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LINEAR MULTIPLETS AND SUPER CHERN-SIMONS FORMS IN 4D-SUPERGRAVITY^{*)}

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A B S T R A C T

We explore general properties of matter-coupled supergravities with antisymmetric tensor fields embedded in linear multiplets. The four-dimensional supersymmetric version of the Green-Schwarz mechanism leads to the introduction of super Chern-Simons form multiplets whose general component expressions are derived. The use of superconformal techniques allows us to discuss several topics relevant to superstring effective actions and in particular to give simple forms of the string-loop corrections to the Kahler potential and the U(1) anomaly cancelling term present in certain superstring compactifications.

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I. INTRODUCTION

An important issue for the field theory limit of four-dimensional superstrings¹ (as well as of ten-dimensional ones² after spontaneous compactification)³ is to understand the full supersymmetric completion of the Green-Schwarz mechanism⁴ to cancel potential gauge and mixed anomalies.

For the $N = 1$ case, this was already done in Ref. 5. However, that answer is not the one which is natural from a geometrical point of view. Indeed, the Green-Schwarz mechanism is based on a 2-form B which changes under Yang-Mills and gravitational gauge transformations in such a way that its improved field strength

$$H = dB + \omega_{3YM} - \omega_{3L} \tag{1.1}$$

is gauge-invariant⁴ and well defined globally.⁶ (Here ω_{3YM} (resp. ω_{3L}) is the Yang-Mills (resp. Lorentz) Chern-Simons 3-form.) Instead, in Ref. 5 we exploited the fact that, in four dimensions, a 2-form is dynamically equivalent to a scalar field. Thus by a duality transformation we obtain a mechanism involving only scalar fields, which is quite easy to supersymmetrize.

Recently a reformulation of the results of Ref. 5 in terms of a two-form was given⁷ by directly performing a duality transformation on the component form of a general matter-coupled supergravity Lagrangian in standard form, of which a particular case is the simplest superstring inspired supergravity Lagrangian obtained sometime ago by Witten.⁸ However, only the bosonic terms of the duality transformed theory, in the presence of the Green-Schwarz mechanism, were obtained.

In the present paper we extend these results and derive the general form of the Chern-Simons Yang-Mills and Lorentz multiplet. Actually these results allow one to derive explicit formulae for all supersymmetric topological invariants. The applications which motivate the present research have a more general character. It seems that the dual formulation of the effective field theory greatly simplifies the understanding of superstring symmetries and gives a better control over terms of higher orders in the curvatures. An application is given here to clarify the global string symmetries⁹ at arbitrary string-loop orders without any reference to individual component fields but rather using pure superspace geometry. In particular, we give a very simple form of the one-loop string corrections to the Kahler potential,¹⁰ f-gauge kinetic¹¹ term and possible "Fayet-Illiopoulos" term¹² in case of a surviving U(1) group with anomalous fermion representations. This analysis is further simplified by use of a superspace "superconformal" geometry in the "minimal formulation" of $N = 1$ four-dimensional supergravity.

The paper is organized as follows: In Section II we write the standard supergravity Lagrangian in the presence of two-form fields, naturally described by "linear" superfields. In Sections III and IV we introduce the Chern-Simons Yang-Mills and Lorentz three-form (dual one-form) and describe their general properties in the framework of superspace (Poincaré) geometry. In Section V the superconformal version of the above results is formulated and an existence proof of Chern-Simons forms is given. In Section VI the application of the superspace formulation of $N = 1$ supergravity, with linear multiplets, is given to simplify the content and the restrictions implied by the global scale superstring symmetries on the effective Lagrangians. Finally in Section VII the explicit component expressions of the linear and Chern-Simons multiplets, and their relation to the duality transformed theory, are exhibited.

II. MATTER COUPLING IN 4D SUPERGRAVITY WITH LINEAR MULTIPLETS AND DUALITY TRANSFORMATIONS

It is well known¹³ that the action for a set of chiral superfields Z_I and a Lie algebra valued real superfield V is given in terms of three arbitrary functions:

- (i) a real function $\Phi(Z_I, \bar{Z}^I)$;
- (ii) an analytic function $g(Z_I)$ called the superpotential; and
- (iii) a symmetric two index analytic function $f_{AB}(Z_I)$

by the standard form:

$$\int d^8\mu \Phi(Z, \bar{Z}e^V) + \text{Re} \int d^6\mu g(Z_I) + \text{Re} \int d^6\mu f_{AB}(Z_I) W^A W^B \quad (2.1)$$

where $d^8\mu$ and $d^6\mu$ are the full and chiral measures in superspace respectively and W^A is the field strength of the superfield V .

A chiral superfield carries a Wess-Zumino multiplet and a real superfield carries a vector one. However, in simple supergravity there is a third basic multiplet called the linear (sometimes tensor) multiplet.¹⁴ This multiplet is carried by a real superfield whose chiral projection vanishes. We will call it a linear superfield and denote it generally by L . So

$$L = \bar{L} \quad (2.2a)$$

$$\Sigma(L) = (\bar{D}\bar{D} - 8R) L = 0 \quad (2.2b)$$

In flat space, Eq. (2.2b) simply reads $DDL = \bar{D}\bar{D}L = 0$ which explains the name of the superfield. Also Eq. (2.2b) implies that there are only three

independent fields contained in L and in particular that the vector component is the field strength of an antisymmetric tensor $b_{\mu\nu}$. This field strength is invariant under the Abelian gauge transformation:

$$b'_{\mu\nu} = b_{\mu\nu} + \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \quad (2.3)$$

for any vector field $\xi_\mu(x)$. Then the fields of a linear multiplet are a real scalar C^L , a Majorana spinor χ^L and $b_{\mu\nu}$:

$$L = [C^L, \chi^L, b_{\mu\nu}] . \quad (2.4)$$

The number of physical degrees of freedom is the same as for a Wess-Zumino multiplet but here we do not have auxiliary fields since the number of fermionic and bosonic degrees of freedom also matches off-shell.

