CERN-TH.4767/87
Ecole Polytechnique
A.7920687

4D FERMIONIC SUPERSTRINGS WITH ARBITRARY TWISTS

I. Antoniadis^{*)}

CERN - Geneva

and

C. Bachas

Centre de Physique Théorique, Ecole Polytechnique
91128 Palaiseau Cedex, France

A B S T R A C T

We present the rules for systematically constructing all consistent four-dimensional string theories, using free world-sheet fermions which pick up arbitrary phases when parallel transported around the string. These rules are necessary and sufficient for multi-loop modular invariance. They lead to theories with general Z_N (GSO-type) projections, whose merits for model-building we discuss. We classify all boundary conditions yielding massless space-time spinors. We show that, in contrast to the case of only real 2d fermions, all possible realizations of world-sheet supersymmetry are now allowed. This opens the way for the construction of a new class of supersymmetric string models.

^{*)} On leave of absence from Centre de Physique Théorique, Ecole Polytechnique, 91128 Palaiseau Cedex, France.

1. - INTRODUCTION

It is at present widely recognized that a unified string theory¹⁾⁻³⁾ cannot be singled out uniquely on the basis of perturbative consistency⁴⁾; many four-dimensional, in particular, models have been shown to pass the tests of invariance under local and global world-sheet superreparametrizations⁵⁾⁻¹⁴⁾. The current belief is of course¹⁵⁾ that all these models correspond to classical solutions of only a few second-quantized theories, whose non-perturbative dynamics, when fully understood, will allow us to make a unique choice. In the meantime, however, it is important to continue the study of these different string vacua, in order to both a) obtain a phenomenologically viable and calculable model with realistic gauge group and chiral quark and lepton families, and b) provide a context in which to study general properties of string theories such as their low-energy symmetries, and the possibility they offer for resolving such long-standing problems like the generation of fermion masses, gauge hierarchies, etc.

In the construction of four-dimensional string models, one usually describes the quantum-number densities on the string either by extra free world-sheet fermions^{6),8),10),11),13)}, or by free world-sheet bosons parametrizing some flat torus or orbifold^{5),7),9),12)}. The two formulations are thought to be equivalent¹⁶⁾, so that the choice is to some extent a matter of taste and convenience. Counting the number of distinct models, or finding the gauge group, is for instance easier in the bosonic language, but we believe that for most other purposes the fermionic language is advantageous. For instance, in the fermionic formulation one may easily classify all possible world-sheet supersymmetries⁶⁾, find the effective low-energy field theory of a string model¹⁷⁾, or prove that space-time supersymmetry implies the vanishing of the cosmological constant to all orders in the loop expansion¹⁸⁾.

In the present work we extend the analysis of four-dimensional fermionic superstrings, presented in Ref. 10), to include free world-sheet fermions which pick up arbitrary phases when parallel transported around the string. We first derive, in Section 3, the set of necessary and sufficient rules which can be used to generate all consistent string models in this framework. Although our starting point is multi-loop modular invariance^{2),19),20)}, reviewed in Section 2, this part of the analysis overlaps considerably with the work of Kawai, Llewellyn and Tye, who have already studied generalized spin structures in Ref. 8). The models one finds have Hilbert spaces obtained by arbitrary $Z_{N_1} \times \dots \times Z_{N_K}$ "truncations and

twistings"; despite some loss of elegance and simplicity when compared with the $Z_2 \times \dots \times Z_2$ case of Ref. 10), this generalization offers some novel possibilities. With regard to realistic model building for example, one may now reduce the rank of the "observable" gauge group, and obtain easily unitary gauge groups and chiral matter fields in arbitrary antisymmetric representations. These facts are illustrated in Section 4, where we construct a theory with $SU(5)$ gauge group, chiral families in the $\overline{5}$ and 10 representations, but no adjoint scalars which could be used to break the grand-unifying group perturbatively.

Potentially more interesting is the fact that generalized spin-structures allow us to evade a "no-go theorem", which restricted the form of string theories with massless space-time spinors in their spectrum, by imposing a particular realization of world-sheet supersymmetry⁶⁾. To be precise, the most general world-sheet supercharge is trilinear in the fermions with coefficients equal to the structure constants of a semi-simple Lie group of dimension 18, i.e., $G = SU(2)^6$, $SU(3) \times O(5)$ or $SU(4) \times SU(2)$. With real (periodic or antiperiodic) world-sheet fermions, only the first of these three choices could lead to a theory with massless space-time spinors; with generalized spin-structures, however, all of the above three choices are allowed. This will be shown in Section 5, where we obtain the exhaustive list of spin-structure assignments yielding massless space-time spinors, by classifying all Lie-algebra automorphisms L which minimize the "mass subtraction" $1/16 \langle L \rangle = -(1/16\pi^2) \text{Tr}[\log(-L)]^2$.

As we will show for all the groups of interest to 4d fermionic superstrings

$$\langle L \rangle_{\text{minimum}} = \frac{1}{3} d_G$$

where d_G is the group's dimension. We believe that this relation, which is precisely what is needed to ensure the existence of massless space-time fermions, is a generic group-theoretic property, but we will not try to establish this conjecture in the present work.

The results of Section 5 open the way for the construction of a new class of superstring theories which will be analyzed elsewhere²¹⁾. Let us also point out that, in light of these results, the arguments of Ref. 14) excluding a realistic type II model should be reviewed carefully.

2. - SUPER-REPARAMETRIZATION INVARIANCE CONDITIONS

Consider a string moving in four-dimensional space-time. Spread out on the string are free fermionic densities which carry all its internal quantum numbers. The precise number of free fermions in each of the left- and right-moving sectors depends on whether we demand world-sheet supersymmetry. A non-supersymmetric sector contains 44 real free fermions ϕ^a . A supersymmetric sector contains in addition to the four superpartners ψ^μ of the space-time co-ordinates, extra 18 real fermions χ^I , among which supersymmetry is non-linearly realized. The supercharge is^(6),22),23)

$$T_F = \psi^\mu \partial X_\mu + f_{IJK} \chi^I \chi^J \chi^K \quad (2.1)$$

where f_{IJK} are appropriately normalized structure constants of a semi-simple Lie group G , with 18 generators. There are thus only three allowed choices for G : $SU(2)^6$, $SU(4) \times SU(2)$ and $O(5) \times SU(3)$. All our results hold equally well for bosonic, heterotic or type II supersymmetric strings meaning respectively that none, one or both of the left- and right-moving sectors have world-sheet supersymmetry. For definiteness, we will restrict ourselves below to the heterotic case, with supersymmetry being realized only among left movers.

The number of free fermions and the form (2.1) of the supercharge are dictated by invariance under local super-reparametrizations of the world-sheet. To complete the construction of a string theory we must also ensure invariance under modular transformations. Consider a genus- g compact Riemann surface M , describing a contribution to the g -loop vacuum to vacuum string amplitude. Let α_i, β_i ($i=1, \dots, g$) be a canonical basis generating the fundamental group $\pi_1(M)$ of the surface. We recall that this is the group of all equivalence classes of closed curves on the surface, passing through some arbitrary base point P ; two curves are equivalent if one can be continuously deformed to the other; the identity element of $\pi_1(M)$ consists of loops contractible to a point.

When parallel transported around some non-trivial loop $\alpha \in \pi_1(M)$, the right-moving fermions transform in general as follows (summation over repeated indices is implied throughout):

$$\phi^a \longrightarrow R(\alpha)^a_b \phi^b$$

where $R(\alpha)$ should be orthogonal in order that the free energy-momentum current be left unchanged. The transformation of the left-movers, on the other hand, must in

addition leave the supercurrent (2.1) invariant up to a sign $\delta_\alpha = \pm 1$. Thus

$$\psi^r \rightarrow -\delta_\alpha \psi^r$$

and

$$\chi^r \rightarrow L(\alpha)^r_j \chi^j$$

where $L(\alpha)$ is an orthogonal matrix obeying the condition

$$L^r_{r'} L^j_{j'} L^k_{k'} f_{ijk} = -\delta_\alpha f_{i'j'k'} \quad (2.2)$$

This means that $-\delta_\alpha L(\alpha)$ is an automorphism of the Lie algebra of G . We will come back to this later.

