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CORRESPONDENCE PRINCIPLE CONSTRAINTS ON QUANTUM GRAVITY

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A B S T R A C T

The construction of effective low energy Lagrangians in quantum gravity is examined. It is concluded that a finite perturbative S-matrix does not uniquely determine the effective theory, and that a non-perturbative approach is essential. A spectrum consisting of massive towers (such as in string theory) is ruled out.

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1. - INTRODUCTION

The remarkable properties of ten-dimensional superstring theory¹⁾ generated a widespread interest in the possibility that it provides a finite and consistent formulation of quantum gravity.

This belief rests on two facts, namely the spectrum of the theory contains a graviton and the one-loop S-matrix elements of the zero-mass particles are finite and satisfy perturbative unitarity²⁾. It is conjectured that these properties persist to all orders of perturbation theory. On this basis it is argued that a well-defined finite and unique "low energy effective theory" can be constructed^{3),4)}. It is this expectation that we wish to examine in what follows.

First, however, it should be emphasized that the on-shell S-matrix is not by itself sufficient for calculating graviton loop corrections to all "low energy" observables. In fact, in order to render finite the divergent gravitational corrections to bound-state energies (hadron masses, positronium, etc.), electroweak form factors, deep inelastic structure functions..., an effective Lagrangian is clearly required. The main properties of this Lagrangian should be:

- 1) finiteness;
- 2) uniqueness;
- 3) the leading terms should be a renormalizable massless theory (preferably containing the standard model) and general relativity;
- 4) the matrix elements of the corrections between massless particle states should be small compared to the leading terms when the momenta are much smaller than the Planck momentum;
- 5) loops calculated with this Lagrangian should be finite.

In superstring theory, the full spectrum contains an infinity of massive particle towers. It is assumed that the mass unit is comparable to the Planck mass. The exchange of these states in loops regulates the S-matrix and renders it finite. Thus, it is expected by integrating out the massive modes, an effective massless theory which satisfies properties (1)-(5) will be produced. We shall see below that these expectations cannot be fulfilled. A systematic construction of the effective Lagrangian leads to either divergences or an infinite arbitrariness. The finiteness of the perturbative S-matrix will be seen to be of no relevance. If unknown non-perturbative effects are invoked, the situation does not improve. In fact the "heavy spectrum" becomes irrelevant and in any event the original massless theory has to be solved non-perturbatively in order to

determine the effective Lagrangian! All the above results are quite general and apply to any theory in which the divergences of quantum gravity are regulated by "massive towers".

The construction and arguments used below rest on very general physical principles which are used only in the domain of applicability of standard quantum field theory and semi-classical general relativity namely, at space-time scales larger than the Planck scale. The results should properly be regarded as constraints imposed by the correspondence principle on the approach to quantum gravity.

2. - EFFECTIVE LAGRANGIAN FROM THE S-MATRIX

The class of theories we wish to examine is characterized by a spectrum which contains a massless multiplet which includes a spin-2 graviton and towers of massive particles starting at a mass M . Since the graviton is present, the massless field theory is non-renormalizable and in d space-time dimensions the gravitational coupling defines a Planck-mass m :

$$G = m^{2-d} \quad (1)$$

For simplicity we take $m \sim M$.

Assume that the loop expansion for the scattering amplitudes of any number of massless particles is given. The aim is to construct an effective theory of the massless fields whose Feynman rules reproduce the on-shell perturbative S-matrix. There are three related questions which need be answered:

- (a) is there a finite solution?
- (b) is the solution unique?
- (c) do the Feynman rules produce finite corrections to off-shell observables?

There are three general properties of the S-matrix which are needed in the construction of the massless Lagrangian:

- (A) finiteness;
- (B) perturbative unitarity;
- (C) analyticity.

