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A NEW APPROACH TO STRING QUANTIZATION
IN CURVED SPACE-TIMES

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A B S T R A C T

We develop a general scheme for solving the equations of motion and constraints of strings in curved space-times, both classically and quantum mechanically. We treat the space-time geometry exactly and the string excitations small as compared with the energy scales of the metric. [The perturbation (dimensionless) parameter is $g = \sqrt{(\pi\alpha')}/R_c$ where $\sqrt{\alpha'} = \ell_{pl}$ is the Planck length and R_c the typical curvature radius of the geometry.] This formalism is particularly well suited to properly consider strings in the context of quantum black holes and cosmology. As an illustration we apply it to de Sitter space-time and found the mass spectrum and vertex operator. The lower mass states are the same as in flat space up to corrections of order g^2 whereas heavy states deviate significantly from the linear Regge trajectories. We find a maximum (very large) value of order $1/g^2$ for the quantum number N and spin J of particles. The critical dimension for bosonic strings is found to be 25 in de Sitter space-time.

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If string theory would provide a theory of quantum gravity, it should give us a proper theory (not yet existent) for describing the ultimate state of quantum black holes and the initial (say very early) state of the Universe. That is, a theory describing the physics (and the geometry) at Planck energies and lengths. Till now gravity has not completely been incorporated in string theory: strings are more frequently formulated in a flat space-time. Gravity appears through massless spin two particles (graviton). One disposes only of partial results for strings in curved backgrounds¹⁻³⁾; these mainly concern the problem of consistency (validity of quantum conformal invariance) through the vanishing of the beta-functions. The non-linear quantum string dynamics in curved space-time has only been studied in the slowly-varying approximation where the field propagator is essentially taken as the flat-space Feynman's propagator. Clearly, such approximations are useless for the computation of physical quantities (finite parts) in strong curvature regimes where quantum gravity effects are important. Our aim is to properly understand strings in the context of quantum gravity. A first step in this program was the investigation by the present authors of string quantization in Rindler space-time (with application to Schwarzschild geometry) and the Hawking-Unruh effect in string theory [see Refs. 4) and 5)]. In this note we develop a general scheme to string quantization in curved space-times taking into account strong curvature effects of the geometry. In our method we start from an exact solution of the string equations in a given curved metric and then we construct a perturbative series around this solution. It must be noticed that the exact zero-order solution fully takes into account gravitational effects including those of the singularities of the geometry. Then, first and second order quantum fluctuations around this solution are computed. (Recall that in the background field method only flat space propagators are used). In our method, the quantum fluctuations lead to physical magnitudes like mass spectrum and scattering amplitudes which fully take into account the effects of the space-time geometry. Our formalism is particularly suitable to incorporate properly (and explicitly) string theory in cosmology and black holes. In this note we apply our method to de Sitter space. Application to general cosmological models and to the Schwarzschild geometry will be reported elsewhere.

The action for a string in an arbitrary D-dimensional Riemannian manifold is given by:

$$S = \frac{1}{2\pi\alpha'} \int d^2x \sqrt{g} g^{\mu\nu} G_{AB}(X) \partial_\mu X^A \partial_\nu X^B. \quad (1)$$

$$0 \leq \mu, \nu \leq 1, \quad 0 \leq A, B \leq D-1.$$

Here G_{AB} and $g_{\alpha\beta}$ stand for the space-time metric and the world sheet metric respectively. $(2\pi\alpha')^{-1}$ is the string tension. Extremizing with respect to the world sheet metric leads to the constraints

$$T_{\mu\nu} = G_{AB}(X) \left[\partial_\mu X^A \partial_\nu X^B - \frac{g_{\mu\nu}}{2} \partial_\sigma X^A \partial^\sigma X^B \right] \simeq 0 \quad (2)$$

