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PION STRUCTURE FUNCTIONS FROM LATTICE QCD

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A B S T R A C T

We compute $\langle x \rangle$ and $\langle x^2 \rangle$ of the valence quarks in a pion, on the lattice using Wilson fermions in the quenched approximation. We find $\langle x \rangle = 0.49 \pm 0.08$ and $\langle x^2 \rangle = 0.20 \pm 0.08$ at a scale $\mu \approx 7$ GeV, in good agreement with experimental data.

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Hadronic structure functions have played a central role in the development of our understanding of Quantum Chromodynamics (QCD), through the phenomenology of hard inclusive reactions. Perturbation theory allows us to predict only the scaling violations. Lattice QCD, on the other hand, provides us with the possibility of computing the structure functions themselves, directly from the QCD Lagrangian. With presently available computers the most practical approach is to proceed by calculating their moments.

Using the operator product expansion and the renormalization group we can express the moments of the structure functions in terms of matrix elements of local operators. For the non-singlet case, and keeping only the contribution from the lowest twist operators, the relation is

$$\int_0^1 dx x^{N-2} F_k^{NS}(x, q^2) = C_{k,N}^{NS}(q^2/\mu^2, g(\mu)) A_N(\mu) \quad (1)$$

where x is the Bjorken variable. In parton language x corresponds to the fraction of momentum carried by the struck partons inside the hadron. The label $k = 1, 2$ or 3 distinguishes between the three possible structure functions and μ is the renormalization scale. The $A_N(\mu)$'s are obtained from the matrix elements of the operators

$$O_N^{\mu_1 \dots \mu_N}(0) \equiv i^N \bar{\psi} \gamma^{\mu_1} \overleftrightarrow{D}^{\mu_2} \dots \overleftrightarrow{D}^{\mu_N} \psi - \text{traces} \quad (2)$$

(with suitable flavour indices and symmetrized over the Lorentz indices) using

$$\langle p | O_N^{\mu_1 \dots \mu_N}(0) | p \rangle \equiv p^{\mu_1} p^{\mu_2} \dots p^{\mu_N} A_N(\mu) \quad (3)$$

where $|p\rangle$ is a hadronic state with momentum p . The coefficient functions $C_{k,N}^{NS}$ are calculable in perturbation theory, provided that μ is sufficiently large (i.e. $\mu \gg \Lambda_{\text{QCD}}$) and contain the light cone dynamics. The "long-distance" non-perturbative physics is contained in the matrix elements (3). In this paper we present the results of a lattice calculation using Wilson fermions¹⁾ for the pion. We have computed the matrix elements for $N = 2$ and 3 non-singlet operators. The main results of our work are

$$\langle x \rangle_{\text{LATT}} \equiv \int_0^1 x V_\pi(x) dx = 0.64 \pm 0.10 \quad (4)$$

and

$$\langle x^2 \rangle_{\text{LATT}} \equiv \int_0^1 x^2 V_\pi(x) dx = 0.26 \pm 0.10 \quad (5)$$

at $\mu = 1/a \approx 2.2 \pm 0.5$ GeV in the lattice renormalization scheme (where a is the lattice spacing). $V_\pi(x)$ is the distribution function of the valence quarks in

the pion ($V_\pi = u_v + \bar{d}_v$). In standard continuum renormalization schemes, (such as $\bar{MS}$) for which the coefficient function ≈ 1 and evolving to $q \approx 7$ GeV we obtain

$$\langle x \rangle_{\text{CONT}} = 0.49 \pm 0.08 \quad (6)$$

$$\langle x^2 \rangle_{\text{CONT}} = 0.20 \pm 0.08 \quad (7)$$

The results of Eqs. (6) and (7) are in good agreement with the experimental results obtained from the Drell-Yan process with lepton pair masses in the continuum region between the J/ψ and the T :

$$\begin{aligned} \langle x \rangle_{\text{exp}} &= 0.4 \pm 0.1 && \text{Ref. 2)} \\ &= 0.35 \pm 0.05 && \text{Ref. 3)} \\ &= 0.4 - 0.5 && \text{Ref. 4)} \\ &= \sim 0.34 && \text{Ref. 5)*)} \end{aligned} \quad (8)$$

and $\langle x^2 \rangle_{\text{CONT}}$ agrees well with the experimental observation that the distribution function $V_\pi(x) \sim (1-x)^a$ as $x \rightarrow 1$ with the exponent a being close to 1. We stress that the results (6) and (7) are obtained from "first principles" with no free parameters or model assumptions. The results are extremely encouraging, especially bearing in mind the fact that certain systematic errors are currently unavoidable, in particular those due to the quenched approximation. In this approximation there is no sea contribution and we expect slightly larger values of $\langle x \rangle$ and $\langle x^2 \rangle$.