Property (A) is obvious. For our purpose it is sufficient to require only a limited version of unitarity which involves only the massless particles: if $T_n(\{p\})$ is the n-loop massless scattering amplitude (regarded as a matrix in momentum space) at a total centre-of-mass energy $\sqrt{s}$, then:

$$\sqrt{s} < M : T_n(s) - T_n^\dagger(s) = i \sum_{k=0}^n T_k^\dagger(s) d\Gamma(s) T_{n-k}(s), \quad (2)$$

where $d\Gamma(s)$ is the phase space measured at centre-of-mass $(\text{energy})^2 = s$ (including the energy-momentum δ -function).

The crucial property is "analyticity" (C), which we proceed to define: for a given set of external massless particles, $T_n(\{p\})$ is expressed in terms of scalar functions,

$$T_n(\{p\}) = \sum_I U_I(\{p\}) A_n^I(\{p_i \cdot p_j\}) \quad (3)$$

The U 's are fixed functions of the external momenta $\{p\}$ (essentially scalar products of free wave functions), while the A 's are scalar functions of the invariants $\{s_{ij} = p_i \cdot p_j\}$. Define a domain D around the origin of these variables by requiring that all sub-energies and momentum transfers be smaller than the lowest mass M :

$$D : |(\sum p_i)^2| < M^2 \quad (4)$$

The analyticity postulate demands that A_n^I be analytic in the domain D except for singularities enforced by the unitarity condition (2) and the kinematic expansion (3).

An immediate corollary of the analyticity postulate and the non-linearity of the unitarity equation (2) is that if T_n^o is any other solution of (A)-(C) and we define

$$T_n^M = T_n - T_n^o, \quad (5)$$

then the scalar amplitudes A_n^{IM} , which are related to T_n^M by Eq. (3) are analytic in the domain D and can be expanded in a power series,

$$A_n^{IM} = \sum_{\{l\}} a_{n\{l\}}^{IM} \prod_{(ij)} \left(\frac{s_{ij}}{M^2} \right)^{l_{ij}}, \quad (6)$$

where $\{l\}$ are integers and a are constants.

Consider now the tree approximation T_0 . Tree unitarity and analyticity imply that the invariant amplitudes A_0^I are composed of a sum over massless poles with factorizable residues and a regular "remainder". Hence,

$$T_0 = \sum \frac{R}{i\Gamma(\sum s_{ij})} + T_0^M \equiv T_0^0 + T_0^M, \quad (7)$$

where $(\sum p_i \cdot p_j)$ are various (sub-energies)². Note incidentally that R and T^M have poles at the massive spectrum - all of which lie outside the domain D .

The factorizability and regularity of the residues imply the existence of an Hermitian Lagrangian $L_0(\phi, \partial\phi, \dots)$ whose on-shell scattering amplitudes constructed by the Feynman rules yield T_0 . More precisely, L_0 is expressed as an (infinite) sum over polynomials in the fields and their derivatives. The Feynman rules generate amplitudes of the form of Eq. (7), where the residues and remainder are power series in the momenta. The unknown coefficients in the Lagrangian are determined by equating the series generated by the Feynman rules to the expansion of R and T^M in Eq. (7). Note that there is no arbitrariness in the procedure since the Lagrangian should determine the tree-graph amplitudes for any number of external legs.

Having constructed L_0 we now turn to the one-loop approximation. By the unitarity postulate T_1 satisfies an equation similar to Eq. (7):

$$T_1 = T_1^0 + T_1^M, \quad (8)$$

where T_1^M is regular in D and T_1^0 contains only poles and cuts due to exchanges of massless particles whose discontinuities are fixed by the unitarity Eq. (2) for $n = 1$. In order to determine L_1 we repeat the preceding procedure: construct the one-loop scattering amplitude by applying the Feynman rules to L_0 , subtract the result from T_1 , and determine a regular remainder whose power series expansion defines L_1 . Obviously we can go on iterating and it seems that the perturbative scattering amplitudes indeed determine the effective massless Lagrangian.

The above reasoning is, however, false. The procedure breaks down since the Lagrangian L_0 is non-renormalizable and therefore generates divergent loop amplitudes. Hence, in order to determine a finite L_1 the Feynman graphs must be subtracted, thus introducing an arbitrariness into the procedure. In fact, the

non-renormalizability of the original massless theory has been recovered! The fact that the loop expansion of the S-matrix is finite (by assumption) provides no help in the determination of the "effective low energy theory".

