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TWO-DIMENSIONAL GRAVITY WITH DYNAMICAL TORSION AND STRINGS

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A B S T R A C T

A two-dimensional theory of gravity with dynamical metric and torsion that was proposed in the context of string theory is considered. We analyse the equations of motion following from a Lagrangian quadratic in curvature and torsion, $L = R^2 + T^2 + \lambda$. The equations of motion are reduced to a system of equations for two scalar fields. In this particular case they yield the Liouville equation. Analysis of the Cauchy problem for the whole system shows that the dynamics is governed by one continuous degree of freedom and also by a set of discrete variables. We have found and analysed stationary solutions with non-trivial torsion. Modification of the string theory due to the addition of the gravity Lagrangian is discussed.

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1. Introduction

Two-dimensional theories of gravity¹⁾⁻⁷⁾ and supergravity⁸⁾⁻¹⁰⁾ have recently attracted growing interest. These theories provide a deeper insight into the theories of strings and superstrings because the latter can be considered as two-dimensional supergravity. At the same time, investigation of simple two-dimensional models leads to a better understanding of four-dimensional gravity and its quantization.

It is well known that the Einstein equations in two dimensions become trivial and one usually adopts the constant curvature equation $R = \text{const}$ as the dynamical equation for the metric. The purely geometric local action yielding the equation $R = \text{const}$ is unknown. Recently the local action containing an additional scalar field was proposed^{1),2),11)}. It was also found that the constant curvature equation arises in semi-classical quantum gravity^{3),5)}. The problem of introducing a dynamics for the metric from a purely geometric viewpoint arises here. In Ref. 12) we have proposed a new bosonic string model with dynamical geometry which solves this problem in the framework of Riemann-Cartan geometry when torsion is present. Torsion is a fundamental geometrical concept and it may be considered as a new independent dynamical field.

Dynamical torsion theories were discussed in four dimensions¹³⁾, [for a review see Refs. 14) and 15)], and are promising in the framework of Kaluza-Klein theories¹⁶⁾. The σ -models with torsion in the target space were considered in Ref. 17) in the context of string theory.

The string world sheet in the model with dynamical geometry¹²⁾ has an internal metric and torsion. A linear connection with torsion is as fundamental a geometrical notion as the metric and it is interesting to consider torsion in string theory and two-dimensional gravity. It was found that dynamical torsion adds new degrees of freedom to the string and considerably modifies the theory. In particular, instead of the Liouville equation which arises in the standard string theory^{18),19)} one already has a modification of the Liouville equation at the classical level. It turns out that the theory of strings with dynamical geometry has no critical dimension and seems to admit quantization in arbitrary dimensional space-time¹²⁾.

In the present paper we analyse the classical equations of motion of two-dimensional gravity with dynamical torsion, following from the most general Lagrangian $L = R^2 + T^2 + \lambda$ quadratic in curvature R and torsion T , and containing a cosmological constant λ . This Lagrangian yields second order equations of motion for the zweibein and Lorentz connection, which we will call the Cartan variables. In particular the equations have a solution of constant curvature $R = \text{const}$ and zero torsion $T = 0$ and thus incorporate two-dimensional gravity model discussed in Refs. 1)-7).

Note that in the Cartan variables, when torsion is considered as an independent dynamical variable, equations of motion following from an R^2 -Lagrangian are of the second order. When one considers R^2 -gravity without torsion, then one gets higher derivative equations of motion for the metric. The quantum properties of such theories were discussed in Ref. 20). It would be interesting to investigate the connection between dynamical torsion theories and R^2 -gravity in detail.

The outline of this paper is as follows. In Section 2 a brief review of Riemann-Cartan geometry is given and the Lagrangian is written. Here we also note a possible generalization of the theory incorporating Weyl or conformal symmetry. In Section 3 we fix the conformal gauge in the Lagrangian and show that in a special case the theory yields an integrable system. In Section 4 we show that a two-dimensional space of constant curvature is a special solution of the equations of motion and analyse the Cauchy problem in conformal gauge. In Section 5, stationary solutions are found and analysed. In Section 6 the large-scale limit resulting in the standard bosonic string action is given for a string with dynamical geometry. In Section 7 we discuss the equations of motion for the string. In conclusion, we outline an operator quantization of a free string with dynamical geometry.

2. The Lagrangian

We begin with a brief review of Riemann-Cartan geometry which provides the basis of our treatment of two-dimensional gravity.

Let us consider a two-dimensional manifold with local co-ordinates ζ^α ($\alpha=0,1$). For example, this may be a two-dimensional world sheet spanned by the evolution of a string. We suppose that the manifold is supplied with the Riemann-Cartan geometry^{21),14)}. That is, metric $g_{\alpha\beta}(\zeta)$ and the metrical

connection $\Gamma_{\alpha\beta}^{\gamma}(\zeta)$ are defined. The metricity condition for a connection can be written as setting the covariant derivative of the metric to zero,

$$\nabla_{\alpha} g_{\beta\gamma} = \partial_{\alpha} g_{\beta\gamma} - \Gamma_{\alpha\beta}^{\delta} g_{\delta\gamma} - \Gamma_{\alpha\gamma}^{\delta} g_{\beta\delta} = 0. \quad (1)$$

The general solution of this equation for $\Gamma_{\alpha\beta\gamma} = \Gamma_{\alpha\beta}^{\delta} g_{\delta\gamma}$ has the following form

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2} (\partial_{\alpha} g_{\beta\gamma} + \partial_{\beta} g_{\alpha\gamma} - \partial_{\gamma} g_{\alpha\beta}) + \frac{1}{2} (T_{\alpha\beta\gamma} + T_{\gamma\alpha\beta} - T_{\beta\gamma\alpha})$$

where $T_{\alpha\beta}^{\gamma} = -T_{\beta\alpha}^{\gamma} = \Gamma_{\alpha\beta}^{\gamma} - \Gamma_{\beta\alpha}^{\gamma}$ is the torsion tensor. The curvature tensor is expressed in terms of the metrical connection as usual

$$R_{\alpha\beta\gamma}^{\delta} = \partial_{\alpha} \Gamma_{\beta\gamma}^{\delta} - \Gamma_{\alpha\gamma}^{\epsilon} \Gamma_{\beta\epsilon}^{\delta} - (\alpha \leftrightarrow \beta).$$

The curvature with lowered indices, $R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma}^{\epsilon} g_{\epsilon\delta}$, is antisymmetric in the first and in the second pair of indices. Symmetry under permutation of the first pair of indices with the second pair is present only in two dimensions and is absent in the general case of arbitrary dimensional spaces. We emphasize that the metric and torsion are independent geometrical objects and can be defined in an arbitrary Riemann-Cartan geometry.