Clearly $R(\alpha)$ and $L(\alpha)$ are matrix representations of the fundamental group $\pi_1(M)$. This is defined by the single constraint

$$\prod_{i=1}^g \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} = 1 \quad (2.3)$$

and thus, for $g > 2$, it is non-Abelian. We could, therefore, allow non-commuting boundary conditions for the fermions on a two- or a higher-loop surface, but the constraints of factorization and multi-loop modular invariance^{19),20)} are, in this case, much more intricate and we do not yet know how to deal with them^{24),*)}. We will therefore limit ourselves here to only commuting boundary conditions. The matrices $R(\alpha)$ and $L(\alpha)$ can then be simultaneously diagonalized for all α in some, in general complex basis, consisting of k real fermions $f_1^r, \dots, f_k^r$, and l complex fermions $f_1^c, \dots, f_l^c$ and their complex conjugates: $f_1^{c*}, \dots, f_l^{c*}$. Of course, $k+2l = 66$, the total number of fermions. For any non-contractible loop α we may now specify the boundary conditions by a $(k+l)$ -component vector (denoted by the same symbol):

$$\alpha = \left[\alpha(f_1^r), \dots, \alpha(f_k^r); \alpha(f_1^c), \dots, \alpha(f_l^c) \right]$$

where under parallel transport around this loop

*) For instance, (2.3) means that the spin-structure assignments on the two donuts of a $g = 2$ surface cannot be chosen independently, even in the degenerate limit where the two donuts are pulled infinitely far apart.

$$f \rightarrow -e^{i\pi\alpha(f)} f \quad (2.4a)$$

Note that for the real fermions, $\alpha(f)$ must be an integer, and that $\delta_\alpha = e^{i\pi\alpha(\psi^\mu)}$, where $\alpha(\psi^\mu)$ is independent of μ , since all ψ^μ as well as the world-sheet gravitino and the supercurrent T_F have the same parallel transport properties.

Strictly speaking, α stands for an equivalence class of vectors, whose components differ by arbitrary even integers. We will pick a "reduced" representative, defined by:

$$\alpha(f_i^r), \alpha(f_j^c) \in (-1, +1] \quad (2.4b)$$

By default, $\lambda\alpha$ and $\alpha+\beta$ will stand, in the sequel, for the reduced representatives of the corresponding vectors. When we occasionally need the unreduced multiple or sum of vectors, we will explicitly specify the reduced representative as: $[\lambda\alpha]$ and $[\alpha+\beta]$ respectively. Furthermore, the usual equality sign will be reserved for equality modulo 2; equality of unreduced vectors will be denoted by $\alpha \equiv \beta$. For instance, $2\alpha \equiv 0$ implies that $\alpha \equiv 0$, while $2\alpha = 0$ simply means that all components of α are integers, i.e., all fermions are either periodic or antiperiodic.

A complete spin-structure assignment to all fermions on a genus- g surface, can be specified by $2g$ vectors

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_g \\ \beta_1 & \beta_2 & \cdots & \beta_g \end{bmatrix}$$

where α_i and β_j are the boundary conditions corresponding to the elements of the canonical homology basis. Exhibiting only spin-structure-dependent terms, we may write the vacuum-to-vacuum amplitude as follows:

$$\begin{aligned} Z_g &= \int_{M_g} [D\Omega] \sum_{\text{spin structures}} c \begin{bmatrix} \alpha_1 \cdots \alpha_g \\ \beta_1 \cdots \beta_g \end{bmatrix} X \begin{bmatrix} \alpha_1(\psi) \cdots \alpha_g(\psi) \\ \beta_1(\psi) \cdots \beta_g(\psi) \end{bmatrix} (\Omega) \cdot \\ &\cdot \prod_{\text{real transverse } f} \Theta^{\frac{1}{2}} \begin{bmatrix} \alpha_1(f) \cdots \alpha_g(f) \\ \beta_1(f) \cdots \beta_g(f) \end{bmatrix} (0, \Omega) \cdot \prod_{\text{complex } f} \Theta \begin{bmatrix} \alpha_1(f) \cdots \alpha_g(f) \\ \beta_1(f) \cdots \beta_g(f) \end{bmatrix} (0, \Omega) \end{aligned} \quad (2.5)$$

We now explain this expression: The Θ -function

$$\Theta \left[\begin{smallmatrix} \epsilon_1 \dots \epsilon_g \\ \epsilon'_1 \dots \epsilon'_g \end{smallmatrix} \right] (\vec{z}, \Omega) = \sum_{\vec{n} \in \mathbb{Z}^g} e^{\left\{ i n(\vec{n} + \frac{\vec{\epsilon}}{2}) \Omega (\vec{n} + \frac{\vec{\epsilon}}{2}) + 2 i n(\vec{n} + \frac{\vec{\epsilon}}{2}) (\vec{z} + \frac{\vec{\epsilon}'}{2}) - \frac{i n}{2} \vec{\epsilon} \cdot \vec{\epsilon}' \right\}} \quad (2.6)$$

is the determinant of the spin-structure dependent chiral Dirac operator, divided by the determinant of the Laplacian²⁰⁾. Its arbitrary overall phase is fixed in (2.6) according to the conventions of Mumford²⁵⁾. The product in Eq. (2.5) runs over all the transverse fermions, that is all fermions in our complex basis with the exception of two ψ^μ 's. The contribution of these latter, as well as of the supersymmetry ghosts and supermoduli is denoted by the factor X . This is a constant at one loop, and it can be dropped; at two or more loops, it represents a spin-structure-dependent correction, due to the non-existence of a globally defined light-cone gauge. The coefficients of the various spin structure assignments in the sum are required by factorization^{19),20)} to be a product of one-loop coefficients:

$$c \left[\begin{smallmatrix} \alpha_1 \dots \alpha_g \\ \beta_1 \dots \beta_g \end{smallmatrix} \right] = c \left[\begin{smallmatrix} \alpha_1 \\ \beta_1 \end{smallmatrix} \right] c \left[\begin{smallmatrix} \alpha_2 \\ \beta_2 \end{smallmatrix} \right] \dots c \left[\begin{smallmatrix} \alpha_g \\ \beta_g \end{smallmatrix} \right] \quad (2.7)$$

Finally the integration in Eq. (2.5) is over a fundamental domain M_g of Teichmüller space, Ω is the period matrix of the surface, and in the integration measure we have absorbed spin-structure independent terms such as determinants of the Laplacian and the contribution of the reparametrization ghosts: for our purposes, it would not be necessary to write these down explicitly.

A comment is in order here, concerning the factorization condition (2.7). In the bosonic string where one need not worry about the factor X , this is a simple consequence of locality: indeed the relative weight of two spin-structure assignments, which differ only on a single donut, should not depend on the remainder of the Riemann surface in the limit where the donut in question is pulled infinitely far apart. The argument can be extended to the heterotic string, provided that the spin-structure dependence of the factor X also factorizes in this limit.

The group of modular transformations of a genus- g surface is generated by $2g$ Dehn twists around the elements of the homology basis, as well as $g-1$ additional Dehn twists mixing spin structures on two neighbouring donuts²⁰⁾. Thanks to factorization, this means that modular invariance to all orders is ensured, if

and only if we impose invariance under the two modular transformations of the torus

$$\Omega \rightarrow \Omega + 1 \quad (a)$$

$$\Omega \rightarrow -\frac{1}{\Omega} \quad (b)$$

as well as under some non-trivial transformation of the double donut, which we here choose to be

$$\Omega \rightarrow \Omega - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (c)$$

We are thus led to the following three necessary and sufficient conditions among the spin-structure coefficients:

$$c \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = e^{i \frac{n}{4} (\alpha \cdot \alpha + 1 \cdot 1)} c \begin{bmatrix} \alpha \\ \beta - \alpha + 1 \end{bmatrix} \quad (2.8a)$$

$$c \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = e^{i \frac{n}{2} \alpha \cdot \beta} c \begin{bmatrix} \beta \\ -\alpha \end{bmatrix} \quad (2.8b)$$

$$c \begin{bmatrix} \alpha \\ \beta \end{bmatrix} c \begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} = \delta_{\alpha} \delta_{\alpha'} e^{-i \frac{n}{2} \alpha \cdot \alpha'} c \begin{bmatrix} \alpha \\ \beta + \alpha' \end{bmatrix} c \begin{bmatrix} \alpha' \\ \beta' + \alpha \end{bmatrix} \quad (2.8c)$$

Here 1 is the vector corresponding to periodic boundary conditions for all fermions; the dot product of two vectors is Lorentzian among left- and right-movers, and puts half weight to the real basis elements:

$$\alpha \cdot \beta = \left\{ \frac{1}{2} \sum_{\substack{\text{real} \\ \text{transv. left}}} + \sum_{\substack{\text{complex} \\ \text{left}}} - \frac{1}{2} \sum_{\substack{\text{real} \\ \text{transv. right}}} - \sum_{\substack{\text{complex} \\ \text{right}}} \right\} \alpha(f) \beta(f) \quad (2.9)$$

The global phase $e^{i\pi/4} 1.1$ in (2.8a) comes from the transformation properties of the spin-structure-independent measure; it equals -1 for the heterotic and +1 for the bosonic or type II supersymmetric strings. All other phases come from the standard transformations of the θ -functions²⁵⁾, with the exception of the sign $\delta_{\alpha} \delta_{\alpha'}$ in (2.8c) which is due to the two-loop light-cone gauge correction factor

X. Although this latter is not known explicitly^{*)}, its modular transformation properties are fixed uniquely²⁷⁾ by the requirement that the ten-dimensional SO(32) heterotic string be modular invariant^{3),4)}. The extra phase $\delta_\alpha \delta_\alpha$, is of course absent in the case of the bosonic string, while for type II theories, where X^μ has both right- and left-moving superpartners ψ^μ and $\bar{\psi}^\mu$, the proper definition is that $\delta_\alpha = -1$ if either $\alpha(\psi^\mu) = 1$ or $\alpha(\bar{\psi}^\mu) = 1$, but not both, and $\delta_\alpha = +1$ otherwise.

