CERN-TH.4733/87

REPARAMETRIZATION INVARIANCE, ANALYTIC MAPPINGS
AND THE HAWKING-UNRUH EFFECT IN STRING THEORY

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A B S T R A C T

We use the reparametrization invariance of string theory, namely the invariance under analytic (holomorphic) mappings of the world sheet co-ordinates, to define the appropriate parametrization for strings in an accelerated frame (or in a curved space-time). We provide a systematic discussion of the Hawking-Unruh effect for (first) quantized strings in terms of analytic mappings. The mappings are solutions of a (zero-energy) Schrödinger equation whose potential determines the ground state of the string. Only the $O(2,1)$ bilinear transformations (corresponding to a zero potential) leave the ground state invariant. Exponential mappings (corresponding to a constant and positive potential) determine thermal states. The Hawking-Unruh temperature arises as the (zero-energy) transmission coefficient and an additional parameter ϵ appears as an integration constant playing the rôle of a fundamental length.

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CERN-TH.4733/87
May 1987

As it is known, the string action in a D-dimensional (generically curved) space-time is given by

$$S = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{g} g^{\alpha\beta} G_{AB}(X) \partial_\alpha X^A \partial_\beta X^B \quad (1)$$

$G_{AB}(X)$ and $g_{\alpha\beta}(\sigma, \tau)$ stand for the space-time and the world-sheet metrics respectively, namely

$$\begin{aligned} dS^2 &= G_{AB}(X) dX^A dX^B & 0 \leq A, B \leq D-1 \\ ds^2 &= g_{\alpha\beta}(x) dx^\alpha dx^\beta & 0 \leq \alpha, \beta \leq 1 \end{aligned} \quad (2)$$

$(2\pi\alpha')^{-1}$ is the string tension (we will take here $\alpha' = 1$).

The equations of motion and the constraints are

$$\square^2 X^A + \Gamma_{BC}^A \partial_\alpha X^B \partial_\beta X^C g^{\alpha\beta} = 0 \quad (3)$$

$$T_{\alpha\beta} = G_{AB} \partial_\alpha X^A \partial_\beta X^B - \frac{1}{2} g_{\alpha\beta} \partial_\sigma X^A \partial^\sigma X^B = 0 \quad (4)$$

$\Gamma_{BC}^A(X)$ stand for the Christoffel connections of the metric G_{AB} . As is well known, the action (1) is invariant under reparametrizations of the world-sheet co-ordinates

$$\delta X^\mu = \varepsilon^\alpha(\xi) \partial_\alpha X^\mu(\xi), \quad \xi^\alpha = (\sigma, \tau)$$

under which the two-dimensional metric $g_{\alpha\beta}$ goes to

$$\delta g_{\alpha\beta}(\xi) = \varepsilon^\gamma \partial_\gamma g_{\alpha\beta} + (\partial_\alpha \varepsilon^\gamma) g_{\gamma\beta} + (\partial_\beta \varepsilon^\gamma) g_{\alpha\gamma}.$$

This reparametrization invariance allows us to choose $g_{\alpha\beta}$ in the conformal form (so called "conformal gauge"), namely,

$$\begin{aligned} g_{\alpha\beta} &= \Lambda(\xi) \eta_{\alpha\beta}, \\ ds^2 &= \Lambda(\sigma, \tau) (d\sigma^2 - d\tau^2) \end{aligned} \quad (5)$$

The choice of the conformal gauge still allows the reparametrizations

$$\begin{aligned} x_+ + \varepsilon &= f(x'_+ + l) \\ x_- + \varepsilon &= g(x'_- + l) \end{aligned} \quad (6)$$

(where $x_{\pm} = \sigma \pm \tau$, $x'_{\pm} = \sigma' \pm \tau'$ and we have introduced the constants ε , l whose meaning will be clear in the sequel). In this way, we can choose the light-cone gauge:

$$V \equiv X_0 + X_1 = p_+ x_+ \quad (7)$$

where the proportionality constant $p_+ > 0$ is the momentum of the centre-of-mass of the string.