An equivalent realization of Riemann-Cartan geometry can be given in terms of the Cartan variables in terms of a zweibein and a Lorentz or spin connection. Let the metric $g_{\alpha\beta}$ have signature $(+,-)$. Define the zweibein field e_{α}^a ($a=0,1$) and the Lorentz connection $\omega_{\alpha}^{ab} = -\omega_{\alpha}^{ba}$ by the following equations

$$e_{\alpha}^a e_{\beta}^b \eta_{ab} = g_{\alpha\beta}, \quad (2a)$$

$$\nabla_{\alpha} e_{\beta}^a = \partial_{\alpha} e_{\beta}^a - \Gamma_{\alpha\beta}^{\gamma} e_{\gamma}^a - \omega_{\alpha}^{ab} e_{\beta}^b = 0 \quad (2b)$$

where $\eta_{ab} = \text{diag}(+,-)$ is the two-dimensional Minkowskian metric. Equation (2b) has the geometrical meaning of covariant constancy of the zweibein field.

The number of components of a Lorentz connection equals that of the torsion. The zweibein has one component more than the metric. This is due to the arbitrariness in the solution of Eq. (2a) associated with the group of local Lorentz rotations which is determined by one arbitrary function in two dimensions.

Curvature and torsion tensors have the following forms in terms of zweibein and Lorentz connections

$$\begin{aligned} R_{\alpha\beta}{}^{ab} &= \partial_\alpha \omega_\beta{}^{ab} - \omega_\alpha{}^{ac} \omega_\beta{}^c{}_b - (\alpha \leftrightarrow \beta), \\ T_{\alpha\beta}{}^a &= \partial_\alpha e_\beta{}^a - \omega_\alpha{}^{ab} e_\beta{}^b - (\alpha \leftrightarrow \beta). \end{aligned} \quad (3)$$

The transformation of Greek indices into Latin ones and vice versa is carried out by the zweibein.

We consider the following Lagrangian, which yields second order equations of motion for the zweibein and the Lorentz connection¹²⁾:

$$L_1 = e \left(-\frac{1}{4} \gamma R_{abcd}^2 - \frac{1}{4} \beta T_{abc}^2 - \lambda \right), \quad e = \det e_\alpha{}^a. \quad (4)$$

The uniqueness of (4) is a particular feature of two dimensions. There is only one independent invariant quadratic in curvature due to the equation

$$R_{abcd} = -\frac{1}{2} R \epsilon_{ab} \epsilon_{cd}$$

where $R = R_{ab}{}^{ab}$ is the scalar curvature and $\epsilon_{ab} = -\epsilon_{ba}$ ($\epsilon_{01} = 1$) is the antisymmetric tensor. Direct calculation shows that all invariants quadratic in torsion are proportional to each other and that there are no pseudoinvariants which lead to equations of motion of second order.

In two dimensions the Lorentz connection can be parametrized by a pseudovector field B_α ,

$$\omega_\alpha{}^{ab} = B_\alpha \epsilon^{ab}, \quad R_{\alpha\beta}{}^{ab} = F_{\alpha\beta} \epsilon^{ab}, \quad F_{\alpha\beta} = \partial_\alpha B_\beta - \partial_\beta B_\alpha.$$

This particular feature of two dimensions will be useful in the following analysis.

We conclude the present section with a brief discussion of a possible extension of the Riemann-Cartan geometry which incorporates the Weyl space-time²³⁾. This is important because it allows one to restore invariance under the conformal or Weyl transformation of the zweibein,

$$e'_\alpha{}^a = e^\sigma e_\alpha{}^a \quad (5)$$

with an arbitrary function $\sigma(\zeta)$, which plays a crucial role in the ordinary string theory.

In general affine space-time the metricity condition (1) is broken and is replaced by the following equation²⁴⁾

$$\nabla_\alpha g_{\beta\gamma} = \partial_\alpha g_{\beta\gamma} - \Gamma_{\alpha\beta\gamma} - \Gamma_{\alpha\gamma\beta} = -Q_{\alpha\beta\gamma}$$

where $\Gamma_{\alpha\beta\gamma}$ is a corresponding affine connection.

If $Q_{\alpha\beta\gamma}$ has the more specific form,

$$Q_{\alpha\beta\gamma} = 2 Q_\alpha g_{\beta\gamma}$$

then one gets a Riemann-Cartan-Weyl space-time. Here Q_α is a Weyl vector field. One can introduce new Cartan variables e_α^a and ω_α^{ab} where the connection ω_α^{ab} is no longer antisymmetric in the indices a and b . The connection ω_α^{ab} can be parametrized by a pseudovector field B_α and a Weyl vector Q_α

$$\omega_\alpha^{ab} = B_\alpha \varepsilon^{ab} + Q_\alpha \eta^{ab}.$$

Curvature and torsion are expressed by the formula (3) in terms of e_α^a and ω_α^{ab} . In terms of B_α and Q_α the curvature reads as follows:

$$R_{\alpha\beta}^{ab} = F_{\alpha\beta} \varepsilon^{ab} + G_{\alpha\beta} \eta^{ab}$$

where $G_{\alpha\beta} = \partial_\alpha Q_\beta - \partial_\beta Q_\alpha$ is the field strength tensor for the Weyl vector.