The one-loop modular invariance conditions (2.8a,b) ensure the absence of all potential anomalies in the effective field theory of the string's massless modes^{**) ,28)}. The two-loop conditions (2.8c) on the other hand ensures, as we will soon see, the spin-statistics connection and unitarity^{27),10)}. This is a marvellous unification of principles: world-sheet (super-) reparametrization invariance encodes by itself some of the most profound consistency conditions of field theories.

3. - THE GENERAL SOLUTION

We turn now to a systematic derivation of the solutions of the modular invariance constraints (2.8a,b,c), i.e., we will find the rules for constructing perturbatively consistent fermionic string theories when spin-structure assignments are mutually commuting. Let us begin by noting that (2.8b) and (2.8c) imply

$$c \begin{bmatrix} \alpha \\ \beta \end{bmatrix} c \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} = \delta_\alpha \delta_\gamma c \begin{bmatrix} \alpha \\ \beta+\gamma \end{bmatrix} c \begin{bmatrix} \gamma \\ 0 \end{bmatrix} \quad (3.1)$$

Setting first $\beta = \gamma = 0$ implies that for all α either $c \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = 0$ or else $c \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \delta_\alpha c \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Since the constraints of modular invariance are homogeneous in the c 's, we may normalize $c \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$ in what follows ($c \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ cannot vanish because so

*) Considerable progress has been made recently towards calculating explicitly the factor X^{26} ; this would allow to verify both its modular transformation properties, and the fact that its spin-structure dependence factorizes on the boundary of moduli space.

**) This has been, strictly speaking, proved only in some special cases of spin-structure assignments.

would all the other coefficients in this case). Now let us define the set of vectors

$$\Xi = \left\{ \alpha \mid c \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \delta_\alpha \right\}$$

Putting $\alpha = 0$ in (3.1) we see that if $\beta, \gamma \in \Xi$ then $\beta + \gamma \in \Xi$, i.e., Ξ is an (Abelian) additive group. Also $1 \in \Xi$, as follows from (2.8a) by setting $\alpha = \beta = 0$. Next we note that the one-loop coefficients $c \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ vanish unless both α and β belong in Ξ ; this follows from (3.1) by setting $\beta = \gamma$ or $\gamma = 0$. Thus contributing spin-structure assignments correspond to pairs of elements in Ξ .

In principle, Ξ need not be finite, nor even denumerable, in which case string-loop amplitudes would be given by infinite sums or by integrals over different spin-structure assignments. Such theories might correspond, in the bosonic language, to compactifications on tori with arbitrary irrational radii. The analysis is, however, in this case more intricate since for example one must study the convergence properties of string-loop integrands. To avoid such complications we will restrict ourselves here to finite groups Ξ . For any vector $\alpha \in \Xi$ all the components $\alpha(f)$ are in this case rational, and there exists a smallest positive integer N_α such that $N_\alpha \alpha = 0$. From the factorization condition (3.1) when $\alpha, \beta, \gamma \in \Xi$:

$$c \begin{bmatrix} \alpha \\ \beta + \gamma \end{bmatrix} = \delta_\alpha c \begin{bmatrix} \alpha \\ \beta \end{bmatrix} c \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} \quad (3.2)$$

it follows easily that

$$c \begin{bmatrix} \alpha \\ \beta \end{bmatrix}^{N_\beta} = \delta_\alpha^{N_\beta} \quad (3.3)$$

and hence $c \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is a pure phase^{*}).

Let us pause here and summarize our conclusions in a

Lemma The one-loop amplitude of a consistent fermionic string theory is a sum over all spin-structures corresponding to pairs of elements of a group Ξ with coefficients which are pure phases.

^{*}) Note that $c \begin{bmatrix} \alpha \\ \beta \end{bmatrix}^* = c \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$ as follows by setting $\beta = -\gamma$ in (3.2). This ensures the reality of the one-loop vacuum amplitude [see Eqs. (2.5) and (2.6)].

We will now characterize the finite groups Ξ that give consistent string theories. Note first that any finite Abelian group is isomorphic to a direct sum of Z_N factors²⁹⁾

$$\Xi \simeq Z_{N_1} \oplus \dots \oplus Z_{N_k} \quad (3.4)$$

This means that there exists a basis $\{b_1, \dots, b_k\}$ generating Ξ , and such that

$$m_1 b_1 + \dots + m_k b_k = 0$$

if and only if $m_i = 0 \pmod{N_i}$ for all i . We shall refer to such a basis as a canonical basis. Of course, not every basis is canonical. There exists, incidentally, a unique decomposition (3.4) and corresponding canonical basis with the property that all N_i 's are powers of prime numbers²⁹⁾, but this will not be necessarily the most convenient one for our purposes. We will in fact be working with canonical bases such that the vector 1, which belongs to Ξ and can therefore be expressed as a linear combination of basis elements, is simply a multiple of the first basis element b_1 . It can be shown that such a canonical basis always exists; furthermore,

$$1 = \frac{1}{N_1} N_1 b_1 \quad (3.5)$$

since N_1 is by definition the smallest integer for which $N_1 b_1 = 0$. In most cases of interest b_1 will simply be equal to 1.

Our strategy for constructing all consistent fermionic string theories will be as follows: we will first derive a set of necessary conditions on the elements of a canonical basis $\{b_1, \dots, b_k\}$ generating Ξ , as well as on the one-loop coefficients $c_{b_j}^{b_i}$. We will then prove that these conditions are also sufficient in the sense that the assignment of coefficients to pairs of basis elements can be extended to any pair of elements of Ξ , in a way consistent with the modular invariance constraints Eqs. (2.8a,b,c). Since the analysis is rather tedious, the reader may want to go directly to the Theorem , which summarizes the results.

First setting $\alpha = b_i$ and $\beta = b_j$ in (2.8b), raising both sides to the power $N_{ij} = \text{least common multiple of } N_i \text{ and } N_j$, and using Eq. (3.3) we find

$$e^{\frac{i\pi}{2} N_{ij} b_i \cdot b_j} = (\delta_{b_i} \delta_{b_j})^{N_{ij}}$$

The right-hand side is always 1, because N_{ij} is even if either δ_{b_i} or δ_{b_j} is -1. Consequently we must have

$$N_{ij} b_i \cdot b_j = 0 \pmod{4} \quad (3.6)$$

for all pairs of canonical basis elements. For $i = j$ this means that $N_i b_i^2 = 0 \pmod{4}$. However, if N_i is even, we must impose the stronger condition

$$N_i b_i^2 = 0 \pmod{8} \quad (N_i \text{ even}) \quad (3.7)$$

To see why, note first that (2.8a) can be rewritten equivalently as

$$c \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = e^{\frac{i\pi}{4} (\alpha \cdot \alpha + 1 \cdot 1)} c \begin{bmatrix} \alpha \\ 1 \end{bmatrix} \quad (3.8)$$

where we have used the factorization Eq. (3.2). Now setting $\alpha = b_i$ and raising both sides to the power N_i , we find (3.7) when N_i is even.

