CERN-TH.4753/87

LIGHT-CONE LATTICE APPROACH TO FERMIONIC THEORIES IN 2D:
THE MASSIVE THIRRING MODEL

C. Destri^{*)} and H.J. de Vega^{**)}
CERN -- Geneva

ABSTRACT

Two-dimensional fermionic field theories are defined on a diagonal lattice obtained by discretizing Minkowski space-time in light-cone co-ordinates. This approach leads to local second-quantized equations of motion on the lattice. The continuum limit is carefully performed, yielding the massive Thirring model whenever fermions without internal structure are considered. The exact eigenstates and eigenvalues constructed in this lattice formalism confirm the known Bethe ansatz equations of the massive Thirring model. The light-cone lattice approach brings a class of integrable fermion models within the general algebraic scheme of the Quantum Inverse Scattering Method.

^{*)} On leave from Dipartimento di Fisica, Università di Parma, Parma, Italy.

^{**)} Permanent address: LPTHE, Université P. et M. Curie, Paris VI, Paris, France.

1. - INTRODUCTION

Integrable quantum field theories (QFT) are those QFT models possessing an infinite number of conserved and commuting charges. They are often exactly solvable to a large extent by means of Bethe ansatz methods essentially based on the existence of a Yang-Baxter algebra¹⁾.

Some of the physically more interesting integrable field theories in two dimensions are relativistic fermionic models with quartic interactions, like the Massive Thirring Model (MTM), the Gross-Neveu Model (GNM) and the chiral fermionic models with internal symmetry Lie groups^{2),3)}. Resolution of these models through the so-called co-ordinate, or x-space, Bethe ansatz in the continuum, provides explicit exact results for on-shell quantities, like the particle spectrum and the S-matrix³⁾. Dynamical mass generation and asymptotic freedom are nicely exhibited by this method. However, this continuum, unregularized Bethe ansatz must be improved in order to compute off-shell physical amplitudes like correlation functions of currents and fields. One needs to unambiguously define the regularized operators of the model before solving it exactly. Therefore, an integrable regularization is required. Space and/or space-time discretization seems the most appealing regulator, since integrability naturally associates to lattices.

From a different point of view, the fermionic models solvable by means of the continuum x-space Bethe ansatz do not fit nicely in the framework for integrable 2D system provided by the Quantum Inverse Scattering Method (QISM)¹⁾. One of the aims of our lattice regularization is to bring integrable fermion theories within the QISM scheme.

The approach followed in this paper consists in a discretization of 2D Minkowski space-time in light-cone co-ordinates $x_{\pm} = x \pm t$. Space-time is thus approximated by a diagonal lattice (see Fig. 1). The analogous construction in 2D classical statistical mechanics is known as the diagonal-to-diagonal transfer matrix method⁴⁾.

In Section 2, the diagonal lattice and its quantum mechanical interpretation are described in detail. It is also shown that the natural objects governing the dynamics are light-cone transfer matrices U_R and U_L . They respectively produce a right or left unit translation on the lattice in the diagonal directions. $U_{R,L}$ are products of equal time microscopic amplitudes associated to each site of the lattice (see Fig. 2). Contact with the notion of integrability follows from the

following observation: the microscopic amplitudes at each site can be assembled into a unitary "bare" scattering matrix, the (quantum) R-matrix; then the dynamics generated by $U_{R,L}$ is completely integrable whenever the R-matrix fulfils the so-called factorizability (or star-triangle, or Yang-Baxter) conditions.

In section 3, we show that the R-matrix, and hence the light-cone transfer matrices $U_{R,L}$, can be expressed in terms of "bare" fermion operators $\psi_{R,n}$ and $\psi_{L,n}$ associated to the equal time links of the lattice (see Fig. 3). At this stage, an explicit choice of R-matrix is needed leading to a particular fermionic model in the continuum. We consider in this work a dynamics with a U(1) conserved charge and fermions without internal degrees of freedom; the corresponding R-matrix is the Minkowski, real time counterpart of the statistical weights of the general six-vertex model (see Fig. 2). Under the assumption of symmetry under left-right exchange (that is parity), the R-matrix is related to an integrable generalization of the symmetric six-vertex model. In the continuum limit, the MTM emerges as the field theory characterizing the scaling behaviour of the dynamics on the lattice. This is explicitly shown at field theory level in Sections 4 and 5 [at the spectrum level, the connection between the six-vertex model "seen diagonally" and the MTM was observed in Refs.5)-7)].

In section 4, we derive the field equations on the lattice [Eqs. (4.6)]. They read in terms of the right and left fields $\psi_{R,n}$ and $\psi_{L,n}$:

$$\begin{aligned} U_R \psi_{R,n-1} U_R^\dagger &= U_L \psi_{R,n} U_L^\dagger = \\ &= b^* \psi_{R,n} + c^* \psi_{L,n} - \left(\frac{b}{\omega} + b^* \right) \psi_{L,n}^\dagger \psi_{L,n} \psi_{R,n} + \left(\frac{c}{\omega} - c^* \right) \psi_{R,n}^\dagger \psi_{R,n} \psi_{L,n} \end{aligned} \quad (1.1a)$$

$$\begin{aligned} U_R \psi_{L,n} U_R^\dagger &= U_L \psi_{L,n+1} U_L^\dagger = \\ &= b^* \psi_{L,n} + c^* \psi_{R,n} - \left(\frac{b}{\omega} + b^* \right) \psi_{R,n}^\dagger \psi_{R,n} \psi_{L,n} + \left(\frac{c}{\omega} - c^* \right) \psi_{L,n}^\dagger \psi_{L,n} \psi_{R,n} \end{aligned} \quad (1.1b)$$

where the complex numbers, b , c and ω parametrize the R-matrix. These are second-quantized "bare" equations, but are perfectly well defined on the light-cone lattice and unitary, since the light-cone transfer matrices $U_{R,L}$ are unitary operators.

The continuum limit of these equations is carefully analyzed in Section 5 where the continuum field Hamiltonian H and momentum operator P are derived. We find that as the lattice spacing a tends to zero:

$$U_A \underset{a \rightarrow 0}{\approx} e^{2i\mu Q} [1 - i a H_A + O(a^2)] \quad , \quad A = L, R \quad (1.2)$$

Here Q is the $U(1)$ conserved charge, while $H_{R,L}$ are light-cone components of the energy and momentum, that is

$$H_R = \frac{1}{2}(H + P) \quad , \quad H_L = \frac{1}{2}(H - P)$$

As anticipated, H and P are the Hamiltonian and momentum of the MTM:

$$H = \int dx \left[-i\psi^\dagger \gamma^5 \partial_x \psi + m \bar{\psi} \psi + \frac{1}{2} g (\bar{\psi} \gamma_\mu \psi)(\bar{\psi} \gamma^\mu \psi) \right] \quad (1.3)$$

$$P = -i \int dx \psi^\dagger \partial_x \psi$$

It must be noticed that our discrete MTM has local lattice field equations (1.1), although the lattice Hamiltonian, which can be defined as logarithm of $U_R U_L$, is non-local. This is why our manifestly chiral representation for the lattice fermion fields (on which chiral rotations act therefore in a strictly local way) does not lead to species doubling.

In Section 6, we construct the exact eigenstates of U_R and U_L by means of a Bethe ansatz, connecting in this way our second quantized formalism with the first quantized amplitudes of Refs. 5)-7). The solution of the corresponding Bethe Ansatz Equations (BAE) leads to the physical spectrum and S-matrix of the theory derived in the continuum approach⁸⁾. On the light-cone lattice we have, in addition, well defined field operators providing a starting point for an exact computation of correlation functions following, for instance, the general approach of Ref. 9). Indeed, the Bethe ansatz regularized on the lattice admits an algebraic formulation of general QISM type, combining row-to-row and light-cone transfer matrices.

2. - DIAGONAL LATTICE AND LIGHT-CONE TRANSFER MATRICES

It is customary to regularize 2D field theories on rectangular lattices where the axis corresponds to discrete space and (real or imaginary, continuous or

discrete) time co-ordinates. In statistical mechanics language this connects to row-to-row transfer matrices.

We shall adopt in this paper an alternative approach based on the diagonal lattice obtained by discretizing Minkowski space-time in light-cone co-ordinates $x_{\pm} = x \pm t$. Given the discretized space-time as a face-centered square lattice of points (world events), the diagonal lattice is reconstructed by emanating a two-dimensional light-cone from each event (Fig. 1). Each site (event) is joined to its four nearest neighbours by light-like links along x_+ and x_- . These diagonal links are then possible world lines for the propagation upwards in time of "bare" massless particles. Particles on right-oriented (R) and left-oriented (L) links are called respectively right and left movers. A horizontal line at fixed time t_0 (see Fig. 1) intersects successively L or R links. The state of the system at t_0 is thus entirely fixed by specifying on which links particles are found. If we assume that at most one particle can be found on a given diagonal link, we can regard our bare particles as fermions without internal degrees of freedom.

Let us associate microscopic amplitudes to each site (world event) where two oppositely oriented world lines cross. These amplitudes describe the different processes that can take place, and must verify general invariance properties like unitarity.