Let the conformal transformation (5) of a zweibein field be accomplished by the following transformation of the vector fields

$$B'_\alpha = B_\alpha, \quad Q'_\alpha = Q_\alpha + \partial_\alpha \sigma. \quad (6)$$

Then the curvature remains unchanged

$$R'_{\alpha\beta}{}^{ab} = R_{\alpha\beta}{}^{ab}.$$

and the torsion transforms as follows

$$T'_{\alpha\beta}{}^a = e^\sigma T_{\alpha\beta}{}^a.$$

Let us remember that the string Lagrangian can be written in the following form⁸⁾

$$L_S = \frac{1}{4\pi\alpha'} e g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\mu$$

where $X^\mu(\zeta)$ is a string position variable ($\mu = 0, 1, \dots, D-1$). This is invariant under the conformal transformation (5), with X^μ unchanged. This invariance plays an important role in string theory. Simple calculations show that the Lagrangian

$$L = -\frac{1}{4} \beta e T_{abc}^2 + \frac{1}{4\pi\alpha'} e g^{\alpha\beta} \partial_\alpha X_\mu \partial_\beta X^\mu \quad (7)$$

has all the invariances of string theory including the conformal symmetry and thus provides a new modification of the string Lagrangian. Conformal invariance is achieved due to the vector field Q_α , which is auxiliary when the curvature squared term is absent. At the classical level, one has algebraic equations of motion for B_α and Q_α which have a solution $T_{abc} = 0$, in which case the theory reduces to the standard string theory. On the other hand, the quantum theory defined by (7) can be different. We will discuss this question and also the string with a general affine geometry elsewhere.

3. Conformal gauge

We begin the analysis of the Lagrangian (4) in a conformally flat gauge. By means of the general co-ordinate and local Lorentz transformations, a zweibein can be transformed into the conformally flat gauge

$$e_\alpha^a = e^\varphi \delta_\alpha^a. \quad (8)$$

The torsion takes the form

$$T_{\alpha\beta}^a = e^\varphi (\partial_\alpha \varphi \delta_\beta^a - B_\alpha \varepsilon_\beta^a - (\alpha \leftrightarrow \beta))$$

and the Lagrangian (4) reduces to the following expression

$$L = \frac{1}{2} \gamma e^{-2\varphi} F_{\alpha\beta}^2 - \frac{1}{2} \beta (\partial_\alpha \varphi \partial^\alpha \varphi + F_{\alpha\beta} \varepsilon^{\alpha\beta} \varphi - B_\alpha^2) - \lambda e^{2\varphi} \quad (9)$$

where summation is carried out with the help of the Minkowskian metric. The Lagrangian (9) describes interacting scalar and pseudovector fields in two-dimensional Minkowskian space-time.

The conformal gauge (8) does not completely specify the choice of co-ordinates. The co-ordinate system is defined by (8) up to an arbitrary transition function satisfying the d'Alembert equation. To prove this, let us

fix the gauge (8) in two steps. Firstly, one chooses the conformal gauge for the metric,

$$g_{\alpha\beta} = e^{2\varphi} \eta_{\alpha\beta} \quad (10)$$

and then goes to (8) by means of the local Lorentz rotation. Let us make a change of co-ordinates $\zeta^\alpha \rightarrow \zeta'^\alpha(\zeta)$ which satisfy the following equations

$$\frac{\partial \zeta'^0}{\partial \zeta^0} = \frac{\partial \zeta'^1}{\partial \zeta^1}, \quad \frac{\partial \zeta'^1}{\partial \zeta^0} = \frac{\partial \zeta'^0}{\partial \zeta^1} \quad (11)$$

Then the metric (10) in the new co-ordinate system remains diagonal and $g_{00} = -g_{11}$. Clearly this transition to a new co-ordinate system reduces to changing the conformal multiplier e^σ . Equation (11) implies that

$$\square \zeta'^\alpha = 0, \quad \square = \partial_0^2 - \partial_1^2.$$

i.e., the transition functions satisfy the d'Alembert equation in the old co-ordinates. To preserve the form of the zweibein (8), this co-ordinate transformation should be accompanied by a local Lorentz rotation. The invariance which we still have after fixing the conformal gauge (8) is important when the Lagrangian (4) is added to the string Lagrangian because it allows one to fix the light cone gauge for the string variables.

Rescaling the fields $B_\alpha \rightarrow B_\alpha/\sqrt{-2\gamma}$, $\phi \rightarrow \phi/\sqrt{-\beta}$ the Lagrangian (5) reduces to the form

$$L = -\frac{1}{4} e^{-\mu\varphi} F_{\alpha\beta}^2 + \frac{1}{2} \partial_\alpha \varphi \partial^\alpha \varphi - \frac{1}{2} m F_{\alpha\beta} \epsilon^{\alpha\beta} \varphi - \frac{1}{2} m^2 B_\alpha^2 - \lambda e^{\mu\varphi}$$

where $\mu = 2/\sqrt{-\beta}$ and $2m^2 = \beta/\gamma$.

In the limit $m = 0$, this Lagrangian gives an integrable system. The corresponding Lagrangian

$$L = -\frac{1}{4} e^{-\mu\varphi} F_{\alpha\beta}^2 + \frac{1}{2} \partial_\alpha \varphi \partial^\alpha \varphi - \lambda e^{\mu\varphi} \quad (12)$$

yields the following equations of motion

$$\partial_\alpha (e^{-\mu\varphi} F^{\alpha\beta}) = 0 \quad (13a)$$

$$-\square\varphi + \frac{1}{4} \mu e^{-\mu\varphi} F_{\alpha\beta}^2 - \lambda \mu e^{\mu\varphi} = 0. \quad (13b)$$

The equation (13a) has the general solution

$$F_{\alpha\beta} = C \varepsilon_{\alpha\beta} e^{\mu\varphi} \quad (14)$$

with an arbitrary constant C. After substitution of (14) into (13b) one gets the one-parameter family of Liouville equations

$$\square \varphi + \mu \left(\frac{1}{2} C^2 + \lambda \right) e^{\mu\varphi} = 0.$$

Thus the Lagrangian (12) yields an integrable system. At the classical level it differs from the Liouville Lagrangian, for example, in setting the Cauchy problem. To get the full initial data set one has to define not only the scalar field $\phi(0, \zeta^1)$ and its derivative $\partial_0 \phi(0, \zeta^1)$ but also the constant C. It is important that Eq. (13) admits a simple vacuum solution

$$\varphi = \varphi_0 = \text{const}, \quad F_{\alpha\beta} = \sqrt{-2\lambda} \varepsilon_{\alpha\beta} e^{\mu\varphi_0}$$

in contrast to the Liouville equation. Quantization of the Lagrangian (12) will be considered in a separate publication.