Consider next the one-loop coefficients $c \begin{bmatrix} b_i \\ b_j \end{bmatrix}$ for pairs of basis elements. From (2.8b) we have

$$c \begin{bmatrix} b_j \\ b_i \end{bmatrix} = e^{\frac{i\pi}{2} b_i \cdot b_j} c \begin{bmatrix} b_i \\ b_j \end{bmatrix}^* \quad (3.9)$$

Thus the coefficients $c \begin{bmatrix} b_i \\ b_j \end{bmatrix}$ for $j > i$ can be determined in terms of those for $j < i$. These latter are constrained from (3.3) and (3.9) to be equal both to δ_{b_i} times an N_j -th root of unity, and $\delta_{b_j} \cdot e^{i\pi/2 b_i \cdot b_j}$ times an N_i -th root of unity:

$$\begin{aligned} c \begin{bmatrix} b_i \\ b_j \end{bmatrix} &= \delta_{b_i} e^{\frac{2\pi i}{N_j} n} \\ &= \delta_{b_j} e^{\frac{i\pi}{2} b_i \cdot b_j} e^{\frac{2\pi i}{N_i} m} \end{aligned} \quad (3.10)$$

Consistency therefore requires that:

$$\frac{N_i}{g_{ij}} n - \frac{N_j}{g_{ij}} m = c_{ij} \mod \left(\frac{N_i N_j}{g_{ij}} \right) \quad (3.11a)$$

where g_{ij} is the greatest common divisor of N_i and N_j , and

$$c_{ij} = \frac{1}{4} N_{ij} b_i \cdot b_j + \frac{1}{4} N_{ij} (\delta_{b_j} - \delta_{b_i}) \quad (3.11b)$$

is an integer thanks to condition (3.6). Using the fact that N_i/g_{ij} and N_j/g_{ij} are relatively prime, it is straightforward to show that Eq. (3.11a) has exactly g_{ij} solutions for the pair (n,m) where n and m are integers modulo N_j and N_i respectively. Consequently there are g_{ij} distinct choices of the coefficients $c_{[b_j^i]}$, for $i \neq j$.

The diagonal coefficients $c_{[b_i^i]}$ for $i \neq 1$ can be expressed in terms of the $c_{[b_1^i]}$ through Eqs. (3.8), (3.5) and (3.2) as follows

$$c \left[\begin{smallmatrix} b_i \\ b_i \end{smallmatrix} \right] = e^{\frac{i\eta}{4} (b_i^2 + 1^2)} c \left[\begin{smallmatrix} b_i \\ b_1 \end{smallmatrix} \right]^{\frac{1}{2} N_1} \quad (3.12)$$

We have here used the fact that $\frac{1}{2}N_1$ is necessarily odd, because $\frac{1}{2}N_1 b_1 = 1$ implies periodic boundary conditions for the real fermions ψ^μ . That Eq. (3.12) is always compatible with (3.9) and (3.10) is not obvious, but can be proved with some algebra, and using also the facts that $\delta_{b_1} = e^{i\pi/4} 1.1$, and $\delta_{b_i} = +1$ whenever N_i is odd. Finally, the coefficient $c_{[b_1^1]}$ is determined by (3.9) up to a sign

$$c \left[\begin{smallmatrix} b_1 \\ b_1 \end{smallmatrix} \right] = \pm e^{\frac{i\eta}{4} b_1^2} \quad (3.13)$$

Let us now extend the assignment of coefficients from the basis to arbitrary pairs of elements in $\mathbb{E}$: $\alpha \equiv \sum k_i b_i$ and $\beta \equiv \sum \ell_i b_i$. Note that the above decompositions are unique if k_i and ℓ_i are in $[0, N_i)$. Using the factorization and transposition properties (3.2) and (2.8b) we find:

$$c \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] = \sum_{\alpha}^{\sum \ell_i - 1} \sum_{\beta}^{\sum k_i - 1} e^{-i\eta r(\alpha) \cdot \beta} \cdot \prod_{i,j} \left(c \left[\begin{smallmatrix} b_i \\ b_j \end{smallmatrix} \right]^{k_i \ell_j} \right) \quad (3.14)$$

where β is here the usual unreduced vector sum $\sum \lambda_i b_i$ (see Section 2), and the "reduction vector" $r(\alpha)$ is defined as the difference of the vector and its reduced representative:

$$2 r(\alpha) \equiv \alpha - [\alpha] \quad (3.15)$$

and hence has integer components. The existence of a consistent string theory would be guaranteed if we verify that the above explicitly constructed coefficients satisfy the modular invariance conditions (2.8a,b,c), which are in fact equivalent to (3.2), (2.8b) and (3.8).

Since this is straightforward, we will only outline the result here. Firstly, the factorization and transposition properties, Eqs. (3.2) and (2.8b) hold^{*)} if and only if, in addition to the previously derived conditions on the coefficients of basis elements, we also have

$$r(\alpha) \cdot r(\beta) = \text{integer} \quad (3.16)$$

for any vector sums α and β . This is automatically satisfied if all world-sheet fermions diagonalizing the boundary conditions are complex. If some of them are real [recall that real fermions contribute to the inner product with a factor $\frac{1}{2}$, cf. Eq. (2.9)], the above condition simply means that the number of real fermions which are simultaneously periodic under any four boundary conditions b_i, b_j, b_k and b_λ is even¹⁰⁾. From (3.16) we find

$$r(\alpha) \cdot \{ r(\alpha) - 1 \} = 0 \text{ mod } 2$$

using also the fact that every term in the above inner product is of the form $I(I-1)$, and hence even. This is all one needs to show that the modular invariance condition (3.8) is also satisfied, and thus there is no further obstruction to consistently assigning coefficients to pairs of elements of $\mathcal{E}$.

We summarize the analysis of this section in the following theorem, which gives the rules for constructing all consistent fermionic string theories, with mutually commuting rational boundary conditions:

*) To prove this we use among other things the fact that

$$r(\alpha + \beta) \equiv r(\alpha) + r(\beta) + r([\alpha] + [\beta])$$

Theorem

A) To any consistent fermionic string theory there corresponds a finite additive group of vectors of boundary conditions

$$\Xi \simeq \mathbb{Z}_{N_1} \oplus \cdots \oplus \mathbb{Z}_{N_k}$$

generated by a basis $\{b_1, \dots, b_k\}$ which we may choose canonical, i.e.

$$(A1) \quad \sum m_i b_i = 0 \iff m_i = 0 \bmod N_i \quad \forall i$$

and which obeys the following conditions:

$$(A2) \quad \frac{1}{2} N_i b_i = \mathbb{1}$$

$$(A3) \quad N_{ij} b_i \cdot b_j = 0 \bmod 4$$

with N_{ij} the least common multiple of N_i and N_j ,

$$(A4) \quad N_i b_i^2 = 0 \bmod 8 \quad \text{if } N_i \text{ is even,}$$

(A5) the number of real fermions which are simultaneously periodic under any four boundary conditions b_i, b_j, b_l, b_m is even, and finally

(A6) the boundary condition-matrix corresponding to each b_i is an automorphism of the Lie algebra(s) defining the world-sheet supercharge(s).

B) For any such group Ξ , there exist $2 \prod_{i>j} g_{ij}$ consistent string theories, where g_{ij} is the greatest common divisor of N_i and N_j . These correspond to the g_{ij} different choices for every coefficient $c \left[\begin{smallmatrix} b_i \\ b_j \end{smallmatrix} \right]$, for $i > j$, such that:

$$(B1) \quad c \left[\begin{smallmatrix} b_i \\ b_j \end{smallmatrix} \right] = \delta_{b_i} e^{\frac{2\pi i}{N_j} n} = \delta_{b_j} e^{\frac{i\pi}{2} b_i \cdot b_j} e^{\frac{2\pi i}{N_i} m}$$

with n and m (i and j -dependent) integers, and to the two possible choices for

$$(B2) \quad c \begin{bmatrix} b_i \\ b_i \end{bmatrix} = \pm e^{\frac{i\eta}{2} b_i^2}$$

Any such choice can be uniquely extended to all pairs of elements of Ξ , using the properties:

$$(B3) \quad c \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = e^{\frac{i\eta}{4} (\alpha \cdot \alpha + 1 \cdot 1)} c \begin{bmatrix} \alpha \\ b_i \end{bmatrix}^{N_{1/2}}$$

$$(B4) \quad c \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = e^{\frac{i\eta}{2} \alpha \cdot \beta} c \begin{bmatrix} \beta \\ \alpha \end{bmatrix}^*$$

$$(B5) \quad c \begin{bmatrix} \alpha \\ \beta + \gamma \end{bmatrix} = \delta_\alpha c \begin{bmatrix} \alpha \\ \beta \end{bmatrix} c \begin{bmatrix} \alpha \\ \gamma \end{bmatrix}$$

The full one-loop amplitude is given by a sum over all spin-structures $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, corresponding to pairs of elements in Ξ , with weight $c \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$.

The above theorem generalizes the analysis of Ref. 10). In this reference, fermions were only allowed to be periodic or antiperiodic, Ξ was a direct sum of Z_2 factors, the elements of Ξ were characterized by the sets of periodic fermions, and vector addition was the corresponding sets' symmetric difference.