In principle, there can be 16 different amplitudes per site, corresponding to the 16 states (occupied/empty) of the four links joining there. We shall, however, consider only U(1) invariant microscopic amplitudes such that the number of particles is conserved at each site. [U(1) transformations act on the link states by $|0\rangle \rightarrow e^{i\lambda}|0\rangle$, $|1\rangle \rightarrow e^{-i\lambda}|1\rangle$, where $|0\rangle \equiv |\text{empty}\rangle$ and $|1\rangle \equiv |\text{occupied}\rangle$.] Overall, there are six U(1)-invariant amplitudes as depicted in Fig. 2. The correspondence with the general, non-symmetric six-vertex model is evident.

Of course, by space-time translation invariance, the amplitudes are the same at all sites of the lattice. It is natural (and causes no loss of generality) to assume the nothing-to-nothing amplitude ω_1 to be 1. Unitarity then requires quite simply

$$|\omega_2|^2 = 1 \quad ; \quad \Omega \Omega^\dagger = 1 \quad , \quad \Omega \equiv \begin{pmatrix} \omega_3 & \omega_5 \\ \omega_6 & \omega_4 \end{pmatrix} \quad (2.1)$$

While ω_3 and ω_4 are naturally interpreted as amplitudes for free propagation (being therefore related to kinetic energies in the continuum limit), ω_5 and ω_6 "look like" mass terms, since they couple right and left movers. ω_2 appears to

be the interaction term, but as we shall see, there is another source of interaction.

Furthermore, if we demand symmetry under parity transformation which exchanges left with right, then $\omega_3 = \omega_4 \equiv b$ and $\omega_5 = \omega_6 \equiv c$. The amplitudes of Fig. 2 correspond now to an integrable six-vertex model¹⁾. The unitarity constraint on b and c reads:

$$|b|^2 + |c|^2 = 1 \quad ; \quad bc^* + b^*c = 0 \quad (2.2)$$

We can organize the microscopic amplitudes at a site into a 4×4 unitary "bare" scattering matrix ($\omega \equiv \omega_2$)

$$R_{\alpha'\beta'}^{\alpha\beta} = \begin{array}{c} \alpha \quad \beta \\ \diagdown \quad \diagup \\ \alpha' \quad \beta' \end{array} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix} \quad (2.3)$$

This is the R-matrix, the central object of our construction.

The amplitude for a global process, from a given state at $t = t_0$ to another given state at a later time, is obtained by summing over the amplitudes of all allowed vertex configurations compatible with initial and final conditions and with boundary conditions. Each of these is given by a product of microscopic amplitudes ω_i . It clearly corresponds to the sum over all possible paths of an arbitrary, but constant in time number of particles. At any instant, a particle can move to the left or to the right at the speed of light. We are thus dealing with a discretization of Feynman path integral for fermions.

We shall now derive an operatorial formalism equivalent to this discrete path integral. Let the diagonal lattice spacing be $a/\sqrt{2}$ (see Fig. 1). While the lattice is clearly left invariant by a translation of any multiple of $a/\sqrt{2}$ in a diagonal direction, it is not invariant under a translation of an odd multiple of $a/2$ in either the time or space directions. Rather, it goes into the dual lattice whose sites are in the centre of the plaquettes of the original lattice. The unit time evolution is therefore defined by the amplitudes associated to a motion by twice $a/2$, that is by a , in the positive direction. With periodic boundary conditions in space, the unit time evolution operator U (the transfer matrix in the language of statistical mechanics) writes as a product

$$U = U_+ U_- \quad (2.4)$$

where the unitary operators U_+ and U_- are depicted in Figs. 3a and 3b. The product follows by summing over the repeated indices $\alpha'_1, \dots, \alpha'_{2N}$ labelling the intermediate state. N stands for the number of sites at a given time, and $|\alpha\rangle \equiv |\alpha_1, \dots, \alpha_{2N}\rangle$, $\alpha_j = 0, 1$, represents a generic state of the system. The operators U_+ and U_- produce each a time translation by $a/2$.

An alternative description is obtained by taking the operators U_+ and U_- in the reverse order which leads to the one-step evolution operator $U' = U_- U_+$ depicted in Fig. 4. These two descriptions are unitarily connected by a half-step translation of $a/2$ to the right or to the left. Let us denote by V the unitary operator associated to this right translation. Its square V^2 , affecting a translation by a , is a symmetry of the problem and we have

$$[V^2, U_{\pm}] = 0 \quad ; \quad [V^2, U] = [V^2, U'] = 0 \quad (2.5)$$

while

$$U_{\pm} = V U_{\mp} V^{\dagger} \quad ; \quad U' = V U V^{\dagger} \quad (2.6)$$

In other words, V takes the lattice into its dual lattice

On the other hand, a time translation by an entire lattice spacing a can be obtained by two successive translations by $a/\sqrt{2}$ in the diagonal directions. There are two equivalent possibilities: first along an R world line (with associated operator U_R) and then along an L world line (with associated operator U_L) or vice versa. We are therefore led to the geometric relations

$$U = U_R U_L = U_L U_R = U_+ U_- \quad (2.7a)$$

$$U_R = U_+ V = V U_- \quad ; \quad U_L = U_+ V^{\dagger} = V^{\dagger} U_- \quad (2.7b)$$

and similarly

$$U' = U'_R U'_L = U'_L U'_R = U_- U_+ \quad (2.8a)$$

$$U'_R = V U_+ = U_- V \quad ; \quad U'_L = V^{\dagger} U_+ = U_- V^{\dagger} \quad (2.8b)$$

Furthermore, we have

$$U_R U_L^\dagger = U_R' U_L'^\dagger = V^2 \quad (2.9)$$

which is also geometrically obvious. U_R and U_L are (unitary) light-cone transfer matrices, that is unit evolution operators on the light-cone.

Relations (2.5) to (2.9) are quite general, relying just on the lattice symmetries. They hold for particles with or without internal symmetries regardless of the particular choice of unitary microscopic amplitudes. Once a choice is made, $U_\pm$ (hence U_R , U_L and U) are completely determined. For instance, we can write for $U_\pm$:

$$\langle \alpha | U_+ | \alpha' \rangle = \prod_{n=1}^N R_{\alpha'_{2n-1}, \alpha'_{2n}}^{\alpha_{2n-1}, \alpha_{2n}} \quad (2.10a)$$

$$\langle \alpha | U_- | \alpha' \rangle = \prod_{n=1}^N R_{\alpha'_{2n}, \alpha'_{2n+1}}^{\alpha_{2n}, \alpha_{2n+1}} \quad (\alpha_{2N+j} \equiv \alpha_j) \quad (2.10b)$$

where the local R amplitudes are given in Eq. (2.3)

3. - LATTICE FERMION FIELDS IN THE CHIRAL REPRESENTATION

We now proceed to the construction of an explicit "second-quantized" representation of the group theoretical relations (2.4) to (2.9), making use of Eqs. (2.3) and (2.10) and of lattice fermion fields, $\phi_{R,n}$ and $\phi_{L,n}$. These are associated to the links stemming upwards from each site "at fixed time", say $t = t_0$ (see Figs. 1 and 5) - $\phi_{R,n}(\phi_{L,n})$ to the $R(L)$ link - and satisfy the usual anticommutation rules:

$$\{\psi_{A,n}, \psi_{B,m}\} = 0 ; \quad \{\psi_{A,n}, \psi_{B,m}^\dagger\} = \delta_{AB} \delta_{nm} , \quad A, B = R, L, \quad n, m = 1, \dots, N \quad (3.1)$$

$\phi_{R,n}$ and $\phi_{L,n}$ can be assembled into a two-component spinor. In this representation, γ_5 is obviously diagonal since chiral rotations act locally on the ϕ 's: $\phi_{R,n} \rightarrow e^{i\lambda} \phi_{R,n}$, $\phi_{L,n} \rightarrow e^{-i\lambda} \phi_{L,n}$; it is the chiral representation. Our lattice fermions are therefore quite different from Kogut-Susskind fermions for which chiral transformations act non-locally on the lattice. In our case the species doubling is avoided, thanks to the non-locality (on the lattice) of the Hamiltonian corresponding to the transfer matrix U .

To simplify our notation, it is convenient to write

$$\psi_{R,n} = \psi_{2n} \quad ; \quad \psi_{L,n} = \psi_{2n-1} \quad ; \quad n = 1, \dots, N \quad (3.2)$$

so that (3.1) is replaced by

$$\{\psi_n, \psi_m\} = 0 \quad ; \quad \{\psi_n, \psi_m^\dagger\} = \delta_{nm} \quad ; \quad n, m = 1, \dots, 2N \quad (3.3)$$

Consider now the bilocal, unitary, even operators

$$R_{nm} = 1 + b K_{nm} + (c-1) K_{nm}^2 + (\omega-1) \psi_n^\dagger \psi_n \psi_m^\dagger \psi_m \quad (3.4)$$

where

$$K_{nm} \equiv \psi_n^\dagger \psi_m + \psi_m^\dagger \psi_n$$

and b, c and ω are the amplitudes introduced in the previous section [Eqs. (2.1)-(2.3)]. In the "bare" particle representation $|\underline{\alpha}\rangle$ we have

$$\langle \underline{\alpha} | R_{nm} | \underline{\alpha}' \rangle = R_{\alpha'_n, \alpha'_m}^{\alpha_n, \alpha_m} \prod_{j \neq n, m} \delta_{\alpha'_j}^{\alpha_j} \quad (3.5)$$

as soon as we make the usual identification $(n_1 < n_2, \dots, < n_M)$:

$$|0 \dots \overset{n_1}{1} \dots \overset{n_2}{1} \dots \overset{n_M}{1} \dots 0\rangle = \psi_{n_1}^\dagger \psi_{n_2}^\dagger \dots \psi_{n_M}^\dagger |00 \dots 0\rangle \quad (3.6)$$

It should perhaps be stressed that this "first-quantized" point of view is fully optional on the lattice: we can manipulate only field operators and never mention "bare" particles with associated wave functions. Of course, the choice of a reference state is fully relevant in the continuum limit when all states like (3.6) with a cut-off independent number M of bare particles may be lifted away from the physical spectrum.