In the present section, the conformally flat gauge was fixed in the Lagrangian. This is not equivalent to fixing the gauge in the equations of motion. In the following we discuss the full equations of motion.

4. Equations of motion

In the present section, we show that the equations of motion for the Lagrangian (4) have a solution $R = \text{const}$, $T = 0$ that was earlier considered as a two-dimensional gravity model¹⁾⁻⁷⁾. Next we analyse the Cauchy problem.

The Lagrangian (4) yields the following equations of motion when varied with respect to B_α and e_α^a

$$-2\gamma \nabla_\beta F^{\beta\alpha} - \beta T^\alpha_{ab} \varepsilon^{ab} = 0, \quad (15a)$$

$$\begin{aligned} \beta \nabla_\beta T^{\beta\alpha}_a + \beta T^{\alpha bc} T_{abc} - \frac{\beta}{4} T^2_{abc} e^\alpha_a - 2\gamma F^{\alpha\beta} F_{a\beta} + \\ + \frac{\gamma}{2} F^2_{bc} e^\alpha_a - \lambda e^\alpha_a = 0. \end{aligned} \quad (15b)$$

Here ∇_α means the covariant derivative with a Lorentz connection ω_α^{ab} when it acts on tensors with Latin indices and the metrical connection without torsion (Christoffel's symbols) when it acts on tensors with Greek indices.

Equations (15) have a solution corresponding to a manifold of constant curvature and zero torsion

$$R = \text{const}, \quad F_{\alpha\beta} = -\frac{1}{2} R \epsilon_{\alpha\beta}, \quad T_{abc} = 0. \quad (16)$$

In this case Eq. (15a) is satisfied and Eq. (15b) defines the curvature in terms of the cosmological constant λ and the coupling constant γ

$$R^2 = 4 \frac{\lambda}{\gamma}.$$

Thus a two-dimensional space of constant curvature¹⁾⁻⁷⁾ is a special case of the theory defined by the Lagrangian (4). But the whole theory of two-dimensional gravity with dynamical torsion is richer and more complicated. In the next section we shall find another (stationary) solution of Eqs. (15) with non-trivial torsion.

Let us analyse the Cauchy problem for Eqs. (15) in the conformal gauge. After fixing the gauge, Eq. (15a) takes the following form

$$-2\gamma \partial_\beta (e^{-2\varphi} F^{\beta\alpha}) - \beta (\epsilon^{\alpha\beta} \partial_\beta \varphi - B^\alpha) = 0$$

This equation has a general solution

$$B^\alpha = \epsilon^{\alpha\beta} \partial_\beta \chi \quad (17)$$

where the scalar field χ satisfies the following equation

$$2\gamma \square \chi + \beta (\varphi - \chi) e^{2\varphi} = 0. \quad (18)$$

Equation (17) is a sufficient condition for the antisymmetric part of Eq. (15b) to be satisfied. The remaining three equations in (15b) can be written as one dynamical equation for ϕ

$$\beta \square \varphi - 2\lambda e^{2\varphi} + \frac{\beta^2}{2\gamma} (\varphi - \chi) e^{2\varphi} + \frac{\beta^2}{2\gamma} (\varphi - \chi)^2 e^{2\varphi} = 0 \quad (19)$$

and two constraints

$$\begin{aligned} & -\beta \partial_1^2 (\varphi - \chi) + \frac{1}{2} \beta (\partial_0 \varphi^2 + \partial_1 \varphi^2 - \partial_0 \chi^2 - \partial_1 \chi^2) - \lambda e^{2\varphi} + \frac{\beta^2}{4\gamma} (\varphi - \chi)^2 e^{2\varphi} = 0 \\ & -\partial_0 \partial_1 (\varphi - \chi) + \partial_0 \varphi \partial_1 \varphi - \partial_0 \chi \partial_1 \chi = 0 \end{aligned} \quad (20)$$

Thus the full equations of motion (15) are equivalent to the dynamical equations (18) and (19) for the scalar fields ϕ and χ and two constraints (20), provided the pseudovector field B_α satisfies (17).

Let us calculate the number of independent functions in the complete set of initial data. Suppose that ϕ and $\partial_0 \phi$ are defined arbitrarily at an initial moment $\zeta^0 = 0$. Then from the constraints (20) one can find initial data for χ and $\partial_1 \chi$ up to some integrating constants, at least locally. We see that χ is not an independent dynamical variable.

The above analysis shows that the Lagrangian (4) describes only one continuous degree of freedom. Note that at the beginning one had six components of e_α^a and B_α which transform under general co-ordinate transformations and local Lorentz rotations defined by three arbitrary functions.

To demonstrate the self-consistency of Eqs. (18), (19) and (20), we write down the non-trivial solution

$$\begin{aligned} \chi &= \varphi + c, \quad c = \text{const}, \\ \square \varphi \pm \sqrt{\frac{\lambda}{8}} e^{2\varphi} &= 0 \end{aligned} \tag{21}$$

which corresponds to zero torsion and constant curvature in (16).

5. Stationary solutions

Stationary solutions of the equations of motion are important from several viewpoints. The stationary solution for constant curvature gravity (11) with a pointlike source can be interpreted as a black hole in two dimensions^{3),4)}. A stationary and spherically symmetric solution in four-dimensional theories with dynamical torsion was discussed for example in Refs. 24) and 25).

Let us show that the equations of motion (18), (19) and (20) can be completely integrated in the stationary limit when ϕ and χ do not depend on ζ^0 .