Observe that the postulates of unitarity and analyticity were used in a limited version and applied only to "low energy" massless particle amplitudes. In particular, S-matrix elements which include massive external legs were not used. We note that in string theory the loop expansion of the latter is ill-defined due to external leg divergences⁵⁾. Thus, perturbative unitarity cannot even be applied for massive external particles without an off-shell theory. Clearly no helpful information can be gleaned from this direction.

To summarize, we have proved that a finite perturbative S-matrix does not define a unique "effective low energy theory" if the starting point contains gravity. In particular, the gravitational corrections to bound states, form factors, etc., remain indeterminate.

3. - NON-PERTURBATIVE EFFECTS

It would seem that we have arrived at a paradox. On the one hand, some consistent low energy theory of gravity should exist. On the other hand, the physical basis which underlies the postulates of unitarity and analyticity (B,C) is the assumption that processes whose total available energy is insufficient to excite the massive spectrum, and which occur in a space-time region whose size exceeds the smallest Compton wavelength associated with the spectrum, should be describable by a weakly interacting massless field theory with correction which vanish when $p/M \rightarrow 0$. Since M is tacitly assumed to be close to the Planck mass, we cannot abandon the above principles without affecting changes in the laws of physics in the "low energy-large distance" domain.

The crucial element in the derivation of the effective Lagrangian from the S-matrix is the assumption of exact analyticity in that below the production threshold the mass M does not affect the location of singularities at all and enters the residues analytically. Inevitably, the massless effective Lagrangian is expressed as a power series in $1/M$ and the Feynman rules generate divergences.

The way out is to invoke non-analyticity, which can only be generated by non-perturbative effects. In fact, if the input S-matrix contains non-analytic contributions, then Eqs. (2), (3), (5) and (6) do not determine the effective Lagrangian since the massless unitarity Eq. (2) does not define any more the full strength of the singularities. Violations of analyticity stem from two different sources related to the mass scales M and m (the Planck mass).

Note that Eq. (4), which defines the domain of regularity D , requires that all sub-energies and momentum transfers be smaller than M . Hence, as the number (N) of external particles increases, the domain D shrinks. When $N \rightarrow \infty$, the contribution from any given set of external lines may come from vanishingly small momenta. In other words, a large number of low momentum particles can produce a real massive state and/or undergo a high momentum transfer short-distance process. In field language, we are considering configurations of strong fields produced by asymptotically smooth initial conditions. If the massless $\leftrightarrow$ massive S-matrix were available, we could try to systematically remove the massive thresholds as N grows. This, however, entails the introduction of local fields associated with the (infinitely many) massive particles. In any event, in string theory the required scattering amplitudes are not well defined. Thus, we are forced to contemplate a full non-perturbative solution to both the massless field theory defined by the effective Lagrangian and the input S-matrix. In the string theory this last problem amounts to a summation over large and infinite genus surfaces.

There is a second source for non-analytic behaviour, which is unrelated to the massive spectrum, and is due to the existence of another scale in the problem, namely m . We have already noted that a large number of low momentum massless quanta can produce a massive state. By the same token, large (Planck) momentum massless particles can also be created. The latter, however, interact strongly and may produce bound states, resonances, etc., which in turn would affect the unitarity-analyticity postulates.

It may or may not be true that the full non-perturbative behaviour is consistent only if the massive towers and their interactions are present. In both cases we would have to solve non-perturbatively a massless theory of quantum gravity in order to obtain the low energy effective theory. Again, the assumed finiteness of the perturbative S-matrix seems to be of no relevance. Might not some purely massless theory be consistent all by itself?

We shall now show that actually the virtual production and decay of short-distance towers of massive states cannot play any rôle in the construction of a theory of gravity.