The shift operator V, affecting one-half translation to the right is now explicitly given by:

$$V = P_{12} P_{23} P_{34} \dots P_{2N-1, 2N} \quad ; \quad VV^\dagger = 1 \quad (3.7)$$

where P_{nm} is the value of R_{nm} at $b = \omega = 1$ and $c = 0$ namely

$$P_{nm} = P_{nm}^{-1} = P_{nm}^\dagger = 1 + K_{nm} - K_{nm}^2 \quad (3.8)$$

One easily verifies that (periodic B.C. means that $\psi_{n+2N} \equiv \psi_n$):

$$P_{nm} \psi_m P_{nm} = \psi_n$$

and

$$V \psi_n V^\dagger = \psi_{n+1} \quad (3.9)$$

According to Eqs. (2.10), U_+ and U_- read:

$$U_+ = R_{12} R_{34} \dots R_{2N-1, 2N} \quad (3.10a)$$

$$U_- = R_{23} R_{45} \dots R_{2N, 1} \quad (3.10b)$$

The exact ordering is actually irrelevant, since the factors clearly commute with one another. The basic relations [see Eqs. (2.5) and (2.6)]

$$U_\pm = V U_\mp V^\dagger \quad (3.11)$$

(which imply $[U_\pm, V^2] = 0$) can now be checked explicitly. Finally, U_R , U_L , U and U'_R , U'_L , U' are defined as in Eqs. (2.7) and (2.8).

A natural connection exists in one space dimension between fermions and spin operators, through the Jordan-Wigner transformation. For the operators in Eq. (3.2), this explicit spin representation reads:

$$\psi_n = \sigma_n^- \prod_{\ell=1}^{n-1} \sigma_\ell^z, \quad \psi_n^\dagger = \sigma_n^+ \prod_{\ell=1}^{n-1} \sigma_\ell^z \quad (3.12)$$

where the σ_n 's are Pauli matrices associated to site n . The operators R_{nm} (the quantum R matrices) of Eq. (3.4) are written in spin language as:

$$R_{nm} = \frac{c+1}{2} + \frac{b}{2} (\sigma_n^x \sigma_m^x + \sigma_n^y \sigma_m^y) + \frac{c-1}{2} \sigma_n^z \sigma_m^z + \frac{\omega-1}{2} (1 - \sigma_n^z)(1 - \sigma_m^z) \quad (3.13)$$

The connection with spin models is now direct.

4. - LATTICE FIELD EQUATIONS

Given the one-step evolution operator U , we can evaluate the field $\phi_n(t)$ at any discrete time $t = t_0 + sa$, $s \in \mathbb{Z}$, in terms of the field at time t_0 , $\phi_n \equiv \phi_n(t_0)$:

$$\psi_n(t) = U^s \psi_n U^{-s}$$

In our conventions the $t = t_0$ line, starting from the left, first cuts an L link (see Fig. 1). Hence the proper time evolution is determined by U rather than U' (see Figs. 3 and 4). The opposite convention (the $t = t_0$ line cutting first an R link) corresponds to the field numbering opposite to that in Eq. (3.2), that is $\phi_{R,n} = \phi_{2n-1}$, $\phi_{L,n} = \phi_{2n}$, and would require U' as proper transfer matrix. These two descriptions which are unitarily related by V [cf. Eqs. (2.6) and (3.9)] become indistinguishable in the continuum limit.

The shortest discrete evolution

$$\psi_n \rightarrow \psi_n(a) = U \psi_n U^\dagger \quad (4.1)$$

replaces on our lattice the time derivative. It is actually possible to define a lattice time derivative and a continuous time evolution by introducing a non-local lattice Hamiltonian via:

$$U = \exp(-ia H_{\text{lat}}) ; \quad H_{\text{lat}} \equiv \frac{i}{a} \ln U \quad (\text{mod } \frac{2\pi}{a}) \quad (4.2)$$

with a suitable choice of the logarithmic branch to ensure the correct properties of the energy spectrum. Typically, a definition like (4.2) leads to a monotonically increasing dispersion relation between energy and modulus of the momentum within each Brillouin zone, with non-smooth joining at the zone boundaries and no danger of unwanted species doubling. As an example, which also illustrates the differences with Kogut-Susskind fermions, consider free massless fermions. In this case the natural definition of free $U_R^{(0)}$ and $U_L^{(0)}$ [giving $U^{(0)}$ by $U_R^{(0)} U_L^{(0)}$] is:

$$U_R^{(0)} = V_R \quad ; \quad U_L^{(0)} = V_L^\dagger \quad (4.3)$$

where $V_R (V_L)$ is a unit shift operator constructed solely with $\phi_{R,n} (\phi_{L,n})$. This means, taking Eq. (3.2) into account:

$$\begin{aligned} V_R &= P_{24} P_{46} \dots P_{2N-2, 2N} \\ V_L &= P_{13} P_{35} \dots P_{2N-3, 2N-1} \\ [V_R, V_L] &= 0 \end{aligned}$$

The "lattice field equations" then read quite simply

$$U^{(0)} \psi_{R,n} U^{(0)\dagger} = V_R \psi_{R,n} V_R^\dagger = \psi_{R,n+1} \quad ; \quad V_R \psi_{L,n} V_R^\dagger = \psi_{L,n} \quad (4.4a)$$

$$U^{(0)} \psi_{L,n} U^{(0)\dagger} = V_L^\dagger \psi_{L,n} V_L = \psi_{L,n-1} \quad ; \quad V_L^\dagger \psi_{R,n} V_L = \psi_{R,n} \quad (4.4b)$$

and are immediately diagonalized by the discrete Fourier transform:

$$B_A(k) = \sum_{n=1}^N \psi_{A,n} e^{ikna} \quad ; \quad A=R,L \quad ; \quad e^{ikL} = 1 \quad ; \quad L \equiv Na \quad (4.5)$$

Thus we obtain the lattice Hamiltonian and momentum operator:

$$H_{\text{latt}}^{(0)} = \frac{i}{a} \ln V_R V_L^\dagger = \sum_k k [B_R^\dagger(k) B_R(k) - B_L^\dagger(k) B_L(k)]$$

$$P_{\text{latt}}^{(0)} = \frac{i}{a} \ln V_R V_L = \sum_k k [B_R^\dagger(k) B_R(k) + B_L^\dagger(k) B_L(k)]$$

where the sum over k runs over all the roots of $e^{ikL} = 1$ within the first Brillouin zone. The dispersion relation

$$E = \pm k \quad (+ \text{ for } R, - \text{ for } L) \quad (\text{mod } \frac{2\pi}{a})$$

has the properties mentioned above and shows no doubling.

On the other hand, free massless fermions in 1+1 dimensions are put on the lattice à la Kogut-Susskind, in the non-chiral representation where γ^0 is diagonal, by associating the upper components to, say, odd sites and the lower components to even sites. The local Hamiltonian

$$H_{KS} = \frac{1}{2ia} \sum_n (\psi_n^\dagger \psi_{n+1} - \psi_{n+1}^\dagger \psi_n)$$

generates the equation of motion:

$$i\dot{\psi}_n = [H_{KS}, \psi_n] = \frac{1}{2ia} (\psi_{n-1} - \psi_{n+1})$$

corresponding to the dispersion relation

$$E = \frac{\sin ka}{a}$$

This exhibits species doubling, as required by the need of reconstructing a two-component object in the continuum limit. Notice that in this second approach a single chiral fermion field, say, $\phi_R(x)$, cannot be discretized consistently, while in the first approach this is manifestly possible since $\phi_{R,n}$ and $\phi_{L,n}$ are completely decoupled.