The second constraint in (20) drops out due to the presence of the time derivative ∂_0 . The remaining three equations for two functions can be written in the following form

$$-\beta f'' - 2\lambda e^{2\varphi} + \frac{\beta^2}{2\gamma} f^2 e^{2\varphi} = 0, \quad (22a)$$

$$-\beta \varphi'' - 2\lambda e^{2\varphi} + \frac{\beta^2}{2\gamma} f e^{2\varphi} + \frac{\beta^2}{2\gamma} f^2 e^{2\varphi} = 0, \quad (22b)$$

$$-\beta f'' + \beta \varphi' f' - \frac{1}{2} \beta f'^2 - \lambda e^{2\varphi} + \frac{\beta^2}{4\gamma} f^2 e^{2\varphi} = 0 \quad (22c)$$

where we have introduced the new variable $f = \phi - \chi$ and Eq. (22c) is the difference between (19) and (18). Multiplying Eq. (22c) by two and taking the difference with (22a), one gets the following equation

$$-\beta f'' + 2\beta \varphi' f' - \beta f'^2 = 0. \quad (23)$$

The solution $f' = 0$ corresponds to zero torsion. In this case, the equations of motion (22) are reduced to the equations

$$\varphi'' \pm \sqrt{\frac{\lambda}{\gamma}} e^{2\varphi} = 0, \quad \chi = \varphi + c, \quad c = \text{const}$$

which can easily be integrated.

Let us consider the case $f' \neq 0$. Then Eq. (23) has the following solution:

$$\varphi = \frac{1}{2} \ln(E f') + \frac{1}{2} f, \quad E = \text{const}. \quad (24)$$

Substitution of this solution into (22a) yields the following equation for f

$$-\beta f'' - 2\lambda E f' e^f + \frac{\beta^2}{2\gamma} E f^2 f' e^f = 0. \quad (25)$$

Equation (22c) is satisfied due to (23). Let us check that Eq. (22b) is also satisfied when Eqs. (24) and (25) hold. Equation (22b) reduces to

$$-\frac{1}{2} \beta \frac{f'' f' - f'^2}{f'^2} - \frac{1}{2} \beta f'' - 2\lambda E f' e^f + \frac{\beta^2}{2\gamma} E f f' e^f + \frac{\beta^2}{2\gamma} E f^2 f' e^f = 0$$

provided Eq. (24) is satisfied. It can be written in the following form

$$\frac{1}{2} \partial_1 \left[-\beta \frac{f''}{f'} - 2\lambda E e^f + \frac{\beta^2}{2\gamma} E f^2 e^f \right] + \frac{1}{2} \left[-\beta f'' - 2\lambda E f' e^f + \frac{\beta^2}{2\gamma} E f^2 f' e^f \right] = 0.$$

We see that the expressions in square brackets are equal to zero due to Eq. (25).

Thus the three equations (22) for two functions ϕ and f reduce to Eq. (25) for f , where ϕ is expressed in terms of f by means of Eq. (24).

Let us consider Eq. (25). Introducing the new variable

$$\zeta' = \frac{2\gamma}{\beta E} x,$$

one gets the following equation

$$-f'' + f'(f^2 - \Lambda)e^f = 0, \quad \Lambda = \frac{4\lambda\gamma}{\beta^2}$$

where the prime denotes differentiation with respect to the x variable. This equation has the general solution

$$f' = (f^2 - 2f + 2 - \Lambda)e^f + A, \quad A = \text{const.} \quad (26)$$

When $A = 0$, Eq. (26) can be integrated in terms of integral logarithm functions²⁶⁾. For example, when $A = 0$, $\Lambda > 1$ and $\lambda_{1,2} = 1 \pm \sqrt{\Lambda - 1}$ are the roots of the equation $f^2 - 2f + 2 - \Lambda = 0$, then

$$x + \tilde{c} = \frac{e^{-\lambda_1} \text{Ei}(\lambda_1 - f) - e^{-\lambda_2} \text{Ei}(\lambda_2 - f)}{\lambda_1 - \lambda_2}, \quad \tilde{c} = \text{const}$$

for $f > \lambda_1, \lambda_2$. Here we use the following definition

$$\text{Ei}(-y) = \int_{\infty}^y \frac{dt}{e^t t}.$$

Equation (26) has qualitatively different solutions for different values of Λ and A . They are shown on Figs. 1 to 9 up to a translation along the x axis. We see that besides the zero torsion solution

$$R = \pm 2\sqrt{\frac{\lambda}{\gamma}}, \quad T = 0, \quad f' = 0$$

one has solutions with non-zero torsion and curvature. It is useful to write down the explicit formulae for the scalar curvature and torsion squared which are geometric invariants in terms of f .

$$R = -\frac{\beta}{\gamma} f,$$

$$T_a^2 = -\frac{\beta}{2\gamma} f' e^{-f} = -\frac{\beta}{2\gamma} (f^2 - 2f + 2 + \Lambda + A e^{-f}).$$

Here Eqs. (24) and (26) were used. For example

$$\begin{aligned} R &= g^{ab} R_{acb}{}^c = -2e^{-2\varphi} \chi_{\zeta\zeta}'' = -2e^{-2\varphi} \left(\frac{\beta E}{2\gamma}\right)^2 \chi'' = -\frac{\beta}{\gamma} f, \\ \text{because } \exp(2\varphi) &= E f_{\zeta}' \exp(f) = E^2 \beta (2\gamma)^{-1} f' \exp(f), \quad \chi = \varphi - f = \\ &= \frac{1}{2} \ln \left(\frac{\beta E^2}{2\gamma} f'\right) - \frac{1}{2} f, \quad \chi' = \frac{1}{2} \frac{f''}{f'} - \frac{1}{2} f' = \frac{1}{2} [(f'' - \Lambda)e^f - f'] = (f-1)e^f + \frac{1}{2} A, \\ \chi'' &= f' f \exp(f). \end{aligned}$$

In the above formulae the prime denotes differentiation with respect to x except where explicitly indicated by ζ .