In principle of course it is possible to attempt a direct calculation of the complete structure function without using the operator product expansion. For such a calculation to be valid, however, the lattice spacing would have to be sufficiently small to resolve the almost light-like separations probed in deep inelastic scattering at a q^2 of a few GeV or more ($1/a \gg q$), and at the same time the lattice would have to be sufficiently large to contain the hadron being studied. Such a calculation is not practicable at present. Fortunately however, we can still obtain a very significant amount of information about the structure functions by computing the matrix elements of the operators which appear in the operator product expansion.

*) Here we have used the distribution $xV_\pi(x) = Ax^\alpha(1-x)^\beta$ with the parameter $\alpha = 0.44$, $\beta = 1.14$ and fixed A by imposing $\int xV_\pi(x)dx = 2$.

The construction of renormalized operators with finite matrix elements from bare lattice operators is a subtle subject⁶⁾. Because Euclidean Invariance (and also chiral symmetry in the case of Wilson fermions) is not a symmetry of the lattice, the operators O_N mix under renormalization with other operators of equal or lower dimension. In the latter case the mixing coefficients diverge as inverse powers of the lattice spacing a . To be specific we consider the transformation properties of the two operators whose matrix elements we wish to compute, $O_2^{\mu\nu}(o)$ and $O_3^{\mu\nu\rho}(o)$, under the hypercubic group. For the case $\mu = \nu$, the traceless operator $O_2^{\mu\mu}$ transforms like one of the three-dimensional irreducible representations of the hypercubic group [in the notation of Ref. 7)] and there is no mixing with operators of lower dimension. In fact, it is precisely the subtraction of the trace, implicit in the definition of $O_2^{\mu\mu}$ which removes the hypercubic group singlet component, and hence is a subtraction of power divergent terms. Such subtractions have in previous calculations led to significant numerical cancellations and hence to large statistical errors⁸⁾. It is possible to avoid such a subtraction by choosing $\mu \neq \nu$. The six operators $O^{\mu\nu}$ with $\mu \neq \nu$ transform like the six-dimensional irreducible representation of the hypercubic group. Again there is no mixing with lower dimensional operators, but, in contrast to the case $\mu = \nu$, there is no numerical subtraction of divergent terms in the calculation itself. Note, however, that in this case (i.e. $\mu \neq \nu$), in order to have a non-vanishing matrix element, the pion must be given a non-zero component of momentum in at least one spatial direction [see Eq. (3)]. We now turn to $O_3^{\mu\nu\rho}(o)$. The transformation properties of these operators under the hypercubic group have been discussed in detail in Refs. 8), and 9). The non-perturbative subtraction of power divergences for the case $\mu = \nu = \rho$ is not practicable. This leaves us with two possibilities, $\mu = \nu \neq \rho$ which again requires an explicit subtraction of divergent terms and $\mu \neq \nu \neq \rho \neq \mu$ in which case there is no such subtraction but the pion must be given a non-zero momentum in at least two spatial directions.