An important step in field as well as string quantization is the definition of positive frequency states and its associated ground state. It is by now well known, in QFT (in either flat or curved space-times) that "canonical states" for different co-ordinate systems are physically different (each timelike vector field leads to a separate indication of what constitutes a positive frequency)^{1),2)}. There are different ways (and there is thus an ambiguity) in choosing such a basis. This makes it possible for a given field theory to have different alternative well-defined Fock spaces (different "sectors" of the theory). To illustrate this feature in string theory, let us consider the simplest case $G_{AB} = \eta_{AB}$, i.e.,

$$dS^2 = -dX^0{}^2 + dX^1{}^2 + \dots + dX^{D-1}{}^2, \quad (8)$$

which means both that the space-time is flat and that the string is described in an inertial frame. Usually the string is described in this frame where

$$\partial_{x_+} \partial_{x_-} X^A = 0, \quad A = 0, 1, \dots, D-1 \quad (9)$$

and then positive frequency modes are defined with respect to the inertial time X^0 . In the light-cone gauge, X^0 is proportional to τ and therefore the modes

$$\overleftarrow{\varphi}_n = \frac{1}{2\sqrt{\pi|n|}} e^{-in x_+}, \quad \overrightarrow{\varphi}_n = \frac{1}{2\sqrt{\pi|n|}} e^{in x_-} \quad (10)$$

define the (inertial) particle states of the string.

Obviously, if we consider the (σ', τ') parametrization of the string world sheet [Eq. (6)], i.e.,

$$\begin{aligned}\sigma' + l &= \frac{1}{2} [G(\sigma + \tau + \varepsilon) + F(\sigma - \tau + \varepsilon)], \quad F \equiv f^{-1} \\ \tau' &= \frac{1}{2} [G(\sigma + \tau + \varepsilon) - F(\sigma - \tau + \varepsilon)] \quad G \equiv g^{-1}\end{aligned} \quad (11)$$

for which we have

$$\partial_{x'_+} \partial_{x'_-} X^A = 0, \quad (12)$$

positive frequency modes with respect to τ' , namely

$$\begin{aligned}\overleftarrow{\phi}_n &= \frac{1}{2\sqrt{\pi|n|}} e^{-i\lambda_n x'_+}, & \overrightarrow{\phi}_m &= \frac{1}{2\sqrt{\pi|m|}} e^{i\lambda_m x'_-} \\ & & & \lambda_m > 0\end{aligned} \quad (13)$$

do not define positive frequency modes with respect to X^0 , i.e., are not inertial particle states of the string. However, they are positive frequency modes with respect to another (accelerated) time X'^0 defined by

$$\begin{aligned}X_1 - X_0 &= l_- \int (X'_1 - X'_0) \\ X_1 + X_0 &= l_+ \int (X'_1 + X'_0)\end{aligned} \quad (14)$$

where f is the same as in Eq. (6). This corresponds to the description of the string in an accelerated reference frame $\{X'_0, X'_1, \dots, X'_{D-1}\}$, with acceleration

$$\tilde{a} = \frac{1}{[f'g']^{1/2}} \partial_{X'_1} [\ln(f'g')]$$

and metric

$$dS^2 = l_+ l_- \int' (X'_1 - X'_0) \int' (X'_1 + X'_0) [dX_0'^2 + dX_1'^2] + dX_2'^2 + \dots + dX_{D-1}'^2 \quad (15)$$

where, for simplicity, we have taken

$$X_{\lambda'}' = X_{\lambda}, \quad (i = 2, \dots, D-1).$$

For the description of the string in this frame we can always choose

$$V' \equiv X'_1 + X'_0 = x'_+ + l \quad (16)$$

as it follows from Eqs. (6), (7) and (14). Thus, Eq. (12) defines the accelerated particle states of the string. [The choice (16) is particularly useful here because of the acceleration points in the X^1 direction]. In this frame, the equations of motion of the string are

$$\partial_{x'_+} \partial_{x'_-} U' + \frac{f''}{f'} \partial_{x'_+} U' \partial_{x'_-} U' + \left(\frac{f''}{f'} - \frac{g''}{g'} \right) \partial_{x'_+} U' \partial_{x'_-} V' = 0 \quad (17)$$

$$\partial_{x'_+} \partial_{x'_-} V' + \frac{g''}{g'} \partial_{x'_+} V' \partial_{x'_-} V' - \left(\frac{f''}{f'} - \frac{g''}{g'} \right) \partial_{x'_+} U' \partial_{x'_-} V' = 0$$

$$(U' \equiv X'_1 - X'_0)$$

$$\partial_{x'_+} \partial_{x'_-} X'^i = 0 \quad (i = 2, \dots, D-1) \quad (18)$$