Using the reparametrization invariance of the world sheet one can choose the metric $g_{\alpha\beta}$ in a conformal gauge. That is

$$g_{\mu\nu} = \rho(\sigma, \tau) \eta_{\mu\nu} \quad (3)$$

Thus, in this gauge, the equations of motion write

$$\partial^2 X^A + \Gamma_{BC}^A(X) \partial_+ X^B \partial_- X^C = 0 \quad (4)$$

where Γ_{BC}^A are the Christoffel connections associated to the metric G_{AB} and $x_\pm = \sigma \pm \tau$. The two non-trivial constraints read now

$$T_{\pm\pm} = G_{AB}(X) \partial_\pm X^A \partial_\pm X^B \simeq 0 \quad (5)$$

The main feature of strings propagating in a curved space-time is that the equations of motion [Eq. (3)] are non-linear in X_A , so right and left movers interact with each other and also with themselves. It must be noticed that purely left modes (or right modes) are exact solutions of Eq. (3), namely

$$X^A = \mathcal{X}^A(\sigma + \tau) \quad \text{or} \quad X^A = \mathcal{X}^A(\sigma - \tau) \quad (6)$$

When G_{AB} is the metric of a symmetric space, the equations of motion possesses an associated linear system and exact solutions can be constructed, by using the inverse scattering method [see for example Refs. 6) and 7)].

We propose in this note a general perturbative scheme to solve the equations of motion and constraints both classically and quantum mechanically. We start from an exact given solution of Eq. (3) and develop in perturbations around. A possible starting point is a solution for the centre of mass motion of the string $q_A(\tau)$ where

$$\ddot{q}_A(\tau) + \Gamma_{BC}^A(q) \dot{q}^B \dot{q}^C = 0 \quad (7)$$

The world sheet time variable τ is here identified with the proper time of the centre of mass trajectory. Another possibility is to take pure left (right) mover solutions [Eq. (6)]. Then we set

$$X_A(\sigma, \tau) = q_A(\sigma, \tau) + \eta_A(\sigma, \tau) + \xi_A(\sigma, \tau) + \dots \quad (8)$$

Here $q_A(\sigma, \tau)$ is an exact solution of Eq. (4) and η_A is a solution of the linearized perturbation around q_A . That is

$$\partial^2 \eta^A + \Gamma_{BC}^A(q) [\partial_- q^B \partial_+ + \partial_+ q^B \partial_-] \eta^C + \partial_C \Gamma_{BD}^A(q) \partial_+ q^B \partial_- q^D \eta^C = 0 \quad (9)$$

$\xi_A(\sigma, \tau)$ fulfils the second order perturbation equation around q_A :

$$\begin{aligned} \partial^2 \xi^A + \Gamma_{BC}^A(q) [\partial_- q^B \partial_+ + \partial_+ q^B \partial_-] \xi^C + \partial_C \Gamma_{BD}^A(q) \partial_+ q^B \partial_- q^D \xi^C = \\ - \Gamma_{BC}^A \partial_+ q^B \partial_- q^C - \frac{1}{2} \eta^D \eta^E \partial_{DE}^2 \Gamma_{BC}^A \partial_+ q^B \partial_- q^C \\ - \eta^D \partial_D \Gamma_{BC}^A (\partial_+ q^B \partial_- + \partial_- q^B \partial_+) \eta^C \end{aligned} \quad (10)$$

One can consider the higher order perturbations but in this note we will restrict ourselves to first and second orders.

It must be noticed that we are treating the space-time metric exactly and taking the string oscillations around its centre of mass $q_A(\tau)$ as perturbation. So our expansion corresponding to low excitations of the string as compared with the energy scales of the metric G_{AB} . For example, this method is exact in flat space-time. In a Schwarzschild geometry it will corresponds to an expansion in ω/M where ω is the frequency mode and M the black hole mass. Thus, our approximation applies to black hole masses larger than the string energy. In other words, this corresponds to a strong gravitational field expansion. This can be equivalently considered as an expansion in powers of $\sqrt{\alpha'}$. Actually, since $\alpha' = \ell_{pl}^2$ (ℓ_{pl} is the Planck length), the expansion parameter turns out to be the dimensionless constant

$$\frac{\gamma}{\sqrt{\pi}} \equiv \frac{\ell_{pl}}{R_c} \approx \frac{1}{\ell_{pl} M} \approx \frac{\omega}{M}$$

where the length R_c characterizes the curvature radius of the space-time under consideration and M its associated mass (the black hole mass in the Schwarzschild geometry, the mass of the Universe in a cosmological model). In most of the interesting situations one clearly has $g \ll 1$.