We now outline the procedure we use to compute the matrix elements (3) on the lattice. Consider the correlation function

$$C(t_x, t_y) \equiv \int d^3x d^3y e^{i p \cdot x} \langle 0 | T \{ J_\pi^+(o) O_N(y) J_\pi(x) \} | 0 \rangle \quad (9)$$

where J_π is an interpolating operator for the pion (in our calculations we take $J_\pi = \bar{\psi} \gamma^5 \psi$) and t_x and t_y are the time coordinates of the four vectors x and y . Inserting complete sets of states between the operators in (9), taking $t_x > t_y > 0$ and using translational invariance, we have, after continuing to

Euclidean space,

$$C(t_x, t_y) = \frac{e^{-Et_x}}{4E^2} \left| \langle \pi | J_\pi(0) | 0 \rangle \right|^2 \langle \pi | O_N(0) | \pi \rangle \quad (10)$$

+ contributions from heavier states,

where $E = \sqrt{m_\pi^2 + p^2}$. By choosing the time intervals t_y and $t_x - t_y$ to be sufficiently large the contribution from the heavier states is exponentially suppressed. Thus from the calculation of $C(t_x, t_y)$, as well as of the pion propagator,

$$K(t_x) \equiv \int d^3x \langle 0 | T \{ J_\pi^+(0) J_\pi(x) \} | 0 \rangle e^{i p \cdot x} \\ \simeq \frac{e^{-Et_x}}{2E} \left| \langle \pi | J_\pi(0) | 0 \rangle \right|^2 \quad (11)$$

we are able to determine the matrix element $\langle \pi | O_N(0) | \pi \rangle$.

We have performed a numerical simulation on a $20 \times 10^2 \times 40$ lattice, using Wilson fermions in the quenched approximation, at a value of the coupling constant given by $\beta = 6/g_0^2 = 6.0$. We have generated 15 $SU(3)_c$ gauge configurations on a $10^3 \times 20$ lattice using the Metropolis algorithm (8 hits) and the Wilson action for the gauge fields. An initial thermalisation of 2000 sweeps from a cold start and a separation of 250 sweeps between configurations were allowed. The lattice was then duplicated in the time and in one space direction. The choice of the parameters of our lattice was dictated by practical considerations. Given that computer time and memory are limited, the optimal choice is to compute all the operator matrix elements between pions with the same momentum. We have chosen to have one longer spatial direction in order to work with a pion with a small, but non-zero, three-momentum $ap_1 = \pi/10 \approx 0.3^*$. In this way we are able to calculate matrix elements of $O_2^{\mu\nu}$ with $\mu \neq \nu$ and avoid the problem of large numerical cancellations. However, we cannot avoid such cancellations in the calculation of matrix elements of $O_3^{\mu\nu\rho}$. On our lattice we have computed the following matrix elements:

$$\langle \pi | O_2^{41} | \pi \rangle \quad (12a)$$

$$\langle \pi | O_2^{11} - O_2^{22} | \pi \rangle \quad (12b)$$

and

$$\langle \pi | O_3^{411} - \frac{1}{2} (O_3^{422} + O_3^{433}) | \pi \rangle \quad (12c)$$

*) Values of $ap \approx 0(1)$ lead to large systematic errors since the physical quantities become sensitive to the granularity of the lattice.

where the pion has momentum $a_p = (\pi/10, 0, 0)$. A priori we expect larger statistical errors in the computation of (12b) than in (12a), but we calculate both of these matrix elements as a check of consistency. The long time direction has the effect of killing spurious effects which are present on a finite periodic lattice, such as those due to the transition amplitude $\langle 0 | O_2^{\mu\nu} | \pi\pi \rangle$.

We have calculated the two and three point correlation functions, $K(t_x)$ and $C(t_x, t_y)$ for three different values of the quark mass, corresponding to the following three values of the Wilson parameter: $K = 0.1515, 0.1530$ and 0.1545 .

The calculation of two point correlation functions such as $K(t_x)$, requires the evaluation of the graph of Fig. 1a, where the quarks are propagating in a background gluon field configuration. The propagators $S(x,0)$, from the origin to an arbitrary lattice point, have been computed by solving the equation

$$\sum_y \Delta(x,y) S(y,0) = \delta_{x,0} \quad (13)$$

using the Gauss-Siedel algorithm. $\Delta(x,y)$ is the kernel of the fermionic action.