Let us go back to the general case and compute the lattice field equations analogous to (4.4a,b). It is fairly straightforward to obtain, from Eqs. (2.7), (3.9) and (3.10)

$$U_R \psi_{2n-2} U_R^\dagger = U_L \psi_{2n} U_L^\dagger = U_+ \psi_{2n-1} U_+^\dagger = R_{2n-1,2n} \psi_{2n-1} R_{2n-1,2n}^\dagger \quad (4.6a)$$

$$U_R \psi_{2n-1} U_R^\dagger = U_L \psi_{2n+1} U_L^\dagger = U_+ \psi_{2n} U_+^\dagger = R_{2n-1,2n} \psi_{2n} R_{2n-1,2n}^\dagger \quad (4.6b)$$

The calculation now reduces to standard anticommuting algebra among the fields in $R_{2n-1,2n}$ and either ψ_{2n-1} or ψ_{2n} . This finally gives:

$$\begin{aligned} U_R \psi_{2n-2} U_R^\dagger &= U_L \psi_{2n} U_L^\dagger = \\ &= b^* \psi_{2n} + c^* \psi_{2n-1} - \left(\frac{b}{\omega} + b^*\right) \psi_{2n-1}^\dagger \psi_{2n-1} \psi_{2n} + \left(\frac{c}{\omega} - c^*\right) \psi_{2n}^\dagger \psi_{2n} \psi_{2n-1} \end{aligned} \quad (4.7a)$$

$$\begin{aligned} U_L \psi_{2n-1} U_L^\dagger &= U_R \psi_{2n+1} U_R^\dagger = \\ &= b^* \psi_{2n-1} + c^* \psi_{2n} - \left(\frac{b}{\omega} + b^*\right) \psi_{2n}^\dagger \psi_{2n} \psi_{2n-1} + \left(\frac{c}{\omega} - c^*\right) \psi_{2n-1}^\dagger \psi_{2n-1} \psi_{2n} \end{aligned} \quad (4.7b)$$

Due to the identification (2.2), these equations are exactly those anticipated in the Introduction [Eqs. (1.1a,b)]. One could now proceed and compute the equations of motion generated by $U = U_R U_L = U_L U_R$, which will involve products up to five fields. Those cumbersome formulae are not really necessary and will be omitted.

Some comments on Eqs. (4.7a,b) are in order. First of all, we would like to stress that these lattice field equations are local, involving only fields at neighbouring points, although the operators U_R and U_L are themselves non-local objects. This must be compared with the commutator of the lattice Hamiltonian $H_{latt.}$ of Eq. (4.2) and a field ϕ_n . It is a very complicated non-local expression containing all powers of the field. The same would happen for a generic time evolution $U(t) \phi_n U(t)^\dagger$, $U(t) = \exp\{-iH_{latt.} t\}$, $t-t_0 \neq sa$, $s \in \mathbb{Z}$. This shows that the light-cone transfer matrices U_R and U_L are the natural objects describing the

dynamics on the diagonal lattice: the usual derivation of the lattice Hamiltonian from row-to-row transfer matrices via continuum limit in time, keeping the space discrete, is not a sensible procedure on the diagonal lattice which has complete symmetry in space and time.

Secondly, we observe that the lattice Eqs. (4.7) have the same structure of continuum field equations derived from a Lagrangian with a quartic fermion interaction. We can regard the terms linear in ψ as "free" part and the cubic terms as interaction. As mentioned earlier, the choice $\omega = 1$ is not enough to eliminate the interaction. If ω, b and c are so chosen, within the unitarity constraint, that no cubic term appears in (4.7) (e.g., $\omega = -1$, $b = b^*$, $c = -c^*$), then we are dealing with discretized free massive fermions. Furthermore, if $b = 1$, then Eqs. (4.7) reduce to Eqs. (4.4), that is $U_{R,L} = U_{R,L}^{(0)}$ and the fermions are free and massless. Matters will perhaps become more transparent after the continuum limit which we discuss in the next section.

We close this section with the equations of motion for the bilinear field $\phi_n^+ \phi_n \equiv (\psi^+ \psi)_n$:

$$U_R(\psi^+ \psi)_{2n-2} U_R^\dagger = U_L(\psi^+ \psi)_{2n} U_L^\dagger = (\psi^+ \psi)_{2n} - |c|^2 M_n \quad (4.8a)$$

$$U_R(\psi^+ \psi)_{2n-1} U_R^\dagger = U_L(\psi^+ \psi)_{2n-1} U_L^\dagger = (\psi^+ \psi)_{2n} + |c|^2 M_n \quad (4.8b)$$

where

$$M_n = (\psi^+ \psi)_{2n} - (\psi^+ \psi)_{2n-1} + \frac{b}{2c} (\psi_{2n-1}^\dagger \psi_{2n} - \psi_{2n}^\dagger \psi_{2n-1})$$

These equations imply a sort of lattice conservation law for the quantity

$$J_n \equiv (\psi^+ \psi)_{2n-1} + (\psi^+ \psi)_{2n} = (\psi_L^\dagger \psi_L)_n + (\psi_R^\dagger \psi_R)_n$$

$$U J_n U^\dagger = J_{n+1} = V^2 J_n V^{-2}$$

(we recall that V^2 affects a unit space translation to the right). In particular we have:

$$[U_R, Q] = [U_L, Q] = [U, Q] = 0$$

where

$$Q = \sum_{n=1}^N J_n = \sum_{n=1}^{2N} \psi_n^\dagger \psi_n$$

(4.9)

is the (bare) $U(1)$ charge or "particle number" operator.

5. - THE CONTINUUM LIMIT

The continuum limit consists in letting $a \rightarrow 0$, $N \rightarrow \infty$, while keeping the total length of the system $L = Na$ fixed. The Hilbert space of the lattice theory which is isomorphic to $(\mathbb{C})^{2N}$, becomes non-separable in this limit. However, only a separable sector of this larger space constitutes the physical Hilbert space, supporting a positive energy representation of the 2D Poincaré group. The physical states are all those with finite energy and momentum relative to the physical vacuum which is the $a \rightarrow 0$ limit of the (unique) lattice ground state. In an interacting theory, as is our case for generic values of b , c and ω , the continuum limit in the physical sector essentially defines the interacting continuum field theory. In particular, the limiting form of the lattice field equation (1.1) or (4.7), once written in terms of composite field operators with finite (i.e., cut-off independent, a^{-1} being the U.V. cut-off) matrix elements in the physical sector, would provide us with renormalized relativistic non-linear field equations.

Therefore, the question naturally arises as how to identify the continuum limit of the lattice theory before solving it explicitly (something that usually cannot be done exactly unless the model at hand is completely integrable on the lattice which is just our case) and/or without performing more or less complicated perturbative subtractions on composite lattice operators. Strictly speaking, the question is of a formal nature, since the value of an interacting continuum field Lagrangian or Hamiltonian is only formal and heuristic before renormalization.

From this point of view, one usually considers the so-called "classical" continuum limit to be performed at the action level on commuting c-numbers in the bosonic case or Grassmann variables in the fermionic case. In our lattice theory, we want to study the continuum limit of the lattice field equations (4.7) which are already quantum in nature and were derived on the basis of quantum anticommutation rules [Eqs. (3.1) and (3.3)]. For the formal identification of the continuum field theory, we shall therefore consider the "bare" continuum limit. This consists in letting the U.V. cut off to infinity, i.e., $a \rightarrow 0$,

$N \rightarrow \infty$, Na fixed, working by hypothesis on states with an arbitrary but cut-off independent, number of bare particles. This corresponds to choosing, as reference state, the "bare vacuum", the state annihilated by all ϕ 's, rather than the would-be physical vacuum which on the lattice contains a number of bare particles of order $N = La^{-1}$. In this bare limit, moreover, the continuum fields $\psi(x)$ are canonical, being related to their lattice counterpart ψ_n by a naïve dimensional factor $\sqrt{a}$, without any anomalous dimension. As it turns out, the "bare" continuum limit itself is already not quite naïve as is usually the case for the "classical" continuum limit.

As a preliminary step, consider the continuum limit of the free equations (4.4). These equations being linear, no need emerges to specify the reference state ("bare" and "renormalized" points of view are trivially related by a single zero-point energy redefinition in this case). The continuum dimensionful Fermi fields are recovered according to:

$$\frac{1}{\sqrt{a}} \psi_{A,n} \rightarrow \psi_A(x) \quad ; \quad A=R,L \quad ; \quad x=na \quad (5.1)$$

when $a \rightarrow 0$ (we use the standard rule that $a^{-1} \delta_{nn'} \rightarrow \delta(x-x')$ if $x=na$, $x'=ma$). On the other hand, the continuum Hamiltonian $H^{(0)}$ and momentum operator $P^{(0)}$ are obtained from $U^{(0)}$ and V^2 according to:

$$H^{(0)} = \lim_{a \rightarrow 0} \frac{i}{a} (U^{(0)} - 1) \quad (5.2)$$

and

$$P^{(0)} = \lim_{a \rightarrow 0} \frac{i}{a} (V^2 - 1) \quad (5.3)$$

Clearly

$$H^{(0)} = H_R^{(0)} + H_L^{(0)} \quad ; \quad P^{(0)} = H_R^{(0)} - H_L^{(0)}$$

if we define

$$H_A^{(0)} = \lim_{a \rightarrow 0} \frac{i}{a} (U_A^{(0)} - 1) \quad ; \quad A=R,L \quad (5.4)$$

With the natural identification

$$\psi_{A,n\pm 1} \sim \sqrt{a} (1 \pm a \partial_x) \psi_A(x)$$

Eqs. (4.4) and (5.4) imply

$$[H_A^{(0)}, \psi_B(x)] = \pm i \delta_{AB} \partial_x \psi_B(x) \quad ; \quad (+ \text{ for } A=R, - \text{ for } A=L)$$

that is

$$H_A^{(0)} = \mp i \int_0^L dx \psi_A^\dagger \partial_x \psi_A \quad (5.5)$$

which is the expected correct result.