The equations of motion of the physical (transverse degrees) of freedom are the same in both frames. The equations of motion of the longitudinal co-ordinates (U', V') are different, but it can be shown³⁾ that, as in the inertial case⁴⁾, they can be eliminated in terms of the transverse co-ordinates by using the constraints

$$T'_{++} = f'(U') g'(V') \partial_{x'_+} U' \partial_{x'_+} V' + \partial_{x'_+} X'^i \partial_{x'_+} X'^i \approx 0$$

$$T'_{+-} = 0 \quad (19)$$

The Poincaré invariance of flat space-time has a non-linear realization in terms of the accelerated co-ordinates, but an explicit realization of the Poincaré algebra can be constructed in terms of the accelerated modes provided (at the quantum level) one sets $D = 26$ and uses symmetric ordering of the operators³⁾.

The manifold (σ', τ') defined by Eq. (11) is the convenient world sheet parametrization for an accelerated string or a string in an accelerated space-time. This is true in either flat or curved space-time [although in the last case, Eq. (12) can only be valid asymptotically for $x'_\pm \rightarrow \pm\infty$].

Let us describe the string in the reference frames (8) and (15) referred in the sequel as (I) and (A) respectively. We consider here a closed string and take for simplicity $f \equiv g$. The boundary conditions are

$$\begin{aligned} X^\mu(0, \tau) &= X^\mu(\pi, \tau) \\ X^\mu(0, \tau') &= X^\mu(L_\varepsilon, \tau') \end{aligned} \quad (20)$$

Notice that

$$\begin{aligned} 0 < \sigma < \pi \\ -\infty < \tau < +\infty \end{aligned} \quad , \quad \begin{aligned} -\infty < X^0 < +\infty \end{aligned} \quad (21)$$

In order to have

$$\begin{aligned} 0 < \sigma' < L_\varepsilon \\ -\infty < \tau' < +\infty \end{aligned} \quad , \quad \begin{aligned} -\infty < X'^0 < +\infty \end{aligned} \quad (22)$$

appropriate boundary conditions on the mappings must be imposed. Condition (20) is automatically satisfied from Eqs. (6) with

$$\begin{aligned} L_\varepsilon &= F(\varepsilon + 2\pi) - F(\varepsilon) \\ l &= F(\varepsilon) \end{aligned} \quad , \quad F \equiv f^{-1} \quad (23)$$

Condition (22) on τ' and X'^0 is not particular to strings. It is required to get a consistent quantization (of fields or strings) in accelerated manifolds and to have a complete in (out) basis²⁾. On the other hand, the manifold A can cover only a (bounded) region (namely $|X^1| > |X^0|$) of the original (I) one. Similarly, the manifold (σ', τ') can cover only a domain ($|\sigma| > \tau$) of the inertial or global world sheet. All these considerations are satisfied by taking mappings such that

$$u_\pm = f(\pm\infty) \quad , \quad (24)$$

where $u_+(u_-)$ are constants which can take independently finite or infinite values and $u_+ > u_-$. This means that the inverse mapping $F \equiv f^{-1}$ has singularities at $u = u_\pm$, i.e.,

$$F(u_\pm) = \pm\infty \quad (25)$$

Singularities of these mappings describe the asymptotic regions of the space-time. Critical points of f , i.e.,

$$f'(\pm\infty) = 0 \quad (26)$$

describe event horizons at $u = u_-$ and $u = u_+$. In this case $u_-(u_+)$ are finite and the manifold A cover the region $u_- < |X^1 + X^0| < u_+$ of the global (I) space-time. If $u_{\pm} = \pm\infty$, there are no horizons. In this approach, the well-known Rindler's space corresponds to the mapping

$$f = e^{\alpha U'} \quad , \quad \alpha = \text{const.} \quad (27)$$

In this case, $u_- = 0$ and $u_+ = +\infty$; $u = u_- = 0$ is an event horizon and the manifold covers the right-hand wedge $|U| > 0$. (In order to cover the whole plane, four Rindler's patches are needed.) For the string world sheet we have

$$x_{\pm} + \varepsilon = e^{\alpha (x'_{\pm} + l)} \quad (28)$$

and we see how the presence of the constant ε is necessary in order to have a finite string period in the manifold (σ', τ') . For the Rindler's mapping (28) we have

$$\begin{aligned} l &= \frac{1}{\alpha} \log \varepsilon \\ L_{\varepsilon} &= \frac{1}{\alpha} \log \left(\frac{2\pi}{\varepsilon} + 1 \right) \end{aligned} \quad (29)$$