Solutions with limited curvature and torsion exist only for $\Lambda > 0$, and particular values of initial data for f and f' . These solutions have asymptotics of zero torsion and constant but different scalar curvatures as ζ^1 goes to $\pm \infty$. Thus we have found all stationary solutions of the full equations of motion (15). There are up to five qualitatively distinct solutions for a definite choice of the parameters α , β and λ in the Lagrangian.

6. Large scale limit

In this section we define the large scale limit for a string with dynamical geometry introduced in Ref. 12). It will be shown that in this limit which describes the low-frequency solutions of the equations of motion, the theory reduces to the usual Nambu string. The large scale limit for a dynamical torsion theory in four dimensions was considered in Ref. 27) where it was shown that the theory reduces to Einstein gravity.

A string with dynamical geometry is described by the following Lagrangian (12)

$$L = \frac{1}{2\pi\alpha'} e g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\mu - \frac{1}{4} \gamma e R_{abcd}^2 - \frac{1}{4} \beta e T_{abc}^2 - \lambda e. \quad (27)$$

which is the sum of the Lagrangian (4) of two-dimensional gravity and the usual bosonic string Lagrangian.

To define the large scale limit, let us attribute the following dimensions to the co-ordinates, fields, and couplings constants

$$[\zeta^\alpha] = [X^\mu] = \ell, \quad [e_\alpha^a] = 1, \quad [B_\alpha] = \ell^{-1}, \\ [\alpha'] = [\gamma] = \ell^2, \quad [\beta] = 1, \quad [\lambda] = \ell^{-2}.$$

Then the action is dimensionless. Scaling co-ordinates and fields in accordance with their dimensions,

$$\zeta^\alpha \rightarrow K \zeta^\alpha, \quad X^\mu \rightarrow K X^\mu, \\ e_\alpha^a \rightarrow e_\alpha^a, \quad B_\alpha \rightarrow K^{-1} B_\alpha.$$

and keeping the coupling constants fixed, we define the large scale limit for (27) as the theory described by the leading terms in the action as k goes to infinity, $k \rightarrow \infty$. The leading terms in the action are as follows

$$I = k^2 \int d^2 \zeta \left[\frac{1}{2\pi\alpha'} e g^{\alpha\beta} \partial_\alpha X_\mu \partial_\beta X^\mu - \lambda e \right] \quad (28)$$

and define the standard bosonic string action.

Thus the large scale limit establishes a connection between the two theories. Namely, the Nambu string describes low-frequency fluctuations of the internal geometry of a string world sheet.

Note that Eq. (27) modifies the standard bosonic string by adding the invariants describing the internal geometry of the string world sheet. In Ref. 28), another modification due to the external geometry was considered, adding the higher derivative terms $(\Delta X^\mu)^2$. This term also does not change the large scale limit.

7. Dynamics of the string

Here we briefly discuss the equations of motion for the string with dynamical geometry.

The Lagrangian (27) yields the following equations of motion

$$-\frac{1}{\pi\alpha'} \nabla_\alpha \nabla_\alpha X^\mu = 0, \quad (29a)$$

$$-2\gamma \nabla_\beta F^{\beta\alpha} - \beta T^\alpha_{ab} \epsilon^{ab} = 0, \quad (29b)$$

$$\begin{aligned} & \beta \nabla_\beta T^\beta_\alpha + \beta T^\alpha_{bc} T_{abc} - \frac{\beta}{4} T_{bcd}^2 e^\alpha_a - 2\gamma F^{\alpha\beta} F_{\alpha\beta} + \\ & + \frac{\gamma}{2} F_{bc}^2 e^\alpha_a - \lambda e^\alpha_a + \frac{1}{2\pi\alpha'} \partial_\beta X_\mu \partial^\beta X^\mu e^\alpha_a - \frac{1}{\pi\alpha'} \partial^\alpha X^\mu \partial_\alpha X_\mu = 0, \end{aligned} \quad (29c)$$

where the symbol of the covariant derivative was defined in Section 4 and $\partial^\alpha X^\mu = g^{\alpha\beta} \partial_\beta X^\mu$, $\partial_a X_\mu = e^\beta_a \partial_\beta X_\mu$.

Equation (29a) coincides, for a string position variable X^μ , with those for the ordinary bosonic string. Equation (29b) coincides with Eq. (15a) and can be integrated as has already been done. Equation (29c) yields the dynamics for a zweibein and the constraints which cause the interaction between X^μ and the internal geometry.

Then in conformal gauge, one has the following equations of motion

$$\square X^\mu = 0, \quad (30a)$$

$$\square \chi + \frac{\beta}{2\gamma} (\varphi - \chi) e^{2\varphi} = 0, \quad (30b)$$

$$\square \varphi - \frac{2\lambda}{\beta} e^{2\varphi} + \frac{\beta}{2\gamma} (\varphi - \chi) e^{2\varphi} + \frac{\beta}{2\gamma} (\varphi - \chi)^2 e^{2\varphi} = 0, \quad (30c)$$

$$-\beta \partial_1^2 (\varphi - \chi) + \frac{1}{2} \beta (\partial_0 \varphi^2 + \partial_1 \varphi^2 - \partial_0 \chi^2 - \partial_1 \chi^2) - \lambda e^{2\varphi} + \quad (30d)$$

$$+ \frac{\beta^2}{4\gamma} (\varphi - \chi)^2 e^{2\varphi} - \frac{1}{2\pi\alpha'} \partial_0 X_\mu \partial_0 X^\mu - \frac{1}{2\pi\alpha'} \partial_1 X_\mu \partial_1 X^\mu = 0, \quad (30e)$$

where $\square = \partial_0^2 - \partial_1^2$. The set of Equations (30) is equivalent to Eq. (29) when $B^a = \epsilon^{ab} \partial_b \chi$.

The equations of motion (30a)-(30c) are the dynamical ones for X^μ , χ and ϕ while Eqs. (30d) and (30e) yield two constraints. In conformal gauge the interaction between the X^μ variable and the fields ϕ , χ comes only from the constraints.