The calculation of $C(t_x, t_y)$ requires the evaluation of the diagram in Fig. 1b. The set of propagators $\{S(x,0)\}$ is not sufficient to evaluate this diagram, since it is not possible to arrange for all the propagators to start or end at the origin. For this reason we compute the set of "generalized" propagators, $\{S_1(y,0;t_x,p)\}^{10),11)}$ with the insertion of a pion source at t_x , given by

$$S_1(y,0;t_x,p) \equiv \sum_{\substack{x \\ t_x \text{ fixed}}} S(y,x) \gamma^5 S(x,0) e^{i p \cdot x} \quad (14)$$

where we have taken $\bar{\psi}(x)\gamma^5\psi(x)$ as the interpolating operator for the pion. The S_1 's are obtained by solving the equation analogous to (13),

$$\sum_z \Delta(y,z) S_1(z,0;t_x,p) = \gamma^5 S(y,0) \delta_{t_y,t_x} e^{i p \cdot y} \quad (15)$$

using the same numerical techniques as for the propagators $S(x,0)$. We have computed the S_1 's for $t_x = 15$ and for $a_p = (\pi/10, 0, 0)$. The value $t_x = 15$ is chosen so that it would be possible to have t_y and $t_x - t_y$ to be large (≥ 5) in Eq. (9), and thus to isolate the pion's contribution to the unitarity sums in the correlation functions. In addition with this choice of t_x we also have room to vary t_y and hence to check that $C(t_x, t_y)$ is indeed independent of t_y [cf. Eq. (10)]. In Fig. 2 we plot $C(t_x = 15, t_y)$ for the operator $O_2(y)$ for t_y in the

range 5 to 10 and for the three different values of the Wilson parameter. The errors were estimated by using the procedure described in Ref. 12) of dividing the 15 configurations for each value of K into 3 sets of five configurations and treating the results from each set as independent^{*)}. Figure 2 clearly confirms that $C(t_x = 15, t_y)$ is independent of t_y . Similar "plateaux" also exist for the other operators studied in this paper. Figure 2 also demonstrates the typical feature that the errors increase as we increase K (i.e. as we decrease the quark mass).

In Fig. 3 we plot the measured values of $C(t_x = 15, t_y)$ corresponding to the operators O_3^{411} and $O_3^{411} - \frac{1}{2}(O_3^{422} + O_3^{433})$. From the Figure we see that the subtraction of the terms in O_3^{411} which are proportional to inverse powers of the lattice spacing, reduces the result by an order of magnitude! A similar feature was observed in Ref. 8), where we calculated the matrix element $\langle 0 | O_3^{411} - O_3^{422} | \pi \rangle$. A welcome outcome is that the relative errors in the case where the power divergences have been subtracted, although significant, are not overwhelming [clearly there is a high degree of correlation between the calculations of the matrix elements of O_3^{411} and $\frac{1}{2}(O_3^{422} + O_3^{433})$] and indeed are somewhat smaller than was the case for the calculation in Ref. 8).

From the computed values of the correlation functions we obtain the operator matrix elements and hence the moments of the structure functions. We express our results in terms of $\langle x \rangle$ and $\langle x^2 \rangle$ where x is the fraction of the pion's momentum (in the infinite momentum frame) carried by the valence partons. These are obtained from

$$C(t_x, t_y) = |\langle \pi | J_\pi(0) | 0 \rangle|^2 \frac{\langle x \rangle_{\text{LATT}} p}{2E} e^{-15E} \quad (16)$$

for the operator O_3^{411}

and

$$C(t_x, t_y) = |\langle \pi | J_\pi(0) | 0 \rangle|^2 \frac{\langle x^2 \rangle_{\text{LATT}} p^2}{E} e^{-15E} \quad (17)$$

for the operator $O_3^{411} - \frac{1}{2}(O_3^{422} + O_3^{433})$,

where $p = \pi/10$ and $E = \sqrt{\frac{m^2 + p^2}{\pi}}$. Strictly speaking Eqs. (16) and (17) should be taken as definitions of $\langle x \rangle_{\text{LATT}}$ and $\langle x^2 \rangle_{\text{LATT}}$, their interpretation in terms of moments of quark distribution functions requires further corrections, to be discussed below. We prefer to present our results in the physically appealing parton language.

*) Note that the points at different values of t , being computed on the same configurations, are strongly correlated.