For $\varepsilon \rightarrow 0$, there is a stretching effect of the string due to the presence of an event horizon in the world sheet.

Let us now consider the string quantization in the frames (I) and (A). The string co-ordinates can be split into left and right movers as

$$X^A(\xi) = \overleftarrow{X}^A(\xi_+) + \overrightarrow{X}^A(\xi_-) \quad (30)$$

where $\xi_{\pm}$ stands for $\sigma \pm \tau$ or $\sigma' \pm \tau'$. As usual, left and right modes of the closed string decouple both in the classical and quantum theories. We consider first the left movers. The wave modes are periodic functions of x_+ and x_- with period 2π . The left modes which are only functions of x_+ will be periodic functions of $x_+ = \sigma' + \tau'$ with period L_{ε} . In the inertial frame (I) we have

$$\begin{aligned} \overleftarrow{X}^i(x_+) &= \frac{q_+^i}{2} + \frac{p_+^i}{2} x_+ + \frac{1}{2} \sum_{n \neq 0} \frac{\overleftarrow{a}_n^i}{\sqrt{|n|}} e^{-in x_+} \\ \overleftarrow{U}(x_+) &= q_- + p_- x_+ + \frac{i}{2} \sum_{n \neq 0} \frac{\overleftarrow{\alpha}_n}{n} e^{-in x_+} \quad , \quad p_- = \frac{\mu_0}{2} \end{aligned}$$

$$\begin{aligned}\overleftarrow{X}^i(x_+ + 2\pi) - \overleftarrow{X}^i(x_+) &= \pi p^i \\ \overleftarrow{U}(x_+ + 2\pi) - \overleftarrow{U}(x_+) &= 2\pi p_- \end{aligned} \quad (31)$$

The independent dynamical variables $\overleftarrow{a}_n^i (n \in \mathbb{Z})$, q_i , q_+ and p_+ obey the canonical commutation relations

$$\begin{aligned}[\overleftarrow{\alpha}_m^i, \overleftarrow{\alpha}_n^j] &= n \delta_{n+m} \delta^{ij}, & \overleftarrow{\alpha}_n^i &= -i\sqrt{|n|} \operatorname{sgn} n \overleftarrow{a}_n^i, \quad n \neq 0 \\ [q^i, \overleftarrow{\alpha}_0^j] &= i \delta^{ij}, & \overleftarrow{\alpha}_0^i &= p^i \\ [q_-, p_+] &= i \end{aligned} \quad (32)$$

Then

$$[\overleftarrow{X}^i(x_+), \overleftarrow{p}^j(y_+)] = \frac{\delta^{ij}}{2} \delta(x_+ - y_+)$$

The energy-momentum tensor, the conformal generators and the mass operator are

$$\begin{aligned}T_{++}(x_+) &= \sum_{n \in \mathbb{Z}} \frac{L_n}{2\pi} e^{-in x_+}, \\ L_n &= \frac{1}{2} \sum_m \overleftarrow{\alpha}_m^i \overleftarrow{\alpha}_{n-m}^i + p_+ u_n \\ L_0 &= \frac{1}{2} \sum_m \overleftarrow{\alpha}_m^i \overleftarrow{\alpha}_{-m}^i + 2 p_+ p_- \end{aligned} \quad (33)$$

and

$$\begin{aligned}M^2 &= p_+ p_- - p_i^2 = \sum_n \frac{n}{2} (a_n^{i+} a_n^i + a_n^i a_n^{i+}) = \\ &= \sum_n n a_n^{i+} a_n^i - \frac{(D-2)}{24} \end{aligned} \quad (34)$$