4. - MODEL BUILDING

The Hilbert spaces of the string theories constructed in the previous section can be read off the full one-loop vacuum to vacuum amplitude:

$$Z_1 = \int \frac{d\tau d\bar{\tau}}{(Im\tau)^2} \sum_{\alpha, \beta \in \Xi} c \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \cdot \left\{ \frac{1}{(Im\tau)(\eta\bar{\eta})^2} \cdot \prod_{\substack{\text{transv.} \\ \text{real}}} \frac{\theta^{\frac{1}{2}} \begin{bmatrix} \alpha(f) \\ \beta(f) \end{bmatrix}}{\eta^{\frac{1}{2}}} \cdot \prod_{\text{complex}} \frac{\theta \begin{bmatrix} \alpha(f) \\ \beta(f) \end{bmatrix}}{\eta} \right\} \quad (4.1)$$

with η the Dedekind eta-function^{*)}. The term in square brackets equals

$$\text{Tr}_{\mathcal{H}_\alpha} \left(e^{in\beta F_\alpha} e^{in\tau H_\alpha} \right) \quad (4.2)$$

Here H_α is the Hamiltonian in the Hilbert space sector $\mathcal{H}_\alpha$, in which each fermion f picks up a phase $-e^{i\pi\alpha(f)}$ when parallel transported around the string; consequently f has oscillator frequencies:

$$\frac{1+\alpha(f)}{2} + \text{integer}$$

while f^* has frequencies:

$$\frac{1-\alpha(f)}{2} + \text{integer}$$

The operator βF_α in (4.2) is

$$\beta F_\alpha \equiv \left\{ \sum_{\text{real+complex left}} - \sum_{\text{real+complex right}} \right\} \beta(f) F_\alpha(f)$$

where $F_\alpha(f)$ is the fermion number operator that counts f once and, if complex, f^* minus once. Our definition of the Θ -function, Eq. (2.6), dictates that $F_\alpha(f)$ is normal ordered so as to vanish on the vacuum $|0\rangle_\alpha$ of the sector $\mathcal{H}_\alpha$, assuming it is not degenerate. When some complex fermion f is periodic, our convention (2.6) implies that $F_\alpha(f)$ takes the values $n = 0$ and $n = -1$ on the components of the vacuum annihilated by the zero modes f_0 and f_0^* respectively.

Using (4.2) and the factorization Eq. (3.2), we may rewrite the one-loop amplitude as follows:

$$\begin{aligned} Z_1 = \int \frac{d\tau d\bar{\tau}}{(Im\tau)^2} \sum_{\alpha \in \Xi} \delta_\alpha \text{Tr} \left\{ \prod_{b_i} \left(1 + \delta_\alpha c \begin{bmatrix} \alpha \\ b_i \end{bmatrix} e^{in_{b_i} F_\alpha} + \right. \right. \\ \left. \left. + \left\{ \delta_\alpha c \begin{bmatrix} \alpha \\ b_i \end{bmatrix} e^{in_{b_i} F_\alpha} \right\}^2 + \dots + \left\{ \delta_\alpha c \begin{bmatrix} \alpha \\ b_i \end{bmatrix} e^{in_{b_i} F_\alpha} \right\}^{N_i-1} \right) e^{in\tau H_\alpha} \right\} \end{aligned} \quad (4.3)$$

Therefore, the Hilbert space of string states contributing to the one-loop amplitude is

*) In our notation we have suppressed the fact that left- and right-movers make complex-conjugate contributions to the amplitude.

$$\mathcal{H} = \bigoplus_{\alpha \in \Xi} \prod_{i=1}^k \left\{ e^{i n b_i F_\alpha} = \delta_\alpha c \left[\begin{smallmatrix} \alpha \\ b_i \end{smallmatrix} \right]^* \right\} \mathcal{H}_\alpha \quad (4.4)$$

where the expression in curly brackets stands for the operator that projects out those states in the sector $\mathcal{H}_\alpha$ that satisfy $e^{i n b_i F_\alpha} = \delta_\alpha c \left[\begin{smallmatrix} \alpha \\ b_i \end{smallmatrix} \right]^*$. This is a generalization to arbitrary $Z_{N_1} \times \dots \times Z_{N_K}$ of the $Z_2 \times \dots \times Z_2$ GSO-type³⁰⁾ projections of Ref. 10). Note that the factor δ_α in Eq. (4.3) ensures the spin-statistics connection, since all the states in a sector $\mathcal{H}_\alpha$ are space-time fermions or bosons according to whether the ψ^μ are periodic or antiperiodic.

The mass of a string state in the sector $\mathcal{H}_\alpha$ is given by the zero-moment Virasoro gauge conditions³¹⁾:

$$\begin{aligned} M^2 &= -c_L + \frac{1}{8} \alpha_L \cdot \alpha_L + \sum_{\text{left-movers}} (\text{frequencies}) \\ &= -c_R + \frac{1}{8} \alpha_R \cdot \alpha_R + \sum_{\text{right-movers}} (\text{frequencies}) \end{aligned} \quad (4.5)$$

where α_L and α_R are the pieces of the vector α corresponding to the left and right movers respectively, $c_L(c_R) = \frac{1}{2}$ or 1 according to whether world-sheet supersymmetry is or is not realized among the left (right) movers, and the summation is over all the oscillators which act on the vacuum $|0\rangle_\alpha$ to create the state in question. Note in particular that there exists a massless state $|\mu, \nu\rangle$ in the sector $\mathcal{H}_0$ (all fermions antiperiodic), where $|\mu, \nu\rangle = (\partial X)_1^\mu (\bar{\partial} X)_1^\nu |0\rangle_0$ in the bosonic, $\phi_{\frac{1}{2}}^\mu (\bar{\partial} X)_1^\nu |0\rangle_0$ in the heterotic and $\phi_{\frac{1}{2}}^\mu \psi_{\frac{1}{2}}^\nu |0\rangle_0$ in the type-II supersymmetric strings. This state which includes the graviton, dilaton and two-index antisymmetric tensor field survives all projections because $\delta_0 c \left[\begin{smallmatrix} 0 \\ b_i \end{smallmatrix} \right]^* = \delta_{b_i}$ always.

Let us now describe a specific example of a fermionic string theory, which will serve to illustrate the rules explained above. We consider the heterotic string, with supercharge

$$T_F = \psi^\mu \partial X_\mu + \sum_{i=1}^6 \xi^i y^i \omega^i \quad (4.6)$$

which equals expression (2.1) for the group $G = SU(2)^6$. The boundary conditions will be diagonal in the basis $\{\psi^\mu, \xi^i, y^i, \omega^i, \phi^a\}$ where ϕ^a are 22 complex right-moving fermions, while all left-moving fermions are real. Consider now a group Ξ generated by $\{b_1 = 1, b_2\}$ with:

$$b_2 = \left[\underbrace{1 \dots 1}_{\psi^r, \xi^i} \underbrace{0 \dots 0}_{y^i, \omega^i} ; \underbrace{\frac{1}{3} \dots \frac{1}{3}}_{\phi^1 \dots \phi^5} \vec{A} \right] \quad (4.7)$$

where the vector $\vec{A}$ that specifies the boundary conditions of the $\phi^6 \dots \phi^{22}$ is chosen so that $\vec{A}^2 = 19/9$, $N_2(\vec{A} \cdot 1 - 5/3) = 0 \pmod{4}$, and none of its components equals $n/3$. Conditions (A3) and (A4) of the Theorem are satisfied, since $b_2 \cdot b_2 = 4/3$ and N_2 is necessarily a multiple of 6. Consider now the massless string states in the sector $\mathcal{H}_0$. In the absence of b_2 , this would contain, besides the graviton and complex dilaton, the gauge bosons of $O(44)$:

$$\psi_{1/2}^r \phi_{1/2}^a \phi_{1/2}^b |0\rangle_0, \quad \psi_{1/2}^r \phi_{1/2}^{a*} \phi_{1/2}^{b*} |0\rangle_0, \quad \psi_{1/2}^r \phi_{1/2}^a \phi_{1/2}^{b*} |0\rangle_0$$

as well as 18 scalars in the adjoint of $O(44)$, obtained by replacing $\phi_{1/2}^\mu$ by one of the other left-moving fermionic oscillators. The b_2 -projection $e^{i\pi b_2 F_0} = \delta_0 c \begin{bmatrix} 0 \\ b_2 \end{bmatrix}^* = -1$, breaks $O(44)$ down to $U(5) \times G(A)$ where the group G depends on the precise choice of the vector $\vec{A}$. The $U(5)$ gauge bosons that survive this projection are:

$$\psi_{1/2}^r \phi_{1/2}^a \phi_{1/2}^{b*} |0\rangle_0 \quad a, b = 1, \dots, 5$$

Among the scalars carrying $U(5)$ quantum numbers, on the other hand, only the six adjoint multiplets $\xi_{1/2}^i \phi_{1/2}^a \phi_{1/2}^{b*} |0\rangle_0$ survive this projection.