In the Table we present our results for $\langle x \rangle_{\text{LATT}}$ (using the operator O_2^{41}) and for $\langle x^2 \rangle_{\text{LATT}}$ [using the operator $O_3^{411} - \frac{1}{2}(O_3^{422} + O_3^{433})$] obtained from our measurements of the corresponding $C(t_x, t_y)$'s and the pion propagator. We average over the results obtained from $t_y = 6$ to 9. The three columns A, B and C correspond to three different ways of eliminating the factor $|\langle \pi | J_\pi | 0 \rangle|^2 e^{-15E}$ of Eqs. (16) and (17). Column A gives the values obtained by determining this factor from the best fit to the pion propagator $K(t_x)$ calculated at zero momentum and using the relation $E^2 = p^2 + m_\pi^2$. For column B we used the values of $|\langle \pi | J_\pi | 0 \rangle|^2$ and E obtained from the best fit to $K(t_x)$ at $\underline{p} = (\pi/10, 0, 0)$. Column C gives the values obtained by dividing directly by the measured value of the pion propagator at $t_x = 15$.

We have also calculated $\langle x \rangle_{\text{LATT}}$ using the operator $O_2^{11} - O_2^{22}$ and as expected the errors are larger than those in the Table. Again the smallest errors are obtained using method A and we find the values 0.70 ± 0.30 for $K = 0.1515$, 0.78 ± 0.38 for $K = 0.1530$ and 0.82 ± 0.42 for $K = 0.1545$. The results are clearly consistent with, but less accurate than those obtained using O_2^{41} .

From the Table we see that the dependence of our results on the pion mass, if present at all, is smaller than our statistical errors. The experimental data for the pion and kaon structure function support the idea that the dependence of the low moments on the pseudoscalar mass is small. For our calculation this effect may be enhanced by the fact that we work in the quenched approximation and cannot produce further $q\bar{q}$ pairs. Our best estimates for $\langle x \rangle_{\text{LATT}}$ and $\langle x^2 \rangle_{\text{LATT}}$, allowing for some uncertainty in the extrapolation to $m_\pi^2 \approx 0$, are given in Eqs. (4) and (5). The numbers in these equations correspond to operators renormalized at a scale $\mu = 1/a = (2.2 \pm .05)$ GeV. The lattice spacing was determined following the usual procedure of measuring the mass of the rho at the critical value of K (i.e., at the value of K for which $m_\pi = 0$).

The final step to be taken is to translate our results from the lattice renormalization scheme at $\mu = 2.2$ GeV to some standard continuum scheme at a scale of $\mu \approx 7$ GeV, a typical scale for which experimental data are available. This can be done using the relation

$$\langle x \rangle_{\text{CONT}} (\mu = 7 \text{ GeV}) = \left(\frac{\alpha_s(\mu = 7 \text{ GeV})}{\alpha_s(\mu = 2.2 \text{ GeV})} \right)^{\delta_1/2b_0} \cdot (1 + \alpha_s C_x) \langle x \rangle_{\text{LATT}} (\mu = 2.2 \text{ GeV}) \quad (18)$$

and a similar equation for $\langle x^2 \rangle$. C_x and C_x^2 have two contributions. The first is the $O(\alpha_s)$ contribution to the coefficient function of Eq. (1). This contribution

in standard renormalization schemes, such as the $\overline{\text{MS}}$ scheme, is negligible compared to our errors. The second, which gives the relation between the lattice and continuum operators has been computed for $O_2^{\mu\nu}$ in Ref. 13) and found also to be negligible^{*)}. For $O_3^{\mu\nu\rho}$, using the tadpole dominance of lattice perturbation theory^{9),13)} we estimate $\alpha_s C_{x2}$ to be 0.13. For the first factor in Eq. (18) we estimate, using $\Lambda_{\text{QCD}} = 0.2 \text{ GeV}$, 0.76 in the case of $\langle x \rangle$ and 0.67 in the case of $\langle x^2 \rangle$. Thus we arrive at the results in Eqs. (6) and (7).

We have also attempted to evaluate the lowest moment of the gluon's distribution function, or equivalently the matrix element of the operator $\text{Tr}(F_{\mu\nu} F_{\nu\rho})$. We found that the signal to noise ratio was so small, that with 15 configurations it was impossible to extract the signal. We expect a similar situation in the unquenched case if one tries to extract the momentum distribution of the sea.