In the accelerated frame (A) we have

$$\begin{aligned}\overleftarrow{X}^i(x'_+) &= \frac{Q^i}{2} + \frac{P^i}{2} \xi + \sum_{n \neq 0} \overleftarrow{C}_n^i \overleftarrow{\Phi}_n(\xi) \\ \overleftarrow{U}(x'_+) &= \mu_0 + \frac{P_-}{2} \xi + \frac{i}{2} \sum_{n \neq 0} \frac{\gamma_n}{n} e^{-i\lambda_n \xi}\end{aligned}\quad (35)$$

with

$$\begin{aligned}\overleftarrow{X}(x'_+ + L_\varepsilon) - \overleftarrow{X}(x'_+) &= \frac{P^i}{2} L_\varepsilon \\ \overleftarrow{U}(x'_+ + L_\varepsilon) - \overleftarrow{U}(x'_+) &= \frac{P_-}{2} L_\varepsilon\end{aligned}\quad (36)$$

and

$$P^i = \frac{2\pi}{L_\varepsilon} p^i, \quad P_- = \frac{2\pi}{L_\varepsilon} p_-$$

The left modes provide the T'_{++} component of the energy-momentum tensor. The accelerated conformal generators (L'_n) follow from the Fourier transform of T'_{++} :

$$\begin{aligned}T'_{++}(x'_+) &= \sum_{n=-\infty}^{\infty} \frac{L'_n}{L_\varepsilon} e^{-i\lambda_n \xi} \\ L'_n &= L_n^{\perp} + L_n^{\parallel} L_\varepsilon \approx 0 \\ L_n^{\perp} &= -\frac{1}{L_\varepsilon} \sum_n \overleftarrow{\beta}_n^i \overleftarrow{\beta}_{m-n}^i, \\ \overleftarrow{C}_n^{i+} &= i \operatorname{sgn} \frac{\overleftarrow{\beta}_n^i}{\sqrt{|n|}}, \quad \overleftarrow{\beta}_{-n}^i = \overleftarrow{\beta}_n^i, \quad \beta_0 = p^i, \quad \gamma_0 = p_- \end{aligned}\quad (37)$$

We find the following Fock representation

$$\begin{aligned}[\overleftarrow{C}_m^i, \overleftarrow{C}_m^j] &= \delta_{nm} \delta^{ij}, \quad [\overleftarrow{\beta}_m^i, \overleftarrow{\beta}_m^j] = n \delta_{n+m,0} \delta^{ij} \\ [P^i, Q_0^j] &= -\left(\frac{2\pi}{L_\varepsilon}\right) i \delta^{ij} \\ [p_+, \mu_0] &= -i\end{aligned}\quad (38)$$

which implies

$$[\overleftarrow{X}^i(\xi), \overleftarrow{P}^j(\xi')] = \frac{i}{2} \delta(\xi - \xi') \delta^{ij} \quad (39)$$

For a string whose centre-of-mass follows an accelerated world line in the longitudinal direction X^1 , of the type

$$x^1 \pm x^0 \pm a_{\pm} = A_{\pm} e^{\pm \alpha z'}, \quad a_{\pm}, A_{\pm} = \text{const.}$$

it is convenient to use the expansion³⁾

$$U = U_0 + \frac{1}{f(x'_+)} \left[\tilde{p}_- + \frac{i}{2} \sum_{n \neq 0} r_n e^{-i \lambda_n \xi} \right], \quad (40)$$

for which the mass formula is given by

$$\begin{aligned} M'^2 &= (\alpha L_{\xi})^2 p_+ \tilde{p}_- - p_i^2 = \sum_1 \frac{n}{2} (C_n^{i+} C_n^i + C_n^i C_n^{i+}) = \\ &= \sum_1 n C_n^i + C_n^i - \frac{(D-2)}{24} \end{aligned} \quad (41)$$

The operators M^2 and M'^2 [Eqs. (34) and (41)] are different but both have the same eigenvalues.