It must be noticed that even for small g , the metric and its derivatives may be very large in some region of the space-time. This shows that our method has a larger domain of applicability than the background field method where one must have everywhere

$$|\sqrt{\alpha'} \partial_c G_{AB}(x)| \ll 1$$

The first order equation (9) describes the interaction between the string modes and the curved space-time geometry. The interactions between the string modes themselves start to appear from the second order perturbation ξ^A [Eq. (10)].

The constraint equations (5) must also be expanded in perturbations. We find up to terms higher than the second order

$$\begin{aligned} T_{\pm\pm} = & G_{AB}(\eta) \partial_{\pm} \eta^A \partial_{\pm} \eta^B + 2 G_{AB}(\eta) \partial_{\pm} \eta^A \partial_{\pm} \xi^B + \eta^C \partial_C G_{AB} \partial_{\pm} \eta^A \partial_{\pm} \eta^B + \\ & + G_{AB}(\eta) \partial_{\pm} \eta^A \partial_{\pm} \xi^B + \frac{1}{2} \eta^C \eta^D \partial_{CD} G_{AB} \partial_{\pm} \eta^A \partial_{\pm} \eta^B + 2 G_{AB} \partial_{\pm} \eta^A \partial_{\pm} \xi^B + \\ & + 2 \eta^C \partial_C G_{AB} \partial_{\pm} \eta^A \partial_{\pm} \xi^B + \xi^C \partial_C G_{AB} \partial_{\pm} \eta^A \partial_{\pm} \eta^B \simeq 0 \quad (11) \end{aligned}$$

This defines the L_n generators as

$$T_{\pm\pm} = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} L_n e^{in(\sigma \pm \tau)} \quad (12)$$

In this second order approximation the constant operators L_n are bilinear in the fields η_A . [One must solve the linear equation (10) for ξ_A , expressing them as a bilinear functional of η_A . Inserting this expression for ξ_A in Eq. (11) give the generators L_n as operators bilinear in the η_A .] Closed solutions of ξ^A , η^A and L_n are given below for the de Sitter universe.

Let us now consider the case where the exact solution q_A describes the centre of mass motion. It fulfils besides Eq. (7)

$$m^2 = G_{AB}(\tau) \dot{\gamma}^A(\tau) \dot{\gamma}^B(\tau) \quad (13)$$

This defines the (mass)² of the string. The fluctuation equations (9) and (10) simplify in this case. One can expand in Fourier series

$$\gamma^A(\sigma, \tau) = \sum_{n \in \mathbb{Z}} e^{in\sigma} C_n^A(\tau) \quad (14)$$

and the C_n^A satisfy decoupled ordinary differential equations

$$\ddot{C}_n^A + m^2 C_n^A + 2 \Gamma_{BC}^A(\gamma) \dot{\gamma}^B \dot{C}_n^C + \dot{\gamma}^B \dot{\gamma}^C \partial_{BD} \Gamma_{BC}^A(\gamma) C_n^C = 0 \quad (15)$$

In this case, the constraint equations read to second order:

$$\begin{aligned} T_{\pm\pm} = m^2 \pm 2 G_{AB} \dot{\gamma}^A (\partial_{\pm} \gamma^B + \partial_{\pm} \xi^B) + G_{AB} \partial_{\pm} \gamma^A \partial_{\pm} \gamma^B + \\ + (\gamma^C + \xi^C) \partial_C G_{AB}(\gamma) \dot{\gamma}^A \dot{\gamma}^B + \frac{1}{2} \gamma^C \gamma^D \partial_{CD} G_{AB}(\gamma) \dot{\gamma}^A \dot{\gamma}^B + \\ \pm 2 \partial_C G_{AB}(\gamma) \gamma^C \dot{\gamma}^A \partial_{\pm} \gamma^B \simeq 0 \end{aligned} \quad (16)$$

here ξ^A is to be expressed in terms of the η_A through the solution of the linear equation (10). This procedure can be iterated to higher orders. One has only to solve ordinary linear differential equations at any step.