In this letter we have presented our results for the first two moments of the quark distribution function in a pion [Eqs. (6) and (7)]. The results are very encouraging, they have reasonably small statistical errors and are in good agreement with experimental data. Clearly the techniques described above can be used to calculate other operator matrix elements, and the results presented here are the first of our research programme to study the structure of hadrons on a lattice. For example we are currently generating the propagators required for the calculation of moments of the up and down quark distribution functions in nucleons.

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*) We disagree on this point with Ref. 9).

	A	B	C	$m_{\pi} a$
$\langle x \rangle_{\text{LATT}}$ at $K = 0.1515$	0.66 ± 0.03	0.66 ± 0.10	0.64 ± 0.10	0.50 ± 0.03
$\langle x \rangle_{\text{LATT}}$ at $K = 0.1530$	0.64 ± 0.05	0.68 ± 0.14	0.64 ± 0.14	0.43 ± 0.03
$\langle x \rangle_{\text{LATT}}$ at $K = 0.1545$	0.57 ± 0.11	0.80 ± 0.26	0.74 ± 0.18	0.36 ± 0.04
$\langle x^2 \rangle_{\text{LATT}}$ at $K = 0.1515$	0.26 ± 0.06	0.25 ± 0.05	0.22 ± 0.04	0.50 ± 0.03
$\langle x^2 \rangle_{\text{LATT}}$ at $K = 0.1530$	0.26 ± 0.07	0.27 ± 0.06	0.26 ± 0.06	0.43 ± 0.03
$\langle x^2 \rangle_{\text{LATT}}$ at $K = 0.1545$	0.21 ± 0.06	0.30 ± 0.08	0.28 ± 0.08	0.36 ± 0.04

TABLE

Values of $\langle x \rangle_{\text{LATT}}$ and $\langle x^2 \rangle_{\text{LATT}}$ obtained from our computation. Columns A,B and C give the values obtained using the three procedures as described in the text. In the last column we also report the pion mass in lattice units.

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FIGURE CAPTIONS

- Fig. 1: Diagrams which have to be evaluated in the calculation of the correlation functions. The crosses x signify the insertion of the pion's interpolating operator J_π , and the closed circle o the insertion of the operator whose matrix element we wish to calculate.
- Fig. 2: The correlation function $C(t_x = 15, t_y)$ for the operator $O_2^{41}(y)$ for the three different values of the Wilson parameter and for t_y in the range 5 to 10.
- Fig. 3: The correlation function $C(t_x = 15, t_y)$ for the operators $O_3^{41}(y)$ and $O_3^{41}(y) - \frac{1}{2}[O_3^{42}(y) + O_3^{43}(y)]$ for $K = 0.1530$ and for t_y in the range 5 to 10.

$$K(t_x) = \sum_{\underline{x}} \quad \begin{array}{c} \text{Diagram: A circle with two vertices marked with 'x'. The left vertex is labeled '0' and the right vertex is labeled 'x'. Two arrows, both labeled 'S', form a loop between the vertices: one arrow points clockwise from '0' to 'x', and the other points counter-clockwise from 'x' to '0'.} \end{array} \quad e^{ip \cdot \underline{x}}$$

Fig. 1(a)

$$\begin{aligned} C(t_x, t_y) &= \sum_{\underline{x}, \underline{y}} \quad \begin{array}{c} \text{Diagram: A circle with two vertices marked with 'x'. The left vertex is labeled '0' and the right vertex is labeled 'x'. A third vertex marked with a dot and labeled 'y' is located at the top of the circle. Two arrows, both labeled 'S', form a loop between '0' and 'x': one arrow points clockwise from '0' to 'x' passing through 'y', and the other points counter-clockwise from 'x' to '0'.} \end{array} \quad e^{ip \cdot \underline{x}} \\ &= \sum_{\underline{y}} \quad \begin{array}{c} \text{Diagram: A circle with two vertices marked with 'x'. The left vertex is labeled '0' and the right vertex is labeled 'y'. Two arrows, both labeled 'S', form a loop between the vertices: one arrow points clockwise from '0' to 'y', and the other points counter-clockwise from 'y' to '0'.} \end{array} \end{aligned}$$

Fig. 1 (b)

Fig. 2

Fig. 3