The same treatment applies to the right movers $\vec{X}^A$ in a gauge defined by

$$\begin{aligned} \vec{U} &= p_- x_- \\ \text{and with } \vec{U}' &= x'_- + l, \\ \text{i.e. } x_- + \varepsilon &= f(x'_- + l) \end{aligned}$$

The (I) and (A) Fock representations are related by a Bogoliubov transformation

$$C_m^i = p^i O_m + \sum_{n=1}^{\infty} (A_{mn} a_n^i + B_{mn} a_n^{i+}), \quad m \neq 0 \quad (42)$$

with Bogoliubov coefficients

$$\begin{aligned} A_{mn} &= \langle \phi_{\lambda_m}, \psi_n \rangle = \frac{1}{2\pi} \sqrt{\frac{n}{m}} \int_0^{2\pi} e^{in\sigma - i\lambda_m} F(\sigma + \varepsilon) d\sigma \\ B_{mn} &= \langle \phi_{\lambda_m}, \psi_n^* \rangle = \frac{1}{2\pi} \sqrt{\frac{n}{m}} \int_0^{2\pi} e^{in\sigma + i\lambda_m} F(\sigma + \varepsilon) d\sigma \end{aligned} \quad (43)$$

and zero-mode contribution

$$O_m = \left\langle \frac{1}{2} \left(x_+ - \frac{2\pi}{L_\epsilon} x'_+ \right), \phi_{\lambda_m} \right\rangle$$

The ground state of the string is defined by

$$\begin{aligned} a_m^i |0\rangle &= 0 \\ p^i |0\rangle &= 0 \end{aligned} \quad \forall n > 1 \quad (44)$$

We also have

$$\begin{aligned} C_m |0'\rangle &= 0 \\ p^i |0'\rangle &= 0 \end{aligned} \quad \forall m > 1 \quad (45)$$

but $C_m |0\rangle \neq 0$. The expectation value of the accelerated number operator in the ground state $|0\rangle$ is equal to (not sum over i)

$$N^i(\lambda_m) \equiv \langle 0 | C_m^i C_m^{i\dagger} | 0 \rangle = N(\lambda_m) = \sum_{n=1}^{\infty} |B_{mn}|^2, \quad (46)$$

which is i -independent. The expectation value of the accelerated M'^2 operator is equal to

$$M'^2 = 24 \sum_1^{\infty} n N(n) - 1 \quad (47)$$

where $N(n)$ is given by Eq. (46) with $\lambda_n = (2\pi/L_\epsilon)n$. If $f(x'_+ + l) = e^{\alpha(x'_+ + l)}$, then:

$$\begin{aligned} B_{mn}(n \gg 1) &= \mathcal{C}(\lambda_m) \frac{n^{-i\lambda_m/\alpha}}{\sqrt{n}} + \frac{b}{\sqrt{n}} + O\left(\frac{1}{n^{3/2}}\right) \\ \mathcal{C} &= \frac{i}{2\pi} \frac{e^{-\frac{\pi\lambda_m}{2\alpha}}}{\sqrt{m}} \Gamma\left(1 + \frac{i\lambda_m}{\alpha}\right), \\ b &= -\frac{i}{2\pi} \frac{(2\pi)^{i\lambda_m/\alpha}}{\sqrt{m}} \end{aligned} \quad (48)$$

and

$$\begin{aligned} N(\lambda_m) &= \frac{1}{(e^{2\pi\lambda_m/\alpha} - 1)} + \frac{\alpha}{2\pi\lambda_m} \\ M'^2 &= 24 \left[\sum_{n=1}^{\infty} \left(\frac{n}{(e^{n/T_0} - 1)} + T_0 \right) \right] - 1 \end{aligned} \quad (49)$$

$$M'^2_{\varepsilon \ll 1} = \frac{1}{2\pi} \ln \left(\frac{2\pi}{\varepsilon} \right) \quad (50)$$

[Recall that in field theory, $B_{\lambda k}(k \gg 1) = \mathcal{C}(\lambda) (k/\sqrt{k})^{-i\lambda/\alpha}$, with $\lambda \in \mathbb{R}$.] $N(\lambda_n)$ [Eq. (49)] is equal to a Planckian spectrum at the Hawking-Unruh temperature

$$T_s = \frac{\alpha}{2\pi} \quad (51)$$

In $N(n)$, the temperature is

$$T_0 = \left(\frac{L_\varepsilon}{2\pi} \right) T_s = \frac{a}{2\pi} \stackrel{\varepsilon \ll 1}{=} \frac{1}{(2\pi)^2} \ln \left(\frac{2\pi}{\varepsilon} \right) \quad (52)$$

The expectation value of M'^2 in the ground state $|0\rangle$ is positive. The tachyon level of this state is filled up with accelerated modes following a thermal distribution. For any other mapping f different from the exponential one [Eq. (28)], $B_{nm}(n \gg 1) \rightarrow e^{-n}$ and $N(\lambda_n)$ and $N(n)$ are non-thermal.