With the help of the dynamical equations (30b) and (30c), the constraint equations can be written in covariant form

$$\beta T_{ab}(\varphi) - \beta T_{ab}(\chi) - \frac{1}{\pi\alpha'} T_{ab}(X) = 0 \quad (31)$$

where

$$T_{ab}(\varphi) = \partial_a \varphi \partial_b \varphi - \frac{1}{2} \eta_{ab} \partial_c \varphi \partial^c \varphi - \partial_a \partial_b \varphi + \frac{1}{2} \eta_{ab} \square \varphi,$$

$$T_{ab}(\chi) = \partial_a \chi \partial_b \chi - \frac{1}{2} \eta_{ab} \partial_c \chi \partial^c \chi - \partial_a \partial_b \chi + \frac{1}{2} \eta_{ab} \square \chi,$$

$$T_{ab}(X) = \partial_a X_\mu \partial_b X^\mu - \frac{1}{2} \eta_{ab} \partial_c X_\mu \partial^c X^\mu.$$

Only two of these equations are independent since $T_{ab} = T_{ba}$ and $T_a^a = 0$.

8. Conclusion

We have considered the equations of motion for two-dimensional gravity with dynamical torsion. These equations result in the system of equations (18) to (20) for the two scalar fields. In a particular case they yield the Liouville equation. Analysis of the Cauchy problem for the whole system shows that dynamics is governed only by one continuous degree of freedom and also by a set of discrete variables. We have analysed stationary solutions of the equations of motion. In this case, one gets three equations (22) for two functions. It is

quite remarkable that this system can be completely integrated. Note that there exist solutions with non-trivial torsion and curvature and without a singularity.

Next we defined the large scale limit for a string with dynamical geometry. It is interesting that this limit is described by the usual bosonic string action.

The equations of motion for a string with dynamical geometry can be decomposed into the dynamical ones and the two constraints for a string position variable X^μ and the two scalar fields. In conformal gauge, the interaction of X^μ with the internal geometry is provided by the constraints and is non-trivial only when the torsion differs from zero. The modification of the usual bosonic string action due to the internal geometry of a string world sheet adds two new continuous bosonic degrees of freedom to the D-2 transverse modes of X^μ . Additional degrees of freedom may hopefully stabilize the string.

The operator quantization of the usual bosonic string which is described by equations (30) when $\phi = \chi = 0$ can proceed as follows. One quantizes the free equation (30a) and then satisfies the constraint $T_{ab}(X) = 0$ which can be written in the form of the Virasoro algebra. A similar line of operator quantization for the string with dynamical geometry would consist of quantizing the dynamical equations (30a) to (30c) and restriction of the Hilbert space by the constraint algebra following from Eq. (31). Then one would be allowed to compute the mass spectrum and angular momentum of the string. This programme of quantization fails at present due to the non-linearity of Eqs. (30b) and (30c).

Note that the symmetry necessary to choose the light cone gauge for the string variable X^μ remains in Eq. (30) as was discussed in Section 3.

It is interesting that the interaction of X^μ modes of the string with the internal modes described by the fields ϕ and χ is provided by non-zero torsion. When the torsion is zero $\phi = \chi \pm 2\sqrt{\gamma\lambda}/\beta^2$, and the equations of motion split into the usual equations for X^μ and the Liouville equation for ϕ .

The problem of operator quantization of the string with dynamical geometry is analogous to that investigated in Ref. 29) where the standard bosonic string action was supplemented by the Liouville action. In Ref. 29) it was shown that the Liouville modes are equivalent to the Brower modes and therefore consistent free open quantum strings can exist in $D < 26$. For closed strings the situation

is more complicated and no completely satisfactory quantum version of the theory has been found.

From our point of view, in order to obtain a fully satisfactory theory of bosonic strings, it is necessary to take into account the effects of the dynamical geometry of the string world sheet that was considered in Ref. 12) and in the present paper. Of course it would be nice to get an explicit solution of the non-linear equations of motion, but it may be sufficient to consider the model in a perturbative framework because the theory admits a simple vacuum solution.

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REFERENCES

- 1) R. Jackiw in Quantum Theory of Gravity, S. Christensen ed. (Hilger, Bristol, 1984); Nucl. Phys. B252 (1985) 343.
- 2) C. Teitelboim, Phys. Lett. 126B (1983) 41; see also his contribution to Quantum Theory of Gravity, S. Christensen ed. (Hilger, Bristol, 1984).
- 3) N. Sanchez, Nucl. Phys. B266 (1986) 487; From black holes to quantum gravity: recent results in two and four dimensions, Meudon preprint (1986).
- 4) J.D. Brown, M. Henneaux and C. Teitelboim, Phys. Rev. D33 (1986) 319.
- 5) R. Balbinot and R. Floreanini, Phys. Lett. 160B (1985) 401;
R. Floreanini, Ann. Phys. 167 (1986) 317.
- 6) T. Fukuyama and K. Kamimura, Phys. Lett. 160B (1985) 259.
- 7) M. Martellini, Ann. Phys. 167 (1986) 437.
- 8) L. Brink, P. Di Vecchia and P. Howe, Phys. Lett. 65B (1976) 471;
S. Deser and B. Zumino, Phys. Lett. 65B (1976) 369.
- 9) J.H. Schwarz, Physics Reports 89 (1982) 223;
M.B. Green, Surveys in High Energy Physics 3 (1983) 127.
- 10) P. van Nieuwenhuizen, Intern. Journ. Mod. Phys. A1 (1986) 155; Critical dimensions and supersymmetric measures, preprint ITP-SB-86-84(1986).
- 11) R. Marnelius, Nucl. Phys. B211 (1983) 14.
- 12) M.O. Katanayev and I.V. Volovich, Phys. Lett. 175B (1986) 413.
- 13) D. Neville, Phys. Rev. D18 (1978) 3535;
E. Sezgin and P. van Nieuwenhuizen, Phys. Rev. D21 (1980) 3269;
K. Hayashi and T. Shirafuji, Prog. Theor. Phys. 64 (1980) 2222;
M. Blagojevic and A. Nikolic, Phys. Rev. D28 (1983) 2455;
M.O. Katanayev, Theor. Math. Phys. 65 (1985) 105.
- 14) F.W. Hehl, Found. Phys. 15 (1985) 451.
- 15) D.D. Ivanenko, P.I. Pronin and G.A. Sardanashvili, Gauge Theory of Gravitation, MGU, Moscow, 1985 (in Russian);
V.N. Ponomarev, A.O. Barvinski and J. Obukhov, Geometrodynamical methods and gauge approach to gravity interactions, Energoizdat, Moscow, 1985 (in Russian).
- 16) M.O. Katanayev and I.V. Volovich, Phys. Lett. 156B (1985) 327; Teor. Mat. Fiz. 66 (1986) 79 (in Russian).
- 17) P.S. Howe and L. Sierra, Phys. Lett. 148B (1984) 451;
T.L. Curtright and C.K. Zachos, Phys. Rev. Lett. 53 (1984) 1799;
S.J. Gates, C.M. Hull and M. Rocek, Nucl. Phys. B248 (1985) 157;
E. Braaten, T. Curtright and C. Zachos, Nucl. Phys. B260 (1985) 630;
T.L. Curtright, L. Mezincescu and C. Zachos, Phys. Lett. 161B (1985) 79.