We have already noted that the relevant initial massless particle states contain an infinity of soft quanta. This means that their evolution can be described by the classical field equations until the massive particles are produced or strong Planck scale fields evolve. Observe now that the relation between a mass M and its Schwarzschild radius (in four space-time dimensions) is given by:

$$R = 2GM \tag{9}$$

As the mass M becomes large compared to the Planck mass, R grows and becomes correspondingly large compared to the Planck length. Thus, the time evolution of an initial soft state destined to produce a massive particle would necessarily create a horizon before the quanta ever got to desired scale. We emphasize that this conclusion depends only on the application of field theory and general relativity at the momentum scale $1/R \ll 1/m$, so that quantum corrections can be neglected.

We conclude that if effects of the presumed massive tower unitarity singularities were to affect low energy large distance physics, this would be tantamount to receiving information out of (arbitrarily large) black holes and constitute a violation of causality on space-time scales large compared to the Planck length.

There is, however, also a positive conclusion to be drawn from the preceding discussion. While large virtual black holes couple very weakly to low momentum quanta (the coupling is described by the Hawking radiation) we may contemplate Planck mass configurations⁶⁾ which might well generate the required non-analyticity and supply an ultra-violet cut-off to graviton loops. The corrections to the Lagrangian will not be parametrized in terms of a power series of polynomials. This, however, is an advantage since higher derivative Lagrangians generate runaway modes and instabilities.

We may conclude that not only is the cancellation of divergences in the perturbative S -matrix by the exchanges of massive towers insufficient for the construction of a consistent quantum gravity - it is also unnecessary.

Clearly, we are forced to confront the problem including black holes in the framework of quantum theory. The issue is inherently non-perturbative and its resolution might supply the mechanism for damping the short-distance singularities due to graviton exchanges.

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APPENDIX - ON THE PHYSICAL SIZE OF STRINGS

The physical spectrum of string states is defined in terms of wave functions whose arguments are the bosonic and fermionic string co-ordinates:

$$\Psi = \Psi[x(\sigma), \chi(\sigma), \dots] \quad (A.1)$$

Among these, the centre-of-mass co-ordinates occupy a privileged position in that the particle interpretation is derived by following their evolution:

$$X = \int_0^\pi \frac{d\sigma}{\pi} x(\sigma) \quad (A.2)$$

The interaction Hamiltonian of all string theories is specified by defining an amplitude for a string functional Ψ to decay suddenly into a product state $\Psi_1 \times \Psi_2$ such that the canonical co-ordinates $x(\sigma)$ and their conjugate momenta are continuous at the division point $\sigma = \eta\pi$. Hence, in perturbation theory, the centre-of-mass co-ordinates of the decay products have initial values which are determined by

$$X_1 = \int_0^{\eta\pi} \frac{d\sigma}{\eta\pi} x(\sigma) \quad , \quad X_2 = \int_{\eta\pi}^\pi \frac{d\sigma}{\pi(1-\eta)\pi} x(\sigma) \quad (A.3)$$

It follows that the spatial flow of physical observables in a final state of massless particles produced by the perturbative decay of some massive state is determined by the probability distribution of the "centre-of-mass" x 's of arbitrary string bits in the initial state. In the zeroth order, this probability measure is necessarily Gaussian and is therefore specified by the mean square fluctuation. Consider thus a closed string where $x(\sigma)$ has the expansion

$$x(\sigma) = X + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} (d_n e^{-2in\sigma} + \tilde{d}_n e^{2in\sigma}) \quad (A.4)$$

Define the centre-of-mass co-ordinate of a string bit through:

$$X(\sigma_1, \eta) = \int_{\sigma_1 - \frac{\eta\pi}{2}}^{\sigma_1 + \frac{\eta\pi}{2}} \frac{d\sigma}{\eta\pi} x(\sigma) \quad (A.5)$$

We find

$$X(\sigma, \eta) = X + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \frac{\sin n\pi\eta}{n\pi\eta} (d_n e^{-2in\sigma} + \tilde{d}_n e^{2in\sigma}) \quad (A.6)$$

A particle state is defined by specifying its mode-occupation numbers $\{N_n\}$:

$$n > 0 : \alpha_{-n} \alpha_n |\{N_n\}\rangle = n N_n |\{N_n\}\rangle \quad . \quad (A.7)$$

We also recall that in the light-cone gauge the mass is given by

$$M^2 = 4 \sum_{n>0} (\alpha_{-n} \alpha_n + \tilde{\alpha}_{-n} \tilde{\alpha}_n) + \dots \quad (A.8)$$

where ... designates contributions from the fermionic oscillators.