Consider next the sectors $\mathcal{H}_{b_2}$ and $\mathcal{H}_{-b_2}$, which give the massless states:

$$\phi_{2/3}^a |0\rangle_{b_2}, \quad \phi_{1/3}^{a*} \phi_{1/3}^{b*} |0\rangle_{b_2} \quad (4.8a)$$

$$\phi_{2/3}^{a*} |0\rangle_{-b_2}, \quad \phi_{1/3}^a \phi_{1/3}^b |0\rangle_{-b_2} \quad (4.8b)$$

for $a, b = 1, \dots, 5$, in the representations $\underline{5}$, $\overline{\underline{5}}$, $\underline{10}$ and $\overline{\underline{10}}$ of $SU(5)$. These are space-time fermions, as well as spinors of an internal $O(6)$ group, since $|0\rangle_{b_2}$ and $|0\rangle_{-b_2}$ should represent the Clifford algebra of eight left-moving zero modes. Let us examine now the effect of the b_2 -projection:

$$e^{i\pi b_2 F_{b_2}} = -c \begin{bmatrix} \pm b_2 \\ b_2 \end{bmatrix}^*$$

on these states. First note that from Eq. (B3)

$$\begin{aligned} c \begin{bmatrix} b_2 \\ b_2 \end{bmatrix} &= -e^{\frac{i\pi}{4} b_2 \cdot b_2} c \begin{bmatrix} b_2 \\ 1 \end{bmatrix} \\ &= -\varepsilon e^{i\pi/3} \end{aligned}$$

where $\varepsilon = c \begin{bmatrix} b_2 \\ 1 \end{bmatrix}$ is an arbitrarily chosen sign. Next, from the transposition property (B4), we find that

$$c \begin{bmatrix} -\alpha \\ \beta \end{bmatrix} = e^{-i\pi r(-\alpha) \cdot \beta} c \begin{bmatrix} \alpha \\ \beta \end{bmatrix}^* \quad (4.9)$$

where the reduction vector $r(-\alpha) \equiv \frac{1}{2}(-\alpha - [-\alpha])$ has entries equal to -1 for periodic fermions, and 0 otherwise. Putting all this together we find that the helicity times $O(6)$ chirality of the fermions is $h\Gamma_6 = \varepsilon$ for the $\underline{5}$ and $\overline{\underline{5}}$, and $h\Gamma_6 = -\varepsilon$ for the $\underline{10}$ and $\overline{\underline{10}}$. There are therefore four families and four antifamilies of $SU(5)$. It is straightforward to check that these states also survive the $b_1 = 1$ projection.

To make this model chiral, we may now introduce a third basis element in E :

$$b_3 = \left[\underbrace{1 \dots 1}_{\psi^i, y^i} \underbrace{0 \dots 0}_{\xi^i, \omega^i} ; \underbrace{0 \dots 0}_{\phi^1 \dots \phi^{10}} \underbrace{1 \dots 1}_{\phi^{11} \dots \phi^{22}} \right] \quad (4.10)$$

The vector $\vec{A}$ should now be chosen so that $N_2 b_2 \cdot b_3 = 0 \pmod{4}$. The effect of the b_3 projection in the sector $\mathcal{H}_0$ is to eliminate all scalars in the adjoint of $SU(5)$. With regard to the fermions the above projection sets

$$h = -c \begin{bmatrix} \pm b_2 \\ b_3 \end{bmatrix}^*$$

From (4.9) we find

$$c \begin{bmatrix} -b_2 \\ b_3 \end{bmatrix} = -c \begin{bmatrix} b_2 \\ b_3 \end{bmatrix}^* = -\tilde{\varepsilon}$$

where since $2b_3 = 0$, $\tilde{\varepsilon}$ is a pure sign. Therefore the states surviving this projection in the sectors $\mathcal{H}_{b_2}$ and $\mathcal{H}_{-b_2}$ have opposite helicities, and form four

chiral families of SU(5). It should be stressed that the two helicity components of a Weyl spinor, transforming in complex-conjugate representations, come from two opposite sectors.

Finally it is possible to render the above model space-time supersymmetric, by adding a fourth basis element¹⁰⁾

$$b_4 = \left[\underbrace{1 \dots 1}_{\psi^\mu, \xi^1, \xi^2, y^3, y^4, \omega^5, \omega^6} \quad \underbrace{0 \dots 0}_{\text{rest}} \right] \quad (4.11)$$

This gives N = 1 supersymmetry but has also the effect of reducing the number of families by two.

The above simple examples serve to illustrate some generic features of the fermionic string model-building, as well as the advantages of generalizing the periodic and antiperiodic boundary conditions. Firstly, since fermions are now complex, unitary gauge groups can be directly obtained, while with real fermions one had to combine adjoint and spinor representations of orthogonal groups. For instance, in this latter case, the adjoint of SU(5) should be obtained from the adjoint, spinor and spinor bar of O(6)×O(2). Regarding the matter fields, the generalized boundary conditions give directly arbitrary antisymmetric representations of unitary groups, in addition to the vector and spinor representations that one obtains with real fermions.

A more serious problem in the old construction was that the chiral fermions transformed under a gauge group of rank 8 or more, which gave rise to a horizontal gauge symmetry or to extra U(1) factors. This difficulty is apparently resolved in the present case, since as we saw in the above example "observable" and "hidden" worlds with the desired rank are easier to obtain. However, one must make sure that massless particles mediating non-gravitational interactions between "observable" and "hidden" sectors do not arise in any of the sectors of the string theory; in general this is a non-trivial exercise. Finally we should point out that in the above-constructed chiral model there are no scalar particles in the adjoint representation which could be used to break SU(5). It looks as if it is a general property that adjoint scalars are incompatible with N = 1 supersymmetry and/or the existence of chiral fermions. If this were the case, the usual perturbative ways of breaking grand unified groups would not be allowed.

5. - ON THE EXISTENCE OF MASSLESS FERMIONS

The requirement that the string theory contains massless space-time fermions, and in particular unbroken supersymmetry at the tree level, puts an important additional constraint besides the consistency conditions discussed up to now^{32),6),14),33)}. For instance, in the case of real periodic or antiperiodic world-sheet fermions it can be shown that only a particular realization of world-sheet supersymmetry based on the group $SU(2)^6$ is allowed⁶⁾. This no-go "theorem" can be evaded, as we will now show, when one allows generalized boundary conditions; in fact all other realizations of world-sheet supersymmetry are also allowed, provided one includes in the group $\mathcal{E}$ of boundary conditions some special vectors with non-integer (but rational) components.

Let us recall that in a sector α giving space-time fermions, the ψ^μ and hence also the supercharge

$$T_F = \psi^\mu \gamma_\mu \chi + f_{IJK} \chi^I \chi^J \chi^K \quad (5.1)$$

should be periodic^{*)}. Thus, as discussed in Section 2, the matrix $L(\alpha)$ specifying the boundary conditions of the χ^I :

$$\chi^I \longrightarrow L(\alpha)^I_J \chi^J \quad (5.2)$$

should be an automorphism of the semi-simple Lie group G whose structure constants enter in the definition of the supercharge. The mass condition, Eq. (4.5), takes the form

$$M^2 = -\frac{1}{2} + \frac{1}{8} \left(\frac{d-2}{2} + \frac{1}{2} \text{Tr} \left[\frac{1}{i\pi} \log(-L) \right]^2 \right) + \sum (\text{frequencies}) \quad (5.3)$$

where the first factor inside the parentheses comes from the contribution of the $(d-2)$ transverse periodic ψ^μ , and the prescription for the multi-valued logarithm follows from definitions (2.4a,b) and Eq. (4.5) when L is diagonalized; it is in fact equivalent to minimizing the trace over all possible sheets. For further reference we will give a name to this quantity:

*) Our discussion in this section applies to the supersymmetric left-moving sector of the heterotic string, or to one of the two supersymmetric sectors of a type II string.

$$\langle L \rangle = \min_{\text{sheets}} -\frac{1}{n^2} \text{Tr} [\log(-L)]^2 \quad *) \quad (5.4)$$

By letting L range over all elements of the automorphism group $\text{Aut}(G)$, we will find that $\langle L \rangle$ at its minimum is a third of the dimension of G :

$$\min_{L \in \text{Aut}(G)} \langle L \rangle = \frac{d_G}{3} \quad (5.5)$$

Since the anomaly cancellation dictates that $d_G = 3(10-d)$, the above value is precisely what is needed to make the vacuum of the sector α massless. We will in fact prove (5.5) for the three groups $\text{SU}(2)^6$, $\text{SU}(4) \times \text{SU}(2)$ and $\text{SU}(3) \times \text{O}(5)$ which are of interest to four-dimensional fermionic strings, and classify the corresponding automorphisms that lead to theories with massless space-time fermions.