- 18) R. Omnes, Nucl. Phys. B149 (1979) 269;
B.M. Barbashov, V.V. Nesterenko and A.M. Chervjakov, Commun. Math. Phys. 84 (1982) 471;
G.P. Pron'ko, A.V. Rasumov and L.D. Soloviev, Elem. Part. and Nucl. Phys. 13 (1983) 558 (in Russian);
A.A. Zheltukhin, Teor. Mat. Fiz. 56 (1983) 785.
- 19) A.M. Polyakov, Phys. Lett. 103B (1981) 207.
- 20) K.S. Stelle, Phys. Rev. D16 (1977) 953;
E.S. Fradkin and A.A. Tseytlin, Nucl. Phys. B201 (1982) 496;
B.L. Voronov and I.V. Tuitin, Yadernaya Fizika 39 (1984) 998 (in Russian);
I. Antoniadis and E.T. Tomboulis, Phys. Rev. D33 (1986) 2756;
I.L. Buchbinder and S.L. Lyahovich, Teor. Mat. Fiz. 72 (1987) 223 (in Russian).
- 21) S. Kobayashi and K. Nomizu, Foundations of differential geometry, Vol. 1, Interscience publishers, New York-London.
- 22) M.O. Katanayev and I.V. Volovich, "Quark - 86" VNU Science Press Holland (1987).
- 23) H. Weyl, Sitzungsber. Preuss. Akad. Wiss. Berlin (1918) 465.
- 24) F.W. Hehl, J.D. McGrea and E.W. Mielke, "Weyl space-times, the dilation current and creation of gravitating mass by symmetry breaking" (1986) 66pp, unpublished.
- 25) R. Rauch, J.C. Shaw and H.-T. Nieh, Gen. Rel. Grav. 14 (1982) 331;
H. Goenner and F. Müller-Hoissen, Class. Quantum Grav. 1 (1984) 651;
J.D. McGrea, Phys. Lett. 100A (1984) 397;
P. Baeckler and P.B. Yasskin, Gen. Rel. Grav. 16 (1984) 1135;
Y.-Z. Zhang, Commun. Theor. Phys. 3 (1984) 551;
H.-J. Lenzen, Gen. Rel. Grav. 17 (1985) 1137.
- 26) Handbook of mathematical functions, M. Abramovitz and I.A. Stegun eds. (1984).
- 27) M.O. Katanayev, Teor. Mat. Phys. 54 (1983) 248; Izv. Vuzov. Fizika 5 (1987) 39 (in Russian).
- 28) W. Helfrich, Z. Naturforsch. C28 (1973) 693; J. de Phys. 46 (1985) 693;
L. Peliti and S. Leibler, Phys. Rev. Lett. 54 (1985) 1690;
D. Förster, Phys. Lett. 114A (1986) 115;
A.M. Polyakov, Nucl. Phys. B268 (1986) 406;
T.L. Curtright, G.I. Grandhour and C. Zachos, preprint ANL-HEP-PR-86-34 (1986).
- 29) L. Johansson and R. Marnelius, Nucl. Phys. B254 (1985) 201;
S. Hwang and R. Marnelius, Nucl. Phys. 13271 (?) (1986) 369;
R. Marnelius, preprint ITP, Göteborg (1986).

$$\Lambda < 1, A \geq 0$$

$$\Lambda = 1, A > 0$$

$$\Lambda > 1, A > 2e^{\sqrt{\Lambda}}(\sqrt{\Lambda}-1)$$

Fig. 1

$$\Lambda \geq 1, A = 2e^{\sqrt{\Lambda}}(\sqrt{\Lambda}-1)$$

Fig. 2

Fig. 3

$$\Lambda > 1, 0 \leq A < 2e^{\sqrt{\Lambda}}(\sqrt{\Lambda}-1)$$

Fig. 4

$$\Lambda \geq 1, -2e^{-\sqrt{\Lambda}}(\sqrt{\Lambda}+1) < A < 0$$

$$0 < \Lambda < 1, -2e^{-\sqrt{\Lambda}}(\sqrt{\Lambda}+1) < A < 2e^{\sqrt{\Lambda}}(\sqrt{\Lambda}-1)$$

Fig. 5 $\Lambda > 0, A = -2e^{-\sqrt{\Lambda}}(\sqrt{\Lambda} + 1)$

Fig. 6
 $\Lambda > 0, A < -2e^{-\sqrt{\Lambda}}(\sqrt{\Lambda} + 1)$
 $\Lambda \leq 0, A < -2(2 - \Lambda)$

Fig. 7 $\Lambda \leq 0, A = \Lambda - 2$

Fig. 8
 $0 \leq \Lambda < 1, 2e^{\sqrt{\Lambda}}(\sqrt{\Lambda} - 1) < A < 0$
 $\Lambda \leq 0, \Lambda - 2 < A < 0$

Fig. 9 $0 < \Lambda < 1, A = -2e^{\sqrt{\Lambda}}(1 - \sqrt{\Lambda})$