In the general case we proceed as follows. Exactly at $a = 0$, the terms cubic in the fields can be dropped relative to the linear terms [they are down by a factor of a , due to (5.1) and the "bare" choice of reference state]. Equations (4.7) then reduce to

$$U_A \psi_B(x) U_A^\dagger = b^* \psi_B(x) + c^* \psi_{B'}(x) ; A = R, L ; B \neq B' = R, L$$

where $U_{R,L}$, b and c are all evaluated at $a = 0$. Without loss of generality we can take $|b| = 1$ and $c = 0$ at $a = 0$. This is the natural choice leading to a direct identification of the right R and L components of the fields on the continuum, without the need of a linear redefinition. We thus see that at $a = 0$

$$U_R = U_L = e^{i\mu Q} ; e^{i\mu} \equiv \lim_{a \rightarrow 0} \frac{b}{|b|} \quad (5.6)$$

where the (bare) $U(1)$ charge Q will now read [cf. Eq. (4.9)]

$$Q = \int_0^L dx (\psi_R^\dagger \psi_R + \psi_L^\dagger \psi_L)$$

in terms of continuum fields. Hence, we are led to the following definition of the continuum Hamiltonian [cf. Eq. (1.2)]:

$$H = \lim_{a \rightarrow 0} \frac{i}{a} (e^{-2i\mu Q} U - 1) \quad (5.7)$$

while for the momentum, we can keep Eq. (5.3)

$$P = \lim_{a \rightarrow 0} \frac{i}{a} (V^2 - 1) \quad (5.8)$$

which is simply the statement that P , like $P^{(0)}$ in the free case, generates space translations. If we define, as in Eq. (5.4):

$$H_A = \lim_{a \rightarrow 0} \frac{i}{a} (e^{-i\mu Q} U_A - 1) \quad (5.9)$$

then [recall (2.9)]

$$H = H_R + H_L \quad ; \quad P = H_R - H_L \quad (5.10)$$

H_R and H_L are the light-cone components of energy and momentum. Notice that P can also be written as $P = H_R^{(0)} - H_L^{(0)}$, that is in a way formally identical to the free case, the operators $H_{R,L}^{(0)}$ being defined as in (5.9) and (4.3) in terms of interacting fields.

Next, from Eqs. (4.7a) and (5.7), we obtain for instance:

$$\begin{aligned} [H_R, \psi_R(x)] = \lim_{a \rightarrow 0} \frac{|b|}{a} \left\{ i(\psi_{R,n-1} - \frac{1}{|b|} \psi_{R,n}) - \frac{ic}{b} \psi_{L,n+1} + \right. \\ \left. - (1 + w^* e^{2i\mu}) \left[i(\psi_L^\dagger \psi_L)_{n+1} \psi_{R,n+1} - \frac{ic}{b} (\psi_R^\dagger \psi_R)_{n+1} \psi_{L,n+1} \right] \right\} \end{aligned} \quad (5.11)$$

Hence, taking the unitarity constraints (2.3) and (5.6) into account, the continuum limit is achieved provided the lattice parameters b and c scale according to

$$b \sim e^{i\mu} (1 + O(a^2)) \quad ; \quad c \sim -\frac{ie^{i\mu}}{2} m a (1 + O(a^2)) \quad (5.12)$$

where μ and the mass parameter m play the role of free parameters characterizing this "bare" scaling limit.

A crucial observation now concerns the scaling limit of the fields themselves. In this interacting case, the simple rule (5.1) is not precise enough: it can be applied directly to isolated terms linear in the fields [as we actually did to obtain the left-hand side of Eq. (5.11)], but not to the cubic terms in the square brackets of the right-hand side of (5.11). It is well known, after all, that special care is needed to define products of operators at the same point in the continuum. From the definition itself of lattice fields (Fig. 5), it is clear that a more precise version of the limit (5.1) reads

$$\psi_{R,n} \sim \sqrt{a} \psi_R(x + \xi a) \quad ; \quad \psi_{L,n} \sim \sqrt{a} \psi_L(x - \xi a) \quad ; \quad x = na \quad (5.13)$$

where $0 < \xi < \frac{1}{2}$. The precise value of ξ is irrelevant in the continuum limit, provided it stays strictly larger than zero. Now consider the binomial $\phi_{2n-1} \phi_{2n} = \phi_{L,n} \phi_{R,n}$ entering the right-hand side of Eqs. (4.7) or (5.11). Using the same

equations of motion (4.7), it is possible to derive the exact lattice field equations for $\psi_{2n-1}\psi_{2n}$. They read, quite simply

$$\begin{aligned} U_R \psi_{2n-2} \psi_{2n-1} U_R^\dagger &= U_R \psi_{2n-2} U_R^\dagger U_R \psi_{2n-1} U_R^\dagger = \omega^* \psi_{2n-1} \psi_{2n} = \\ &= U_L \psi_{2n} \psi_{2n+1} U_L^\dagger \end{aligned} \quad (5.14)$$

With this result and Eqs. (5.9) and (5.13), it is fairly straightforward to obtain:

$$\begin{aligned} [H_R, \psi_R(x-0) \psi_L(x)] &= [H_L, \psi_R(x) \psi_L(x+0)] = \\ &= \lim_{a \rightarrow 0} \frac{-i}{a} (\omega^* e^{2i\mu} \psi_{R,n-1} + \psi_{R,n}) \psi_{L,n} \\ &\simeq \lim_{a \rightarrow 0} \frac{-i}{a} \{ \omega^* e^{2i\mu} \psi_R(x-0) + \psi_R(x-0) \} \psi_L(x) \end{aligned} \quad (5.15)$$

with the conventional notation $\phi(x \pm 0) \equiv \lim_{\epsilon \rightarrow 0} \phi(x \pm \epsilon)$. Since both $H_{R,L}$ and $\psi_R(x) \psi_L(x')$, $\forall x, x'$, are meaningful operators in this "bare" continuum limit [$\psi_R(x) \psi_L(x')$ would be so even in a "renormalized" continuum limit, but $H_{R,L}$ would not, lacking at least the naïve normal-ordering], we conclude that the combination in the curly brackets of Eq. (5.15) must vanish in the (dense) domain of definition of $H_{R,L}$. Stated differently, the domain of $H_{R,L}$ in any sector with a given, cut-off independent number of bare particles, is composed of piece-wise differentiable wave functions with prescribed discontinuities at coincident arguments relative to opposite chirality particles.

Rearranging a little, one can cast this last result in the form:

$$\frac{1}{a} \psi_{R,n} \psi_{L,n} \rightarrow \frac{2}{1 - \omega^* e^{2i\mu}} (\psi_R \psi_L)(x) \quad (5.16)$$

where $(\psi_A \psi_{A'}) (x)$ is defined as the symmetric sum:

$$(\psi_A \psi_{A'}) (x) \equiv \frac{1}{2} \{ \psi_A(x+0) \psi_{A'}(x-0) + \psi_A(x-0) \psi_{A'}(x+0) \} \quad (5.17)$$

which is the unique choice compatible with the antisymmetry requirement

$$(\psi_A \psi_A)(x) = -(\psi_A \psi_A)(x).$$

Once (5.16) is inserted into Eq. (5.11) and the equations analogous to it involving H_L and $\psi_{R,L}$, we finally find, taking (5.12) into account:

$$[H_R, \psi_R(x)] = i\partial_x \psi_R(x) - \frac{\omega}{2} \psi_L(x) - g(\psi_L^\dagger \psi_L \psi_R)(x) \quad (5.18a)$$

$$[H_R, \psi_L(x)] = -\frac{\omega}{2} \psi_R(x) - g(\psi_R^\dagger \psi_R \psi_L)(x) \quad (5.18b)$$

and

$$[H_L, \psi_R(x)] = -\frac{\omega}{2} \psi_L(x) - g(\psi_L^\dagger \psi_L \psi_R)(x) \quad (5.19a)$$

$$[H_L, \psi_L(x)] = -i\partial_x \psi_L(x) - \frac{\omega}{2} \psi_R(x) - g(\psi_R^\dagger \psi_R \psi_L)(x) \quad (5.19b)$$

where

$$g = 2i \frac{\omega + e^{2i\mu}}{\omega - e^{2i\mu}} = -2\cot(\mu - \mu_0) \quad ; \quad \omega \equiv e^{2i\mu_0} \quad (5.20)$$

and, e.g.,

$$(\psi_R^\dagger \psi_R \psi_L)(x) \equiv \psi_R^\dagger(x) (\psi_R \psi_L)(x).$$

$(\psi_R \psi_L)(x)$ is defined in (5.17) and the product in the right-hand side of this last expression is well defined in the "bare" continuum limit.