It can be noted that the acceleration as well as the temperature T_s appearing in $N(n)$ are re-scaled by the factor $(L_\varepsilon/2\pi)$, i.e., the ratio of the string period in the inertial and in the accelerated frame. Thus, the dimensionless temperature T_0 does not depend on the acceleration parameter α of Rindler's space, but on the parameter

$$a = \left(\frac{L_\varepsilon}{2\pi} \right) \alpha = \frac{1}{2\pi} \ln \left(\frac{2\pi}{\varepsilon} \right) \stackrel{\varepsilon \ll 1}{=} \left(\frac{L_\varepsilon}{2\pi} \right) \alpha \quad (53)$$

We see that the parameter ε [$\ell \equiv \ell(\varepsilon)$] introduced in the mappings on the world sheet, plays a fundamental rôle in the string context. ε acts as a regulator or cut-off to avoid the presence of an event horizon in the world sheet and to get a finite string period. Its magnitude is of the order of the Planck length.

We see that the transformations (6) are not without consequences but change the ground state and in general, the quantum states, except for mappings belonging to the $O(2,1)$ group. Such invariance mappings are the Möbius or bilinear transformations. Under the transformations (6) and (14), the states transform as

$$| \rangle \rightarrow |' \rangle = e^{i \hat{G}} | \rangle \quad (54)$$

where $a_n | \rangle = 0$, $c_n |' \rangle = 0$, $\forall_n$ and

$$\hat{G} = \sum_n \Theta_n (c_n c_n - c_n^+ c_n^+)$$

[this is an operatorial representation for the Bogoliubov transformation, Eq. (42), with Bogoliubov coefficients $\cosh \theta_n$ and $\sinh \theta_n$].

The vacuum expectation value of $T_{\mu\nu}$ transforms as

$$\langle T_{\mu\nu} \rangle = \langle T_{\mu\nu}' \rangle + \oplus_{\mu\nu} + P_{\mu\nu} \quad (55)$$

$P_{\mu\nu}$ is any conserved traceless tensor taking into account the dependence of $\langle T_{\mu\nu} \rangle$ on the quantum state. It represents the non-local part of $T_{\mu\nu}$. $\oplus_{\mu\nu}$ depends on the mapping and represents the local part. In the conformal gauge we have

$$P_{--} = \mathcal{U}_-(x'_-) \quad , \quad P_{++} = \mathcal{U}_+(x'_+) \quad , \quad P_{+-} = P_{-(56)} = 0$$

$$\begin{aligned} \oplus_{--} &= -\frac{1}{12\pi} \sqrt{f'} d^2_{x'_+} \left(\frac{1}{\sqrt{f'}} \right) \\ \oplus_{++} &= -\frac{1}{12\pi} \sqrt{g'} d^2_{x'_-} \left(\frac{1}{\sqrt{g'}} \right) \\ \oplus_{+-} &= \oplus_{-+} = 0 \end{aligned} \quad (57)$$

$\mathcal{U}_-$ and $\mathcal{U}_+$ are arbitrary functions of the indicated variables. Equations (56) derive from the conditions $\nabla^\nu P_{\mu\nu} = 0$ and $P_\nu^\nu = 0$. Therefore, the constraints

$$\langle T_{\mu\nu} \rangle = 0 \quad , \quad \langle T_{\mu\nu}' \rangle = 0$$

yield to the equations

$$\begin{aligned} \sqrt{f'} d^2_{x'_+} \left(\frac{1}{\sqrt{f'}} \right) - 12\pi \mathcal{U}_-(x'_-) &= 0 \\ \sqrt{g'} d^2_{x'_-} \left(\frac{1}{\sqrt{g'}} \right) - 12\pi \mathcal{U}_+(x'_+) &= 0 \end{aligned} \quad (58)$$

These are zero-energy Schrödinger equations

$$\frac{d^2}{dx'^2} \psi_{\pm} - 12\pi U_{\pm}(x'_{\pm}) \psi_{\pm} = 0 \quad (59)$$

for the mappings f and g . By giving the potentials $U_{\pm}$, Eqs. (59) determine the wave functions

$$\psi_{-} = \frac{1}{\sqrt{f'}}, \quad \psi_{+} = \frac{1}{\sqrt{g'}} \quad (60)$$

Because $U_{\pm}$ are arbitrary functions [compatible with the boundary conditions (24)], Eqs. (59) do not yield additional constraints on the mappings f and g , but a way of connecting the mapping to a potential problem.