Non-physical degrees of freedom are to be eliminated by imposing the constraints [Eq. (16)]. In addition, the mass spectrum follows from the L_0 component of these constraints. It must be noticed that one must use the solution of the equation of motion of second order (ξ^A) in order to construct physically interesting $T_{\pm\pm}$. It is easy to check that this $T_{\pm\pm}$ is conserved order by order in the perturbation expansion. This perturbative scheme can be consistently extended to any order of perturbation and around any exact solution of the equations of motion.

As an application of our formalism we consider now a conformally flat space-time .

$$G_{AB}(X) = \Omega^2(X^0) \gamma_{AB} \quad (17)$$

where

$$\gamma_{AB} = \text{diag}(-1, +1, \dots, +1)$$

is the flat Minkowski metric.

The exact solution for the centre of mass motion Eq. (13) can be written in terms of quadratures:

$$q^i(\tau) - q^i(0) = p_i \int_0^\tau \frac{dz'}{\Omega^2(q^0(z'))} \quad (18)$$

$$\tau = \int_{q_0(0)}^{q_0(\tau)} \frac{\Omega(q_0) dq_0}{\sqrt{m^2 + n^2/\Omega(q_0)}}, \quad n^2 \equiv \sum_{i=1}^{D-1} n_i^2$$

Let us recall that the proper time of the geometry is given by

$$dt = \Omega(q^0) dq^0 = d\tau \sqrt{m^2 + n^2/\Omega(q_0)} \quad (19)$$

For $\Omega \rightarrow \infty$, we have

$$dt \approx m d\tau \quad (20)$$

Then, in the infinite radius limit, the proper time of the geometry is proportional to the string time τ . This identification is important in order to define the positive frequency (particle) states of the string in this metric (q_0 is the so-called "conformal time").

Let us now consider the particular interesting case

$$\Omega(X^0) = R_0 / X^0 \quad (21)$$

corresponding to the de Sitter geometry (here $R_0 = H^{-1} = 3/\sqrt{\Lambda}$) and the curvature scalar $R = HD(D-1)$ is constant. In this case, the solution Eq. (18) is equal to

$$q^0(\tau) = \frac{m}{p \sinh(m\tau)}, \quad q^1(\tau) = r^1 - \frac{m}{p} \coth(m\tau)$$

$$q^i(\tau) = r^i, \quad 2 \leq i \leq D-1. \quad (22)$$

here we chose the axis 1 parallel to the momentum p . $r_1, \dots, r_{D-1}$ are constants.

The first order perturbation equations (9) read in this case

$$\partial^2 z^0 + \frac{2}{q^0} (\dot{q}^0 \dot{z}^0 + \dot{q}^1 \dot{z}^1) - \frac{\dot{q}^0{}^2 + \dot{q}^1{}^2}{(q^0)^2} z^0 = 0,$$

$$\partial^2 z^1 + \frac{2}{q^0} (\dot{q}^0 \dot{z}^1 + \dot{q}^1 \dot{z}^0) - \frac{2 \dot{q}^0 \dot{q}^1}{(q^0)^2} z^0 = 0,$$

$$\partial^2 z^i + \frac{2}{q^0} \dot{q}^0 \dot{z}^i = 0, \quad 2 \leq i \leq D-1. \quad (23)$$

It is remarkable that this set of equations can be solved exactly in closed form. The general solution is given by

$$\begin{aligned} z^0(\sigma, \tau) &= \frac{1}{p \sinh^2(m\tau)} [A_1(\sigma, \tau) + B(\sigma, \tau) \cosh(m\tau)] \\ z^1(\sigma, \tau) &= -\frac{1}{p \sinh^2(m\tau)} [B(\sigma, \tau) + A_1(\sigma, \tau) \cosh(m\tau)] \\ z^i(\sigma, \tau) &= \frac{A_i(\sigma, \tau)}{p \sinh(m\tau)}, \quad 2 \leq i \leq D-1, \end{aligned} \quad (24)$$