From the equation of motion in the continuum (5.18) and (5.19), we can now recognize in $H = H_R + H_L$ the Hamiltonian of the massive Thirring model with bare mass m and coupling constant g :

$$\begin{aligned} H &= \int_0^L dx \left[-i(\psi_R^\dagger \partial_x \psi_R - \psi_L^\dagger \partial_x \psi_L) + \omega(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R) + 2g \psi_R^\dagger \psi_R \psi_L^\dagger \psi_L \right] \\ &= \int_0^L dx \left[-i \psi^\dagger (\gamma^5 \partial_x + i m \gamma^0) \psi + \frac{1}{2} g J_\mu J^\mu \right] \end{aligned} \quad (5.21)$$

where $\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \quad ; \quad J_\mu \equiv \bar{\psi} \gamma_\mu \psi$

and

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} ; \quad \gamma^r = \gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This completes the formal identification of the continuum field theory describing the scaling dynamics on our diagonal lattice. Of course, the possibility of exactly solving the theory on the lattice offers the exciting possibility of retaining the full control, at field theory level (that is off-shell as well as on-shell), over the "renormalized" continuum limit taken in the physical sector. At the spectrum level, such a limit can be taken using the results of Refs. 5)-7) obtained via a Bethe ansatz formulated in first quantized language. We will make connection with this formalism in the next section, hoping to return to the problem of the "renormalized" field-theoretic continuum limit in the future.

For the moment, we close this section with the following observations.

The MTM has zero β -function, that is, the coupling constant g is not (perturbatively) renormalized; hence the four-field product in the Hamiltonian (5.21) can be rendered finite, apart from multiplicative, scheme-dependent wave function renormalization factors, by simply normal-ordering the $U(1)$ current J_μ (that is the bilinears $\phi_R^\dagger \phi_R$, $\phi_L^\dagger \phi_L$) in the interaction representation. The corresponding subtraction does not depend on the mass m , as far as the leading singularity is concerned, and is therefore unaffected by mass renormalization. With the lattice cut-off, this leading singularity is of order a^{-1} . The precise proportionality factor can be determined simply by looking at massless free fields. Quite naturally, it turns out that the combination $\phi_A^\dagger \phi_A - 1/2a$ is free of leading a^{-1} divergences. This corresponds on the lattice to the "normal-ordering"

$$\psi_n^\dagger \psi_n \rightarrow \frac{1}{2} [\psi_n^\dagger, \psi_n] = \psi_n^\dagger \psi_n - \frac{1}{2} \quad (5.22)$$

This seems the first thing to do when changing the reference state from the state killed by ψ_n to a state with the sea of negative energy levels filled up. Let us then "normal order" in this sense the right-hand side of the lattice field equation (4.7). The first of them will read, for instance:

$$\begin{aligned} U_R \psi_{2n-2} U_R^\dagger &= \\ &= \frac{1}{2} \left(b^* - \frac{b}{\omega} \right) \psi_{2n} + \frac{1}{2} \left(c^* + \frac{c}{\omega} \right) \psi_{2n-1} - \frac{1}{2} \left(b^* + \frac{b}{\omega} \right) [\psi_n^\dagger, \psi]_{2n-1} \psi_{2n} - \frac{1}{2} \left(c^* - \frac{c}{\omega} \right) [\psi_n^\dagger, \psi]_{2n} \psi_{2n-1} \end{aligned}$$

$$\begin{aligned} \approx_{a \rightarrow 0} & (-i\bar{e}^{i\mu_0} \sin \mu) \left\{ \psi_{2n} + \frac{i}{2} m a \psi_{2n-1} + \right. \\ & \left. + \frac{i}{2} g ([\psi^\dagger, \psi]_{2n-1} \psi_{2n} + \frac{i}{2} m a [\psi^\dagger, \psi]_{2n} \psi_{2n-1}) \right\} \end{aligned} \quad (5.23)$$

It is now very tempting to write

$$U_R \approx_{a \rightarrow 0} (-i\bar{e}^{i\mu_0} \sin \mu)^{:Q:} \left[1 - i a :H_R: + O(a^2) \right] \quad (5.24)$$

where $:Q:$ and $:H_R:$ are respectively $U(1)$ charge and right light-cone component of the Hamiltonian normal ordered (at fixed time) in the continuum. By fixed time normal-ordering we mean normal-ordering in the interaction representation, where fields can still be split in positive and negative frequency parts just like free fields. Alternatively, such normal-ordering can be written, say for $\psi^\dagger \psi$, in the equivalent way $:\psi^\dagger(x)\psi(x): = \frac{1}{2}[\psi^\dagger(x+0)\psi(x) - \psi(x)\psi^\dagger(x-0)]$ which is just the continuum analogue of the lattice equation (5.22). A comparison of (5.23) and (5.24) shows that indeed $:H:$ is the normal ordered form of the H given in (5.21), provided we take the naïve limit (5.1) on the lattice fields ϕ_n , simply setting $\phi_{2n-1}\phi_{2n} \sim a\phi_L(x)\phi_R(x)$, rather than the discontinuous limit characterized by (5.13) and (5.16) [Eq. (5.15) now puts no real constraint on $\phi_{2n-1}\phi_{2n}$, since the change of reference state implied by (5.22) now makes $H_{R,L}$ divergent as $a \rightarrow 0$]. In other words, the fields seem somewhat "smeared" by the filling of (approximate) negative energy levels. The above argument is not completely rigorous. We have completely ignored mass and wave-function renormalization, as well as all problems connected with the normal-ordering of the non-local, and in the continuum limit non-polynomial operators $U_{R,L}$. The price paid is that, in order to reproduce the limiting form of Eq. (5.23) in the simplest possible way, we had to give up (at least for $\mu \neq \pi/2$) the manifest unitarity of $U_{R,L}$. The non-unitary prefactor in Eq. (5.24), however, does not concern the Hamiltonian $:H_R:$ which is Hermitian. At any rate, Eqs. (5.23) and (5.24) indicate that the MTM emerges in the continuum limit also when the inherently infinite bare particle structure of relativistic theories is (approximately) taken into account.

6. - BETHE ANSATZ

We construct in this section common eigenstates of the light-cone transfer matrices U_R and U_L introduced in Section 2. We connect in this way our second-quantized formalism with the first-quantized approach of Refs. 5)-7).

Let us start with one particle states. We can write, following Eq. (3.6):

$$|F_1\rangle = \sum_{n=1}^{2N} F(n) \psi_n^+ |0\rangle \quad (6.1)$$

where $|0\rangle \equiv |0\dots 0\rangle$ is the state empty of particles, $\psi_n|0\rangle = 0$. Acting with U_R on $|F_1\rangle$ and using the lattice field equations (4.7) yields the first-quantized form u_R of the operator U_R :

$$U_R |F_1\rangle = \sum_{n=1}^{2N} (u_R F)(n) \psi_n^+ |0\rangle \quad (6.2)$$

$$\begin{cases} (u_R F)(2n+1) = b F(2n+1) + c F(2n) \\ (u_R F)(2n+2) = b F(2n) + c F(2n+1) \end{cases} \quad (6.3)$$

with the periodic boundary conditions

$$F(n+2N) = F(n) \quad ; \quad n \in \mathbb{Z}$$

The eigenvalue equation $U_R |F_1\rangle = \Lambda_R |F_1\rangle$ is solved by the setting:

$$F(2n-1) = F(1) e^{ik(n-1)} \quad ; \quad F(2n) = F(2) e^{ik(n-1)} \quad (6.4)$$

This yields two solutions

$$\Lambda_R^{(\varepsilon)} = \Lambda(k, \varepsilon) e^{-ik/2} \quad ; \quad \Lambda(k, \varepsilon) = \frac{b}{|b|} e^{-\frac{i}{2}\varepsilon e(k)} \quad ; \quad \varepsilon = \pm 1 \quad (6.5)$$

with the energy-momentum dispersion relation

$$e(k) = 2 \arccos \left(\frac{\cos k/2}{\sqrt{1-c^2/b^2}} \right) \quad (6.6)$$

which has angular points at the Brillouin boundaries $k = \pm\pi$.

$|F_1\rangle$ given by (6.1), (6.4) is also an eigenstate of U_L with eigenvalue

$$\Lambda_L^{(\epsilon)} = \Lambda(k, \epsilon) e^{ik/2} \quad (6.7)$$

A relativistic spectrum follows in the continuum limit discussed in the previous section [see Eq. (5.12)]. One finds

$$\Lambda_{R,L}^{(\epsilon)} \sim e^{i\mu} \left[1 - \frac{i\partial}{2} (\epsilon \sqrt{p^2 + m^2} \pm p) + O(\partial^2) \right] \quad (6.8)$$

where $p = k/a$ and $\pm \sqrt{p^2 + m^2}$ are naturally identified with the momentum and the energy on the continuum following Eqs. (5.9) and (5.10). So, the one particle eigenstates (6.1), (6.4) provide relativistic massive particle states in the scaling limit.

