact results [21] for this non-linear system. This lends credibility to the general features of the semiclassical results that we have presented.

In order to simplify the present calculations, we have considered only a very special topologically stabilized background. An obvious open question is what can be

learned by performing semiclassical quantization about a classical solution that corresponds to the expected leading Regge trajectory of the supermembrane, such as that considered in Ref.[9] in the bosonic case. Furthermore, in this paper we have only considered the case where the eleven-dimensional superspace is flat. Since, as discussed in [4],[5], the Siegel symmetry leads to superspace constraints that are precisely the field equations of $d = 11$ supergravity, all solutions of $d = 11$ supergravity [see, for example, 22] are possible backgrounds for the supermembrane. Indeed, one might expect a close analogy between strings moving on the group manifold and membranes moving on the seven-sphere.

A more general question concerns the influence of the membrane's topology on its dynamics. There is a possibility of a richer structure here than in string theory, where all closed strings have the same topology [S^1]. By contrast, a closed membrane can have the topology of an arbitrary Riemann surface of any genus. Furthermore, since all orientable Riemann surfaces are cobordant, the possibility exists for the topology of the membrane to evolve. For example, a world volume could have two boundaries: one a sphere, the other a torus. This raises a further question, concerning the loop expansion for the quantized membrane. For strings, there is a one-to-one relation between n -loop diagrams and Riemann surfaces of genus n . For membranes, on the other hand, it is not clear what, if any, topological characterization of 3-manifolds corresponds to the characterization of loop diagrams. It is perhaps worth remarking here that the problem of giving a complete topological classification of 3-manifolds remains unsolved [23]. Optimistically, one might hope that since membranes are extended objects they will yield ultraviolet-finite scattering amplitudes, but it is unlikely that a proof will be any easier than that for strings. Note that this is a problem over and above that of the non-renormalizability of the three-dimensional σ -model.

In the absence of an anomalous symmetry it is difficult to establish eleven as the critical dimension for the supermembrane in the same way that ten is the critical dimension for the superstring. Hence it is difficult to set up a criterion by which to judge the quantum consistency. One could check the commutation relations for the non-linearly realized Lorentz generators, but at least in the case of the bosonic membrane it is difficult to see what could go wrong. For the supermembrane, it is conceivable that there could be an anomaly in the Siegel symmetry, and it would be interesting to investigate this further.

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APPENDIX

Our conventions are

$$\eta_{\mu\nu} = \text{diag}(-1, 1, \dots, 1), \quad \eta_{ij} = \text{diag}(-1, 1, 1), \quad (A1)$$

$$\varepsilon^{012} = -1 \quad (A2)$$

$$\{\Gamma^\mu, \Gamma^\nu\} = -2\eta^{\mu\nu}. \quad (A3)$$

In eleven dimensions, the charge conjugation matrix C is antisymmetric, as are $C\Gamma^{\mu\nu\rho}$ and $C\Gamma^{\mu\nu\rho\sigma}$, while $C\Gamma^\mu$, $C\Gamma^{\mu\nu}$ and $C\Gamma^{\mu\nu\rho\sigma\lambda}$ are symmetric. A crucial identity for the existence of the eleven-dimensional supermembrane is

$$\eta_{\nu\rho}(C\Gamma^\nu)_{(\alpha\beta}(C\Gamma^{\mu\rho})_{\gamma\delta)} \equiv 0, \quad (A4)$$

where $\alpha, \beta, \gamma, \delta$ are the 32-component spinor indices. A useful consequence of this identity can be succinctly written in terms of four Majorana spinors χ, λ, ψ_a ($a = 1, 2$) as

$$(\bar{\chi}\Gamma^\nu\lambda\bar{\psi}_a\Gamma_{\mu\nu}\psi_b - 2\bar{\chi}\Gamma^\nu\psi_a\bar{\lambda}\Gamma_{\mu\nu}\psi_b)\varepsilon_{ab} = -(\bar{\chi}\Gamma_{\mu\nu}\lambda\bar{\psi}_a\Gamma^\nu\psi_b - 2\bar{\chi}\Gamma_{\mu\nu}\psi_a\bar{\lambda}\Gamma^\nu\psi_b)\varepsilon_{ab}. \quad (A5)$$

In light-cone coordinates, the Γ^μ become $(\Gamma^+, \Gamma^-, \Gamma^1, \Gamma^2, \Gamma^I)$, where

$$\Gamma^\pm = \frac{1}{\sqrt{2}} (\Gamma^0 \pm \Gamma^{10}), \quad (A6)$$

$$(\Gamma^+)^2 = (\Gamma^-)^2 = 0, \quad \{\Gamma^+, \Gamma^-\} = 2. \quad (A7)$$

A representation of the Γ -matrices appropriate to the $11 = 9 + 2$ split that we are making in this paper is:

$$\begin{aligned} \Gamma^I &= \gamma^I \otimes \sigma_3 \otimes \sigma_3, \quad I = 3, \dots, 9, \\ \Gamma^\pm &= 1 \otimes \sigma_\pm \otimes 1, \quad \sigma_\pm = \frac{1}{\sqrt{2}} (\sigma_1 \pm i\sigma_2), \\ \Gamma^a &= 1 \otimes \sigma_3 \otimes i\sigma_a, \quad a = 1, 2. \end{aligned} \quad (A8)$$

The eleven-dimensional charge conjugation matrix C is given by

$$C = 1 \otimes \sigma_2 \otimes \sigma_1$$

where in the seven-dimensional space the charge conjugation matrix is the unit matrix and Majorana spinors are purely real.

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