Let us first recall⁶⁾ why the automorphisms L obeying (5.5) cannot be diagonalized in a real basis except possibly for the group $\text{SU}(2)^6$; this is the reason why we had to use this realization of supersymmetry when we only allowed periodic or antiperiodic world-sheet fermions. Indeed if the eigenvalues of L are real (± 1), L is an automorphism if and only if periodic fermions correspond to the generators of a subgroup H , such that G/H is a symmetric space. In this case $\langle L \rangle = d_H$ which is always bigger than $d_G/3$ except possibly for the group $\text{SU}(2)^6$.

We now proceed to the generic case. The group of automorphisms of the Lie algebra of a semi-simple group G , is equal to the product of the inner automorphisms (which are the elements of G in the adjoint representation), and a discrete factor group of symmetries of the associated Dynkin diagram³⁴⁾:

$$\text{Aut}(G) \simeq \left(\frac{G}{\text{center}} \right) \times \Gamma$$

The discrete factor Γ is the permutation group P_6 for $\text{SU}(2)^6$, a Z_2 charge conjugation for $\text{SU}(3)$ and $\text{SU}(4)$, and trivial for $\text{O}(5)$.

Let us begin with the group $\text{SU}(2)^6$. The general automorphism is a block-diagonal matrix, corresponding to an inner automorphism of each of the six $\text{SU}(2)$'s, times an arbitrary permutation of blocks. Resolving the permutation into commuting cycles, we note that the automorphism-matrix becomes superblock-diagonal; each superblock corresponds to a given cycle and can be separately diagonalized. Consider then a cyclic permutation $(1 \rightarrow 2 \rightarrow \dots \rightarrow n \rightarrow 1)$, times a block diagonal matrix

*) Note that $\langle L \rangle$ is left invariant by similarity transformations since $\log(-ULU^{-1}) = U[\log(-L)]U^{-1}$.

$$L(n) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \\ 1 & & & & 0 \end{pmatrix} \begin{pmatrix} A_1 & & & & \\ & A_2 & & & \\ & & \ddots & & \\ & & & A_n & \end{pmatrix} \quad (5.6)$$

where A_i are arbitrary 3×3 rotation matrices [in the adjoint of $SU(2)$] and 1 is the identity. By a similarity transformation we may bring this matrix in the form

$$U^{-1} L(n) U = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \\ V^{-1} A_1 \dots A_n V & & & & 0 \end{pmatrix} \quad (5.7a)$$

with

$$U = \begin{pmatrix} 0 & & & & V \\ A_1 \dots A_n V & 0 & & & \\ & A_2 \dots A_n V & \ddots & & \\ & & \ddots & 0 & \\ & & & A_n V & 0 \end{pmatrix} \quad (5.7b)$$

and V a unitary matrix that diagonalizes the product $A_1 \dots A_n$. We may therefore write

$$V^{-1} A_1 \dots A_n V = e^{i\pi\theta J_3} = \begin{pmatrix} e^{i\pi\theta} & & \\ & e^{-i\pi\theta} & \\ & & 1 \end{pmatrix} \quad (5.7c)$$

where the three eigenvalues correspond to the three generators $J_{\pm}$ and J_3 in a Cartan-Weyl basis. Now if we label the $3n$ fermions corresponding to the generators of $SU(2)^n$ by $\chi_{\pm}^i, \chi_3^i$ ($i=1, \dots, n$) the matrix (5.7a) describes the composition of the following two transformations:

$$\chi_{\pm}^i \rightarrow e^{\pm i\pi\theta} \chi_{\pm}^i, \quad \chi_3^i \rightarrow \chi_3^i, \quad \chi_{\pm,3}^i \rightarrow \chi_{\pm,3}^i \quad (i=2, \dots, n) \quad (5.8a)$$

$$\chi_{\pm,3}^1 \rightarrow \chi_{\pm,3}^2 \rightarrow \dots \rightarrow \chi_{\pm,3}^n \rightarrow \chi_{\pm,3}^1 \quad (5.8b)$$

Therefore the $3n$ eigenvalues λ are all n -th roots of 1, $e^{i\pi\theta}$ and $e^{-i\pi\theta}$ with corresponding eigenvectors:

$$u_3(\lambda) = \lambda^{-1} \chi_3^1 + \lambda^{-2} \chi_3^2 + \dots + \lambda^{-n} \chi_3^n \quad (\lambda^n = 1) \quad (5.9a)$$

$$u_{\pm}(\lambda) = e^{\mp i n \theta} \lambda^{-1} \chi_{\pm}^1 + \lambda^{-2} \chi_{\pm}^2 + \dots + \lambda^{-n} \chi_{\pm}^n \quad (\lambda^n = e^{\pm i n \theta}) \quad (5.9b)$$

The contribution to the mass subtraction Eq. (5.3) is therefore

$$\begin{aligned} \langle L(n) \rangle &= \sum_{m=1}^n \left(\frac{2m}{n} - 1 \right)^2 + 2 \sum_{m=1}^n \left(\frac{2m}{n} - 1 - \frac{\theta}{n} \right)^2 \\ &= n + \frac{2}{n} (1-\theta)^2 \end{aligned} \quad (5.10)$$

where we must choose $0 < \theta < 2$, following the prescription (5.4). Expression (5.10) is minimized for $\theta = 1$, its value at the minimum being n . The components of the vector corresponding to this boundary condition [cf. Eq. (2.4a)] take the values $(2m/n)-1$ and $(2m-1/n)-1$, and cannot all be integer for a non-trivial cycle ($n > 1$).

Going back now to the original automorphism $L = p.g$ of the entire $SU(2)^6$, where p is a permutation and g an inner automorphism, we have shown that for any given p decomposable into cycles of lengths $n_1, \dots, n_k$ there exists a unique g_p that minimizes the mass subtraction

$$\langle p.g_p \rangle = n_1 + \dots + n_k = 6 \quad (5.11)$$

This is precisely the value announced in (5.5). Note that there are 11 distinct vectors of boundary conditions yielding this minimum mass subtraction; they correspond to the 11 conjugacy classes (i.e., choices of the cycle-lengths) of P_6 . Only the one that corresponds to the identity-conjugacy-class has integer components. It is the only vector allowed if world-sheet fermions are real.

We now proceed to the other two groups of interest, namely $SU(3) \times SO(5)$ and $SU(4) \times SU(2)$. We first examine the inner automorphisms for every factor group separately. A general group element can always be written in the form

$$g = \exp \left(i \pi \vec{\theta} \cdot \vec{H} \right) \quad (5.12)$$

where H_i are the mutually commuting Cartan generators of a Cartan-Weyl basis $\{H_i, E_a\}$. In the adjoint representation g is diagonal, with eigenvalues 1 for all

H_i , and $e^{i\pi\vec{\theta}\cdot\vec{a}}$ for each E_a , where $\vec{a}$ is the corresponding root vector. We will denote the fermions in this basis by χ_i and χ_a , the χ_i are real while $\chi_a^* = \chi_{-a}$. The contribution to the mass subtraction following the prescription (5.4) is

$$\langle g \rangle = r + \min_{\ell_a \in \mathbb{Z}} \sum_a \left(\vec{\theta} \cdot \vec{a} - 1 + 2\ell_a \right)^2 \quad (5.13)$$

where r is the rank of the group. To find the lowest mass subtraction we must therefore minimize the above expression over all vectors $\vec{\theta}$.

For instance, in the case of $SU(3)$ whose root diagram is shown in Fig. 1a, (5.13) becomes

$$\begin{aligned} \langle g \rangle_{SU(3)} = 2 + \min_{\ell_1, \ell_2, \ell_3 \in \mathbb{Z}} \left\{ 2 \left(\theta_1 - 1 + 2\ell_1 \right)^2 + 2 \left(\frac{1}{2} \theta_1 + \frac{\sqrt{3}}{2} \theta_2 - 1 + 2\ell_2 \right)^2 + \right. \\ \left. + 2 \left(-\frac{1}{2} \theta_1 + \frac{\sqrt{3}}{2} \theta_2 - 1 + 2\ell_3 \right)^2 \right\} \end{aligned} \quad (5.14)$$

A minimum of this expression is at $(\theta_1, \theta_2) = (2/3, 2/\sqrt{3})$, and it is equal to $2+2/3$ in accordance with Eq. (5.5). The corresponding boundary condition is

$$\chi_i \rightarrow \chi_i$$

and

$$\chi_a \rightarrow -e^{i\pi/3} \chi_a$$

for three complex χ_a that correspond to root vectors with 120° relative angles. This is schematically depicted by labelling the root diagram as shown in Fig. 1a. The minimizing boundary condition is described by the vector:

$$S_{SU(3)} = \left(1, 1 ; \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \quad (5.15)$$

and it is unique up to reflections and rotations that leave the (unlabelled) root-diagram unchanged; in this case the only additional boundary condition is $-S_{SU(3)}$, which is obtained by a 60° rotation of the root diagram.