where A_α and B are free fields satisfying the following equations

$$(\partial_\tau^2 - \partial_\sigma^2) B(\sigma, \tau) = 0 \quad (25)$$

$$\begin{aligned} (\partial_\tau^2 - \partial_\sigma^2 - m^2) A_\alpha(\sigma, \tau) &= 0, \quad 1 \leq \alpha \leq D-1 \\ \text{i.e.,} \quad A_\alpha &= \sum_{n \in \mathbb{Z}} [\gamma_n^\alpha e^{i(m\sigma - \omega_n \tau)} + \gamma_n^{\alpha+} e^{-i(m\sigma - \omega_n \tau)}] \\ B &= \sum_{n \in \mathbb{Z}} [\beta_n e^{in(\sigma - \tau)} + \beta_n^+ e^{-in(\sigma + \tau)}], \quad \beta_m^+ = \beta_{-m}, \quad \beta_m^- = \beta_{-m}^+ \end{aligned}$$

Therefore, the frequency of the A_α fields reads

$$\omega_m = \sqrt{n^2 - m^2} \quad (26)$$

The existence of imaginary frequencies for

$$n < |m|$$

exhibits an instability of the exact solution Eq. (24) in the de Sitter space.

It is convenient to parametrize the second order perturbations in the following way

$$\begin{aligned} \xi_0 &= \frac{1}{\eta \sinh^2(mz)} \left[D(\sigma, z) + R(\sigma, z) \cosh(mz) \right] \\ \xi_1 &= -\frac{1}{\eta \sinh^2(mz)} \left[R(\sigma, z) + D(\sigma, z) \cosh(mz) \right] \\ \xi_i &= \frac{\omega_i}{\eta \sinh(mz)} \end{aligned} \quad (27)$$

In the second order approximation, the constraints are expressed as

$$\begin{aligned} T_{\pm\pm} &= m^2 \pm 2m \partial_\pm (B+R) - \frac{(A_1 + B \cosh mz)^2}{\sinh^2(mz)} \\ &\pm \frac{2m}{\sinh(mz)} \left[\partial_\pm B (A_1 + B \cosh mz) + \partial_\pm A_1 (B + A_1 \cosh mz) \right] \\ &- \frac{1}{m^2} \left[\partial_\pm A_i \mp \coth(mz) A_i \right]^2 + (\partial_\pm B)^2 - (\partial_\pm A_1)^2 + \\ &+ m^2 (B^2 - A_1^2) \simeq 0 \end{aligned} \quad (28)$$

The field R is determined by the solution of one of the second order equations (10):

$$\begin{aligned}
 (\partial_z^2 - \partial_\sigma^2) R = & \coth(mz) [A_1'^2 - \dot{A}_1^2 + B'^2 - \dot{B}^2] + \\
 & + \frac{2}{\sinh^2(mz)} (B + A_1 \cosh mz)(\dot{B} + \dot{A}_1 \cosh mz) - \frac{\coth(mz)}{\sinh^2(mz)} (B + A_1 \cosh mz)^2 \\
 & + \coth(mz) [A_i'^2 - (\dot{A}_i - A_i m \cosh mz)^2] + \frac{2}{\sinh(mz)} [A_1' B' - \dot{A}_1 \dot{B}]
 \end{aligned} \quad (29)$$

Inserting the solution R of Eq. (29) in Eq. (28) for $T_{\pm\pm}$ yields after calculation

$$T_{\pm\pm} = (m \pm \partial_\pm B)^2 + \frac{1}{2\pi} \sum_{\alpha=1}^{D-1} L_0^\alpha \quad (30)$$

where

$$L_0^\alpha = -2\pi \sum_n (2n^2 - m^2) (\gamma_n^\alpha \gamma_n^{\alpha+} + \gamma_n^{\alpha+} \gamma_n^\alpha) \quad (31)$$

It must be noticed that the "massive" fields A_α ($\alpha = 1, \dots, D-1$) only contribute to $T_{\pm\pm}$ through a constant term. This is so since they cannot give rise at this quadratic order to a non-trivial dependence on $(\sigma \pm \tau)$. Higher orders will clearly mix the fields B and A_α .