Two-particle states are written

$$|F_2\rangle = \sum_{n_1, n_2} F(n_1, n_2) \psi_{n_1}^\dagger \psi_{n_2}^\dagger |0\rangle \quad (6.9)$$

where

$$F(n_1, n_2) = -F(n_2, n_1) = F(n_1, n_2 + 2N)$$

Requiring $|F_2\rangle$ to be an eigenvector of U_R yields

$$(u_R^{(12)} F)(n_1, n_2) = \Lambda_R^{(12)} F(n_1, n_2) \quad (6.10)$$

where the two-body operator $u_R^{(12)}$ reads, using the field Eqs. (4.7)

$$\begin{aligned} (u_R^{(12)} F)(n_1, n_2) = & (u_R^{(1)} u_R^{(2)} F)(n_1, n_2) + \\ & + 2b\Delta \sum_{n=1}^N F(2n-2, 2n-1) [\delta_{n_1, 2n-1} \delta_{n_2, 2n} - (n_1 \neq n_2)] \end{aligned} \quad (6.11)$$

and

$$\Delta = \frac{\omega + b^2 - c^2}{2b} \quad (6.12)$$

Equation (6.10) can be solved by a Bethe ansatz for $F(n_1, n_2)$:

$$F(n_1, n_2) = A_{12} F_1(n_1) F_2(n_2) + A_{21} F_2(n_1) F_1(n_2) ; n_1 < n_2 \quad (6.13)$$

where $F_1(n)$ and $F_2(n)$ are both eigenfunctions of the one-body operator u_R of Eq. (6.3), with momentum k_1 and k_2 respectively, and relative eigenvalues $\Lambda_R(k_1)$, $\Lambda_R(k_2)$ given as in (6.5). The ratio of the two amplitudes A_{12} and A_{21} is fixed by the extra condition on $F(n_1, n_2)$ implied by the interaction term in (6.11) whenever $(n_1, n_2) = (2n-1, n)$. It reads

$$\Lambda_R^{(12)} F(2n-1, 2n) = \omega F(2n-2, 2n-1) \quad (6.14)$$

where now

$$\Lambda_R^{(12)} = \Lambda_R(k_1) \Lambda_R(k_2) \quad (6.15)$$

Inserting Eq. (6.13) in (6.14) yields the solution [$\Lambda_j = \Lambda_R(k_j)$ for brevity]

$$\frac{A_{21}}{A_{12}} = - \frac{\Lambda_1 \Lambda_2 - 2\Delta \Lambda_1 + 1}{\Lambda_1 \Lambda_2 - 2\Delta \Lambda_2 + 1} \quad (6.16)$$

In the scaling limit, Eq. (6.8) with the identification (5.20) of the coupling constant g , one recovers the two-body phase shift for pseudoparticles of Ref. 8). These authors, working directly in the continuum, need to define the distribution product $\delta(x)\theta(x)$. Their choice, $\delta(x)\theta(x) \equiv \frac{1}{2}\delta(x)$ is exactly equivalent to our definition (5.17) of the field product $(\phi_R \phi_L)(x)$.

Multiparticle states are the obvious generalization of Eq. (6.9):

$$|F_M\rangle = \sum_{n_1, \dots, n_M} F(n_1, \dots, n_M) \psi_{n_1}^+ \dots \psi_{n_M}^+ |0\rangle \quad (6.17)$$

with $F(n_1, \dots, n_M)$ given by a Bethe ansatz. The Yang-Baxter relations satisfied by the R-matrix ensure the compatibility of the eigenvalue equations. In conclusion, we exhibited through Eqs. (6.1)-(6.17) the connection of our second-quantized formalism based on the lattice field equations (4.7) with the first-quantized equations for amplitudes derived in Refs. 5)-7). We do not need to repeat here the construction of the renormalized vacuum and excited states done in Ref. 7). In the scaling limit, their result for the spectrum agree with the x-space, unregularized approach of Refs. 8) and 10).

For our purposes, it is more interesting to sketch how the eigenstates of U_R and U_L can be built by means of an algebrized procedure¹⁾. This is a very interesting possibility, since it opens the way to the calculation of exact correlation functions following the general procedure of Ref. 9). Let us then associate an auxiliary two-dimensional space to the $t = t_0$ horizontal line which cuts alternatively into L and R links (see Fig. 1). At each R(L) intersection local operators $L_n^{(+)}(L_n^{(-)})$ act. $L_n^{(\pm)}$ is called a local vertex. It is a two-by-two matrix whose entries are operators in the two-dimensional Hilbert space attached to the link $2n-1$ or $2n$. Thus $L_n^{(\pm)}$ can be regarded as a 4×4 numerical matrix. Its structure is fixed by factorizability, that is by the requirement that two neighbouring local vertices $L_{2n-1}^{(-)}, L_{2n}^{(+)}$ and the (quantum) R-matrix acting at size n , $R_{2n-1, 2n}$, satisfy the star-triangle relations (here $n = 1$ for brevity)

$$(L_1^{(-)})_{\beta\beta_1}^{\alpha\alpha_1} (L_2^{(+)})_{\alpha'\beta_2}^{\beta\alpha_2} (R_{12})_{\alpha'_1\alpha'_2}^{\beta_1\beta_2} = (R_{12})_{\beta_1\beta_2}^{\alpha_1\alpha_2} (L_1^{(+)})_{\beta\alpha'_1}^{\alpha\beta_1} (L_2^{(-)})_{\alpha'_2\alpha_2}^{\beta\beta_2} \quad (6.18)$$

which have the graphical interpretation of "pulling a line through a vertex", see Fig. 6. Since R itself satisfies factorizability, the local vertex $L_n^{(\pm)}$ is "built like R". At this stage it is convenient to introduce the so-called uniform parametrization for the microscopic amplitudes b , c and ω entering the R-matrix (2.3). This reads

$$b = b(\theta) = \frac{\text{sh} \theta}{\text{sh}(\theta - i\mu)} ; \quad c = c(\theta) = \frac{\sin \mu}{i \text{sh}(\theta - i\mu)} ; \quad \begin{matrix} 0 < \theta < \infty \\ 0 < \mu < \pi \end{matrix} \quad (6.19)$$

and without loss of generality for the MTM, $\omega = 1$ [cf. Eq. (5.20)]. From the statistical mechanics point of view, we are therefore dealing with the symmetric six-vertex model, although with complex statistical weights. Notice that the parametrization (6.19) is consistent with the scaling limit (5.12) provided $\theta \rightarrow \infty$ as

$$\theta \sim \ln \left(\frac{4 \sin \mu}{m a} \right) = \ln a^{-1} + \text{const.}$$

as $a \rightarrow 0$. In the "renormalized" continuum limit, one should keep fixed the physical mass m_R , rather than the bare mass m as $a \rightarrow 0$. From the exact

solution⁷⁾, one finds $m_R \sim a^{-1} \exp(-\pi/2 \theta/(\pi-\mu))$.

In terms of the uniform parametrization (6.19), it is natural to set for $L_n^{(\pm)}$, as 4×4 numerical matrices

$$(L_n^{(\pm)})_{\beta\beta_n}^{\alpha\alpha_n} = R(\theta_0 \pm \frac{1}{2}\theta)_{\beta_n\beta}^{\alpha\alpha_n} \quad (6.20)$$

where $R(\theta)$ means the R-matrix (2.3) with $\omega = 1$ and b, c parameters given by Eq. (6.19). θ_0 is the spectral parameter associated to the $t = t_0$ horizontal line. As 2×2 matrix with operator entries, $L_n^{(\pm)}$ reads:

$$L_n^{(\pm)} \equiv L_n^{(\pm)}(\theta_0) = \begin{pmatrix} b + (1-b)\psi_n^+\psi_n & c\sigma_n^- \\ c\sigma_n^+ & 1 - (1-b)\psi_n^+\psi_n \end{pmatrix} \quad (6.21)$$

where $b = b(\theta_0 \pm \frac{1}{2}\theta)$, $c = c(\theta_0 \pm \frac{1}{2}\theta)$ and $\sigma_n^\pm$ is a spin-operator related to ψ_n by the Jordan-Wigner transformation (3.12). To make $L_n^{(\pm)}$ fully local in terms of fermion operators, one can perform a local (quantum) gauge transformation¹⁾

$$L_n \rightarrow W_n^+ L_n W_{n+1} \equiv \tilde{L}_n \quad (6.22)$$

with

$$W_n = \begin{pmatrix} w_n & 0 \\ 0 & 1 \end{pmatrix} ; \quad w_n = \prod_{m=1}^{n-1} \sigma_m^z = \prod_{m=1}^{n-1} (1 - 2\psi_m^+\psi_m)$$

Then

$$\tilde{L}_n^{(\pm)} = \begin{pmatrix} b - (1+b)\psi_n^+\psi_n & c\psi_n^+ \\ -c\psi_n & 1 - (1-b)\psi_n^+\psi_n \end{pmatrix} \quad (6.23)$$

We can now rewrite the Yang-Baxter algebra (6.17) in purely operatorial terms:

$$R_{n,n+1} L_n^{(+)} L_{n+1}^{(-)} = L_n^{(-)} L_{n+1}^{(+)} R_{n,n+1} \quad (6.24)$$

where $R_{n,m}$ is given by Eq. (3.4). Rather than a c-number R-matrix acting on the tensor product of two auxiliary spaces, we have here a quantum R-matrix which is trivial in auxiliary space. A similar point of view was taken, in a different context, in Ref. 11). Under the gauge transformation (6.22) the central relation (6.24) holds unchanged, with L_n simply replaced by $\tilde{L}_n$, since the quantum R-matrix has the appropriate invariance

$$W_n^\dagger R_{n,n+1} W_n = W_{n+2}^\dagger R_{n,n+1} W_{n+2} = R_{n,n+1} \quad (6.25)$$