Let us now evaluate the mean squared fluctuation of our string bit co-ordinate:

$$\begin{aligned} & \langle \{N_n\} | [X(\sigma, \eta) - X]^2 | \{N_n\} \rangle = \\ & = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(2N_n+1)}{n} \cdot \frac{\sin^2 n\pi\eta}{(n\pi\eta)^2} \quad . \end{aligned} \quad (A.9)$$

A rigorous upper bound on (A.9) may be obtained by dividing the sum into two parts:

$$\sum_{n=1}^{\infty} = \sum_{n=1}^{\eta^{-1}} + \sum_{\eta^{-1}}^{\infty} \quad . \quad (A.10)$$

This leads to:

$$\langle [\Delta X(\sigma, \eta)]^2 \rangle \leq \frac{1}{2} \left(\sum_{n=1}^{\eta^{-1}} \frac{2N_n+1}{n} + \frac{1}{\pi^2 \eta^2} \sum_{\eta^{-1}}^{\infty} \frac{2N_n+1}{n^3} \right) \quad . \quad (A.11)$$

Let us choose $\eta \ll 1$ and designate by $n_0 \ll \eta^{-1}$ the lowest excited mode. We then easily get:

$$\langle [\Delta X(\sigma, \eta)]^2 \rangle \lesssim \frac{1}{2} \ln \eta^{-1} + \frac{M^2}{4n_0^2} + \text{const.} \quad (A.11)$$

Conversely, if $\eta \sim 0(1)$, only the second term in (A.9) contributes and we find:

$$\langle [\Delta X(\sigma, \eta)]^2 \rangle \lesssim \frac{1}{\pi^2 \eta^2} \cdot \frac{M^2}{4n_0^2} + \frac{1}{4\pi^2} \quad . \quad (A.12)$$

Both (A.11) and (A.12) imply that for sufficiently excited states, any string bit which is not too small ($\ln \eta^{-1} \lesssim 0(M^2)$) will satisfy our inequality

$$\langle [\Delta X(\eta)]^2 \rangle \lesssim \frac{M^2}{4n_0^2} \quad . \quad (A.13)$$

Observe now that in four space-time dimensions, M is related to its Schwarzschild radius through:

$$2GM = R \quad (\text{A.14})$$

we thus have (in units $2\alpha' = 1$):

$$\langle (\Delta X)^2 \rangle \lesssim \frac{R^2}{(4G\eta_0)^2} . \quad (\text{A.15})$$

We conclude that any massive string state whose lowest excited mode satisfies

$$4G\eta_0 > 1 \quad (\text{A.16})$$

is "smaller" than its own Schwarzschild radius.

In other words, any prediction about the decay of such a state into a set of massless quanta which involves the evolution of string bits whose total η satisfies the inequality

$$\eta \gg e^{-M^2} \quad (\text{A.17})$$

is inconsistent with macroscopic causality since it extracts information from the interior of a horizon. We note that the inequality (A.17) is extremely mild. In fact, if a string bit carries a longitudinal momentum fraction $\sim \eta$, then its transverse mass is bounded by

$$p_\perp^2 < \eta M_\perp^2 . \quad (\text{A.18})$$

Thus, if the string bit is to have a finite mass when $\eta \rightarrow 0$, we need

$$p_\perp^2 < \sigma(\eta) \quad (\text{A.19})$$

Since the transverse phase space vanishes like $p_\perp^2$, the contributions of intermediate states which satisfy Eq. (A.19) to physical processes may in general be neglected.

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