The conjugate canonical momenta to the X_A fields

$$\pi_A = \frac{1}{\pi \alpha'} G_{AB}(X) \dot{X}^B$$

read in the linear approximation

$$\begin{aligned}
 \pi_0 &= -\frac{\pi}{m^2 g^2} [\dot{A}_1 + \dot{B} \cosh(mz) + m B \sinh(mz)] \\
 \pi_1 &= -\frac{\pi}{m^2 g^2} [\dot{B} + \dot{A}_1 \cosh(mz) - m A_1 \sinh(mz)] \\
 \pi_i &= \frac{\pi}{m^2 g^2} [\dot{A}_i \sinh(mz) - m A_i \cosh(mz)], \quad 2 \leq i \leq D-1
 \end{aligned} \quad (32)$$

where $g^2 = \frac{\pi \alpha'}{R_0^2} = \frac{\pi \ell_P^2}{R_0^2}$

γ_n^α and β_n satisfy the commutation rules

$$\begin{aligned} [\gamma_m^\alpha, \gamma_{m'}^{\beta\dagger}] &= \frac{m^2}{4\pi\omega_m} \delta^{\alpha\beta} \delta_{mm'} \\ [\beta_m, \beta_{m'}] &= \frac{m^2}{4\pi m} \delta_{m, -m'} \end{aligned} \quad (33)$$

all other commutators vanish. Hence, we have a Fock representation of the canonical commutation rules:

$$[\gamma^\alpha(\sigma, z), \pi_\beta(\sigma', z)] = \delta_\beta^\alpha \delta(\sigma - \sigma') \quad (34)$$

The Virasoro algebra holds to this order with a central charge equal to 1 due to the contribution of the B-field:

$$[\hat{L}_m, \hat{L}_n] = (m-n) \hat{L}_{m+n} + \frac{n(m^2-1)}{12} \delta_{m, -n} \quad (35)$$

where

$$\hat{L}_m = L_m - \delta_{m,0} \sum_{\alpha=1}^{D-1} L_0^\alpha$$

are conserved operators associated to the B-field.

Using the conformal invariance we can fix the field B to be

$$B = \alpha m \tau,$$

where α is a constant. The excitation described by this solution is a zero mode associated to dilations of p and m in the classical centre of mass solution. This can be easily checked from Eqs. (24) and (22). Therefore, these non-physical modes will be excluded. The $n = 0$ modes associated to the fields A_α ($\alpha = 1, \dots, D-1$) will also be excluded for the same reasons.

Finally, we find (D-1) components of physical string excitations, all of them associated to states of positive norm in the Fock space. [Recall that in flat space-time one has (D-2) components.] From Eqs. (30) and (31) we find the mass formula:

$$\frac{\pi m^2}{2g^2} = (D-1) E(m^2) + \sum_{n \neq 0} \Omega_n(m^2) \sum_{\alpha=1}^{D-1} a_n^{\alpha\dagger} a_n^{\alpha} + O(g^2) \quad (36)$$

where

$$\Omega_n(m^2) = \sqrt{m^2 - m_n^2} + \frac{m^2}{2\sqrt{m^2 - m_n^2}}$$

$$E(m^2) = \frac{1}{2} \sum_{n \neq 0} \Omega_n(m^2) \quad (37)$$

and

$$a_n^{\alpha} = 2 \frac{\sqrt{\pi \omega_n}}{m} \gamma_n^{\alpha}, \quad [a_n^{\alpha}, a_{n'}^{\beta\dagger}] = \delta^{\alpha\beta} \delta_{nn'}.$$

Equation (36) implicitly determines the mass spectrum in the quadratic approximation. The series Eq. (37) yields a finite function after regularization

$$E(m^2) = -\frac{1}{12} + \frac{m^2}{2} \int_0^{\infty} \frac{ds}{e^s - 1} I_2(ms) \quad (38)$$

where $I_2(x)$ is a modified Bessel function.