The first term of Eq. (58) is the Schwarzian derivative of f :

$$D[f] = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2,$$

which is invariant under the Möbius or bilinear transformations. Under these transformations, f becomes a new function but $D[f]$ is invariant determining the same ground state of the string. In particular, $U_{+} = U_{-} = 0$ determine $f(g)$ as

$$f = \frac{\alpha\mu + \beta}{\gamma\mu + \delta}, \quad (\alpha\delta - \beta\gamma) = 1$$

The ground state defined by this mapping can be considered as a reference or "minimal" state at zero temperature with respect to which other ground states corresponding to non-zero potentials $U_{\pm}$ appear as excited or thermal ones. If $U_{+} = U_{-} = U_0 = \text{constant} > 0$, then

$$\psi_{\pm} = A e^{-K x'_{\pm}}, \quad f = \frac{1}{\alpha A^2} e^{\alpha x'_{+}} + \varepsilon \quad (61)$$

where A is a normalizing constant (we choose $A = e^{-\alpha\lambda/\alpha}$) and K is the zero-energy transmission coefficient

$$K \equiv \frac{\alpha}{2} = \sqrt{12\pi U_0},$$

For $\varepsilon \rightarrow 0$, the mapping (61) defines an event horizon at $x_{\pm} = 0$ ($x'_{\pm} = -\infty$) and carries an intrinsic temperature $T_s = \alpha/2\pi$, as can be seen by putting $t = i\tau$ ($x_{+} = x + i\tau$) and then $0 < \tau' < 2\pi/\alpha$. The temperature appears related to the height of the potential, namely

$$T_s = \sqrt{\frac{12}{\pi} u_0}$$

and the parameter ϵ arises naturally as an integration constant. This temperature T_s characterizes the spectrum $N(\lambda_n)$, Eq. (49).

Similar equations to (58) also appear in the so-called "back reaction problem" in two dimensions⁵⁾ [as a consequence of the $(\pm\pm)$ components of the semi-classical Einstein equations, the $(+-)$ component giving rise to the Liouville equation for the geometry because of the conformal anomaly]. Equations (58) can also be derived in the context of conformal field theories on higher genus Riemann surfaces (the potential playing the rôle of the zero-point energy) in connection with the approach of Ref. 6).

We have shown that with appropriate boundary conditions, the holomorphic mappings of reparametrization invariance of string theory can be interpreted as a change of co-ordinate frame in the space-time in which strings are embedded. These mappings change the ground state in the quantized theory except for transformations belonging to the $O(2,1)$ group. This allowed us to discuss in a systematic way the Hawking-Unruh effect in string theory. The transformations describing the world sheet of an accelerated string need the introduction of an additional parameter ϵ with respect to those describing the trajectories of accelerated point particles.

ACKNOWLEDGEMENTS

I thank G. Veneziano for his interest and discussions.

REFERENCES

- 1) See, for example: N.D. Birrell and P.C.W. Davies - "Quantum Fields in Curved Space", Cambridge University Press, Cambridge, England, 1982, and references therein.
- 2) N. Sánchez - Phys.Rev.24D (1981) 2100;
N. Sánchez and B.F. Whiting - Phys.Rev. 34D (1986) 1056.
- 3) H.J. de Vega and N. Sánchez - CERN Preprint TH. 4681 (1987).
- 4) P. Goddard, J. Goldstone, C. Rebbi and C.B. Thorn - Nucl.Phys. B56 (1973) 109.
- 5) N. Sánchez - Nucl.Phys. B266 (1986) 487.
- 6) M. Martellini and Sánchez - CERN Preprint TH. 4680 (1987).