The analysis for the groups $SO(5)$ and $SU(4)$ is similar, and is a straightforward exercise in analytic geometry. In Figs. 1b and 1c, we give, for the reader's convenience, the root diagram of $SO(5)$, and the simple roots

generating the three-dimensional root-diagram of $SU(4)^{34}$. For $SO(5)$ the minimal mass subtraction is $10/3$, and it is obtained for the boundary-condition-vector:

$$S_{SO(5)} = \left(1, 1 ; 0, \frac{1}{3}, -\frac{2}{3}, \frac{1}{3} \right) \quad (5.16a)$$

which is depicted by the labelled diagram in Fig. 1b. A 90° rotation yields the additional vector

$$S'_{SO(5)} = \left(1, 1 ; \frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3} \right) \quad (5.16b)$$

while all other reflections and rotations give back the same vectors or their negatives. For $SU(4)$, the minimized mass subtraction is $15/3$, and it is obtained for the boundary-condition-vectors:

$$S_{SU(4)} = \left(1, 1, 1 ; \frac{1}{2}, -\frac{1}{2}, 0, 0, \frac{1}{2}, -\frac{1}{2} \right) \quad (5.17a)$$

$$S'_{SU(4)} = \left(1, 1, 1 ; 0, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, 0 \right) \quad (5.17b)$$

$$S''_{SU(4)} = \left(1, 1, 1 ; \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{2} \right) \quad (5.17c)$$

or their negatives.

Finally we examine non-trivial outer automorphisms, which are not equal to a group element in the adjoint representation. Recall that $SU(2)$ and $SO(5)$ have only inner automorphisms and need not, therefore, be considered; for $SU(3)$ and $SU(4)$ on the other hand we have

$$\text{Aut}(G) \simeq (G/\text{center}) \times Z_2$$

It follows that the square of any automorphism L is necessarily an inner automorphism, and hence we have $L^2 = g^2$ for some inner automorphism g . Setting $L = Cg$ we therefore find: $CgC = g$, from which we easily deduce by diagonalizing C that:

$$C^2 = 1 \quad \text{and} \quad Cg = gC.$$

In other words, we may always write the generic automorphism of $SU(N)$ as the product of an involution C times an inner automorphism g commuting with C .

The involution C has eigenvalues ± 1 ; the Lie algebra elements with eigenvalue $+1$ must therefore generate a proper (if $C \neq 1$) maximal subgroup H of G . For $G = SU(3)$, H is either $SU(2) \times U(1)$, or $SO(3)$. In the first case it is straightforward to show that C , and hence also $L=Cg$, is an inner automorphism and these have been already examined above. We need only preoccupy ourselves therefore with the case $H = SO(3)$. Let then χ_+, χ_-, χ_3 be the three fermions corresponding to the generators of this maximal $SO(3)$, while $\tilde{\chi}_2, \tilde{\chi}_1, \tilde{\chi}_0, \tilde{\chi}_{-1}, \tilde{\chi}_{-2}$ correspond to the remaining generators of $SU(3)$ which transform in the 5

representation of $S(3)$. The involution C sends χ_i to χ_i and $\tilde{\chi}_j$ to $-\tilde{\chi}_j$, while the inner automorphism g commuting with C is necessarily an $S(3)$ rotation . Putting all this together we find the generic non-trivial outer automorphism of $S(3)$ in the form:

$$L = C \cdot g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{2i\theta} \end{pmatrix}$$

It is now straightforward to calculate the mass subtraction $\langle L \rangle$ and minimize over the angle θ ; the answer is $\langle L \rangle_{\min} = 8/3$ for $\theta_{\min} = \pm 1/6$ or $\pm 5/6$. To our list of boundary conditions yielding massless space-time spinors we must thus add the vectors :

$$S_{S(3)}^{\text{out}} = \left(1, 0 ; -\frac{5}{6}, \frac{1}{3}, \frac{1}{6} \right) \quad (5.18a)$$

$$S_{S(3)}^{\prime \text{out}} = \left(1, 0 ; \frac{1}{6}, \frac{1}{3}, -\frac{5}{6} \right) \quad (5.18b)$$

and their negatives , where here the entries correspond to the real fermions $\chi_3, \tilde{\chi}_0$ and the complex fermions $\chi_+, \tilde{\chi}_2, \tilde{\chi}_1$ in this order. We should point out that these non-trivial outer automorphisms cannot be diagonalized in a Cartan-Weyl basis , and consequently do not commute with the inner automorphism described by the vector (5.15) .

The analysis for the group $G=SU(4)$ is very similar; the maximal subgroup whose generators are left unchanged by the involution C is $H = SO(4) = SU(2) \times SU(2)$. The generators of $SU(4)$ transform in the $(3,1) + (1,3) + (3,3)$ representation of H ; accordingly we label the fifteen fermions, in a self explanatory notation, as follows: $\chi_I, \tilde{\chi}_i$ and $\hat{\chi}_{Ii}$, where the indices I and i range over the values $+, -, 3$. The general non-trivial outer automorphism can always be brought to the form $L=Cg$, with g an inner automorphism corresponding to two arbitrary $SU(2)$ rotations*: $g = \exp(i\theta J_3) \exp(i\tilde{\theta} \tilde{J}_3)$. The minimized mass subtraction is $\langle L \rangle_{\min} = 5$, and is obtained for the values $(\theta, \tilde{\theta}) = \pm(1/4, 1/4), \pm(1/4, -1/4), \pm(3/4, 3/4)$ and $\pm(3/4, -3/4)$. The corresponding boundary condition vectors are given by :

$$S_{SU(4)}^{\text{out}} = (1, 1, 0; \theta+1, \tilde{\theta}+1, \theta+\tilde{\theta}, \theta-\tilde{\theta}, \theta, \tilde{\theta}) \quad (5.19)$$

where here the entries correspond to the real fermions $\chi_3, \tilde{\chi}_3, \hat{\chi}_{33}$ and the complex $\chi_+, \tilde{\chi}_+, \hat{\chi}_{++}, \hat{\chi}_{+-}, \hat{\chi}_{+3}, \hat{\chi}_{3+}$, in this order, and they should as usual be reduced to lie in the interval $(-1, +1]$. This then completes our exhaustive list of all spin structure assignments yielding massless space-time spinors in the corresponding string theories. Note incidentally that our demonstration that $\langle L \rangle \geq d_G/3$ implies as a corollary that there are no negative-mass fermionic string excitations; this is of course to be expected for the consistency of string theories.

* These two rotations could have been followed by a transposition of the two $SU(2)$'s, an operation commuting with C . It is straightforward, though slightly tedious to show that in a different basis this can be always rewritten in a form where the transposition is absent. This other basis corresponds to a different embedding of the maximal $SO(4)$ in $SU(4)$.

We conclude with some comments: firstly, it is remarkable that the group-theoretic quantity $\langle L \rangle$, Eq. (5.4), takes always at its minimum the value $d_G/3$, which is precisely what is needed for the existence of massless fermionic string excitations. Although we only proved this here for the particular groups that interested us, it looks as if (5.5) is a general property; it might be interesting to try and prove this for all groups. Secondly, let us note that all automorphisms minimizing the mass-subtraction transform the world-sheet fermions with rational phases. This is both curious and fortunate, because it means that we can use them in string theories with finite E 's. The use of these boundary conditions in the construction of new superstring models, as well as their implications, will be reported elsewhere²¹⁾. Among other things, one should recheck the arguments of Ref. 14) which seem to exclude realistic type II models.

ACKNOWLEDGEMENTS

We thank L. Alvarez-Gaumé, C. de Callan, F. Delduc, J. Ellis, J. Lascoux, K. Narain, D. Olive, P. Sorba, T. Tomaras and particularly C. Kounnas for useful conversations. C.B. acknowledges the hospitality of the CERN Theory Group, where part of this work was completed.

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FIGURE CAPTIONS

Fig. 1a The root diagram of $SU(3)$; the numbers labelling the root vectors are the phases acquired by the corresponding fermions, for the particular automorphism minimizing the mass subtraction, as is explained in the text.

Fig. 1b The root diagram of $SO(5)$, labelled as for 1a.

Fig. 1c The three simple roots which generate the root diagram of $SU(4)$.

Fig. 1a

Fig. 1b

Fig. 1c