For $m^2 \rightarrow 0$ and $m^2 \rightarrow 1$, $E(m^2)$ behaves as

$$E(m^2) \underset{m^2 \rightarrow 0}{=} -\frac{1}{12} + \frac{m^4}{8} \zeta(3) + O(m^6)$$

$$E(m^2) \underset{m^2 \rightarrow 1}{=} \frac{1}{2\sqrt{1-m^2}} - \frac{1}{12} + O(\sqrt{1-m^2}) \quad (39)$$

The eigenvalue equation for the ground state

$$\frac{\pi m_0^2}{g^2} = (D-1) E(m_0^2) \quad (40)$$

has as solution a tachyonic state

$$m_0^2 = - (D-1) \frac{g^2}{12\pi} + O(g^4) \quad (41)$$

From Eq. (36) it is easy to check that spin two states exist in $D = 25$ with zero mass up to $O(g^4)$. This is a graviton with a very small mass linked to the finite size of the de Sitter universe. Therefore, the critical dimension for bosonic strings in de Sitter space-time is $D = 25$. An independent evaluation of this critical dimension by heat kernel techniques yields the same result. In other words, the dilaton β -function in D -dimensional de Sitter space-time must be

$$\beta^\phi = \frac{D+1-26}{\alpha' 48\pi^2} + O(1)$$

The lower mass states are the same as in flat space-time up to small corrections $O(g^4)$ whereas for $m^2 \rightarrow 1$ the spectrum deviates significantly from the linear Regge trajectories. By analyzing the eigenvalue equation for an excited state $n = N$:

$$\frac{\pi m_N^2}{2g^2} = 24 E(m_N^2) + N \Omega_N(m_N^2) \quad (42)$$

we find real mass solutions only for

$$N \leq N_{\text{MAX}} = \frac{\pi}{2g^2} + O(g^{-2/3}) \quad (43)$$

This N_{MAX} corresponds to very heavy particles with masses close to the critical value $0 < 1-m^2 = O(g^{4/3})$. Moreover, for states in the leading Regge trajectory

$$\frac{\pi m_J^2}{2g^2} = 24 E(m_J^2) + J \Omega_1(m_J^2) \quad (44)$$

we find that the mass monotonously increases with J up to the value

$$\begin{aligned} J &= J_{\text{MAX}} = 0.9/g^2 + O(1) \\ m_J^2 &= m_{\text{MAX}}^2 = 0.76 + O(g^2) \end{aligned} \quad (45)$$

Beyond J_{MAX} , the masses become complex. These complex solutions correspond to unstable states, already present here at the tree (zero handle) level.

It is also possible to compute here the scattering amplitudes. The tachyon vertex operator in curved space must fulfil the equation⁸⁾

$$(\square^2 - \mu^2) T(X) = 0 \quad (46)$$

where

$$T(X) = f(X_0) e^{i\vec{k} \cdot \vec{X}}$$

and μ^2 is the tachyon mass. For de Sitter space

$$\square^2 = X_0^2 (\partial_0^2 - \partial_i^2) - (D-2) X_0 \partial_0$$

Eq. (46) has as solution

$$f(X_0) = X_0^{\frac{D-1}{2}} J_\nu(k X_0)$$

$$\nu = \pm \sqrt{\mu^2 + \left(\frac{D}{2} - 1\right)^2}$$

If $X_0 \rightarrow \infty$, $f(X_0) \simeq X_0^{\frac{D}{2}-1} e^{\pm i k X_0}$

As is known, quantum conformal invariance in string theory is ensured by the vanishing of the beta-functions for the background fields. The perturbative computation (in power of α' , but for zero handle surfaces) of these beta-functions can be found in Refs. 1-3). If we insert the de Sitter metric in the perturbative expression of the beta-functions, we find a non-zero result in the absence of the dilaton field. However, as suggested by Fischler and Susskind⁹⁾ [see also Ref. 10)], contributions from higher genus surfaces may combine with the zero handle beta-function eventually yielding a vanishing result for the de Sitter metric.

The results obtained here for the de Sitter metric show that our perturbative method is well suited to investigate the quantum dynamics of strings in curved space-time fully taking into account strong curvature effects.

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