Following the general QISM procedure¹⁾, one defines the monodromy matrix

$$\tau^{(out)}(\theta_0; \theta) = L_1^{(-)} L_2^{(+)} L_3^{(-)} \dots L_{2N-1}^{(-)} L_{2N}^{(+)} \quad (6.26)$$

which is depicted in Fig. 7. Figure 8 shows instead the "in" monodromy matrix:

$$\tau^{(in)}(\theta_0; \theta) = L_1^{(+)} L_2^{(-)} L_3^{(+)} \dots L_{2N-1}^{(+)} L_{2N}^{(-)} \quad (6.27)$$

which corresponds to the horizontal line "pulled down through the row of vertices". The basic star-triangle relation (6.24) allows one to relate the "in" and "out" monodromy matrices [recall Eqs. (2.7) and (3.10)]:

$$U_R (V^\dagger \tau^{(in)} V) = \tau^{(out)} U_R \quad (6.28a)$$

$$U_L (V \tau^{(in)} V^\dagger) = \tau^{(out)} U_L \quad (6.28b)$$

If we write the 2×2 matrices $\tau^{(in)}$, $\tau^{(out)}$ in the conventional way:

$$\tau = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Then we find

$$A^{(out)} + D^{(out)} = V^\dagger (A^{(in)} + D^{(in)}) V \equiv T(\theta_0) \quad (6.29)$$

which is the so-called row-to-row transfer matrix. Hence Eqs. (6.28) and $[T, V^2] = 0$ give:

$$[U_R, T(\theta_0)] = 0 = [U_L, T(\theta_0)] \quad (6.30)$$

and it appears now evident that the common eigenvectors of U_R , U_L and T can be constructed by means of the algebrized Bethe ansatz:

$$|\theta_1, \dots, \theta_M\rangle = V^\dagger B^{(in)}(\theta_1) \dots B^{(in)}(\theta_M) |0\rangle \quad (6.31)$$

Indeed, whenever the parameters $\theta_1, \dots, \theta_M$ satisfy the Bethe ansatz equations which make (6.31) an eigenstate of T , one explicitly verifies that^{5),6)}:

$$B^{(out)}(\theta_1) \dots B^{(out)}(\theta_M) |0\rangle = \Lambda_R(k_1) \dots \Lambda_R(k_M) |\theta_1, \dots, \theta_M\rangle \quad (6.32)$$

where $\Lambda_R(k_j)$ are one-particle eigenvalues. The momenta k_j and "rapidity" θ_j are related by means of the uniform parametrization of $\Lambda_R(k)$

$$\Lambda_R(k(\theta_0)) = \frac{\text{sh} \left[\frac{1}{2} (\theta_0 + \theta + i\mu) \right]}{\text{sh} \left[\frac{1}{2} (\theta_0 + \theta - i\mu) \right]} \quad (6.35)$$

Thanks to Eqs. (6.28), (6.32) and

$$V^2 |\theta_1, \dots, \theta_M\rangle = \left(\prod_{j=1}^M e^{-ik_j} \right) |\theta_1, \dots, \theta_M\rangle \quad (6.34)$$

(we recall that V^2 is the unit space translation), one obtains

$$\begin{aligned} U_R |\theta_1, \dots, \theta_M\rangle &= U_R V^\dagger B^{(in)}(\theta_1) \dots B^{(in)}(\theta_M) |0\rangle \\ &= B^{(out)}(\theta_1) \dots B^{(out)}(\theta_M) |0\rangle \\ &= \Lambda_R(k_1) \dots \Lambda_R(k_M) |\theta_1, \dots, \theta_M\rangle \end{aligned}$$

and [cf. Eq. (6.7)]

$$\begin{aligned} U_L |\theta_1, \dots, \theta_M\rangle &= U_L V V^{-2} B^{(in)}(\theta_1) \dots B^{(in)}(\theta_M) |0\rangle \\ &= \left(\prod_{j=1}^M e^{ik_j} \right) B^{(out)}(\theta_1) \dots B^{(out)}(\theta_M) |0\rangle \\ &= \Lambda_L(k_1) \dots \Lambda_L(k_M) |\theta_1, \dots, \theta_M\rangle \end{aligned}$$

which verifies the algebrized Bethe ansatz for U_R , U_L .

CONCLUSIONS

The light-cone lattice approach presented here for the massive Thirring model can be generalized for fermion models having internal quantum numbers as well as Z_4 fermions associated to eight-vertex models. For the first generalization, multistate integrable vertex models are needed¹⁾.

It must be noticed that the spectral parameter θ plays a different role here and in the usual row-to-row approach¹⁾. One obtains spin Hamiltonians and then by Jordan-Wigner transformation fermion Hamiltonians from row-to-row transfer matrices by expanding them around $\theta = 0$. In such an approach θ is therefore vanishing with vanishing lattice spacing in the time direction. In the present light-cone lattice approach space and time lattice constants are necessarily equal and vanishing simultaneously in the continuum limit. This leads to $\theta \rightarrow \infty$ instead of $\theta \rightarrow 0$. It must be noticed that one can describe the massive Thirring model (MTM) in the light-cone lattice approach starting from the six-vertex model whereas one needs to consider the eight-vertex model in the row-to-row approach¹²⁾. That is, the two parameters m and g of the MTM arise from the two parameters θ and μ (here $\omega = 1$) contained in U . In the row-to-row approach one essentially loses the parameter θ and hence one needs a vertex model with two parameters besides θ (γ and the elliptic modulus)¹²⁾.

In the light-cone lattice approach transfer matrices U_R and U_L do not commute for different values of θ . They only provide energy, momentum and the $U(1)$ charge for the MTM. Higher conserved non-local charges follow from the row-to-row transfer matrix $T(\theta_0)$ [e.g., (6.29)]. Since $T(\theta_0)$ commutes with U_R , U_L and $T(\theta_0')$ for any value of θ_0 and θ_0' , one gets an infinite set of commuting operators by expanding $T(\theta_0)$ around a given value of θ_0 . So, contrary to the usual situation¹⁾, the momentum and the Hamiltonian do not arise from the same generating functional as the higher conserved charges.

REFERENCES

- 1) For reviews, see:
 - a) L.D. Faddeev, Soc. Sci. Rev. C1 (1980) 107 and in Recent Advances in Field Theory and Statistical Mechanics, Proc. Les Houches School, 1982, (North Holland, Amsterdam, 1984), p. 561.
 - b) L.D. Faddeev and L.A. Takhtadzhyan, Russ. Mth. Surveys 34 (1979) 11;
 - c) P.P. Kulish and E.K. Sklyanin, in Springer Lecture Notes in Physics, 151 (1981) 61.
 - d) H.B. Thacker, Rev. Mod. Phys. 53 (1981) 243.
 - e) R.J. Baxter, "Exactly solved models in statistical mechanics", (Academic Press, New York, 1982).
 - f) H.J. de Vega, "Integrable QFT and statistical models", LPTHE preprint 85-54 (1985).
- 2) D.J. Gross and A. Neveu, Phys. Rev. D10 (1974) 3235;
H.J. de Vega, H. Eichenherr and J.M. Maillet, Phys. Lett. B132 (1983) 337;
N. Andrei and C. Destri, Nucl. Phys. B231 (1984) 445.
- 3) N. Andrei and J.H. Lowenstein, Phys. Rev. Lett. 43 (1979) 1698 and Phys. Lett. 90B (1980) 106;
A.A. Belavin, Phys. Lett. 87B (1979) 117;
N.Y. Reshetikhin, Nucl. Phys. B259 (1985) 565.
- 4) H.N.V. Temperley and E.H. Lieb, Proc. Roy. Soc. A322 (1971) 251;
R.Z. Bariev, Theor. Math. Phys. 49 (1981) 1021 and 58 (1984) 207.
- 5) T.T. Truong and M.D. Schotte, Nucl. Phys. B220 (1983) 77 and B230 (1984) 1.
- 6) T.T. Truong, in Non-Linear Equations in Classical and QFT, ed. N. Sanchez (Springer Lectures in Physics), vol. 226 (1985).
- 7) M.F. Weiss and M.D. Schotte, Nucl. Phys. B225 (1983) 247.
- 8) H. Bergknoff and H.B. Thacker, Phys. Rev. D19 (1979) 3666.
- 9) A.G. Izergin and V.E. Korepin, Comm. Math. Phys. 94 (1984) 67 and 93 and 99 (1985) 271.
- 10) V.E. Korepin, Theor. Math. Phys. 41 (1979) 953.
- 11) V.O. Tarasov, L.A. Takhtadzhyan and L.D. Faddeev, Theor. Math. 57 (1984) 1059.
- 12) M. Lüscher, Nucl. Phys. B117 (1976) 475.

Fig. 1

$| \equiv | 0 > \equiv | \text{empty} > ; \quad \uparrow \equiv | 1 > \equiv | \text{occupied} >$

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8