A linear superfield can be solved in terms of a prepotential L_α , which is a chiral superfield with an external undotted spinor index, as follows:

$$L = \mathcal{D}^\alpha L_\alpha + \bar{\mathcal{D}}_{\dot{\alpha}} \bar{L}^{\dot{\alpha}} \quad (2.5a)$$

with

$$\bar{\mathcal{D}}_{\dot{\beta}} L_\alpha = 0, \quad \mathcal{D}^\beta \bar{L}^{\dot{\alpha}} = 0 . \quad (2.5b)$$

Strictly speaking, the field $b_{\mu\nu}$ appears as such only in L_α while in L appears only through its field strength, even though in (2.4) we listed $b_{\mu\nu}$ as the third field of the linear multiplet.

The superfield generalization of the transformation (2.3) is given by

$$L'_\alpha = L_\alpha + i(\bar{D}\bar{D} - 8R) D_\alpha \Theta \quad (2.6)$$

where Θ is an arbitrary real superfield. L is invariant under the transformation (2.6) by virtue of the identity:¹⁵

$$D^\alpha (\bar{D}\bar{D} - 8R) D_\alpha \Theta = \bar{D}_\alpha (\bar{D}\bar{D} - 8R) \bar{D}^{\dot{\alpha}} \Theta. \quad (2.7)$$

In order to obtain the action for a theory including linear superfields we will perform a duality transformation¹⁶ on an action of the standard type (2.1), as outlined in Ref. 7. To proceed, let us split the chiral superfields

$$Z_I = X_j, S_i$$

where the subset S_i satisfies the following requirements:

- (i) the S_i are gauge inert;
- (ii) they appear in the function Φ only in the combination $S_i + \bar{S}^i$ (which means that the first component of the Φ superfield depends only on $\text{Re } s_i$);
- (iii) the superpotential function g is independent of S_i (up to a redefinition of the supervierbein which at most allows an exponential dependence of g in terms of S_i) and
- (iv) the function f_{AB} is at most linear in S_i .

That is:

$$\Phi(Z, \bar{Z} e^V) = \Phi(X, \bar{X} e^V, S_i + \bar{S}^i) \quad (2.8a)$$

$$g(Z_I) = e^{c^i S_i} \hat{g}(X_j) \quad (2.8b)$$

$$f_{AB}(Z_I) = a^i_{AB} S_i \delta_{AB} + \hat{f}_{AB}(X_j) , \quad (c^i, a^i \text{ are real coefficients}) . \quad (2.8c)$$

When these conditions are satisfied, the corresponding theory can be transformed via a duality transformation into a theory where the chiral superfields S_i have been replaced by linear ones L_i .

In order to perform the duality transformation, one must write the Lagrangian in such a way that the superfields S_i appear only through the combination $S_i + \bar{S}^i$. At this point one considers $S_i + \bar{S}^i$ as unconstrained real superfields U^i and then adds the Lagrange multiplier terms:

$$-\int d^8\mu U^i L_i \quad (2.9)$$

where the L_i are linear superfields. Varying with respect to L_i just gives us:

$$U^i = S_i + \bar{S}^i \quad \text{with} \quad \bar{D}_{\dot{\alpha}} S_i = 0 \quad (2.10)$$

and replacing this result back forces the term (2.9) to vanish and we obtain the old theory. On the other hand, by varying with respect to U_i , solving for it from its equation of motion and replacing the result back into the Lagrangian, one obtains the dual theory in terms of L_i . For this procedure to work it is necessary that:

$$2 \operatorname{Re} \int d^6\mu S W_A W^A = \int d^8\mu (S + \bar{S}) \Omega(V) \quad (2.11a)$$

which demands the existence of a real superfield $\Omega(V)$ whose chiral projection is:

$$\Sigma[\Omega(V)] = (\bar{T}\bar{T} - 8R) \Omega(V) = w_A w^A. \quad (2.11b)$$

In the next section we show that this is true for Abelian as well as Yang-Mills gauge fields in superspace formalism while in Section VII we will give the necessary expressions in component language. Thus we have the action in terms of the superfields U^i :

$$\int d^8\mu \Phi(x, \bar{x}e^V, U^i) + \int d^8\mu a_i U^i \Omega(V) - \int d^8\mu U^i L_i \quad (2.12)$$

where we have dropped the pieces which do not depend on U^i . Then we obtain

$$\frac{\partial \Phi}{\partial U^i} = L_i - a_i \Omega(V) = \hat{L}_i. \quad (2.13)$$

Inverting this relation and replacing the result back in (2.12), we get the action of the dual theory in terms of $\hat{L}_i$:

$$\int d^8\mu M(x, \bar{x}e^V, \hat{L}_i). \quad (2.14)$$

In order to go from a formulation of the theory in terms of linear superfields to the dual one we consider now L_i as an unconstrained real superfield and add to the action (2.14) the Lagrange multiplier term:

$$-\int d^8\mu L_i (S_i + \bar{S}^i) \quad (2.15)$$

where S_i is chiral. Varying with respect to S_i just gives us the conditions (2.2) for L_i to be a linear superfield and replacing the result back forces (2.15) to vanish and we are just left with (2.14). On the other hand, varying

with respect to L_i , we get

$$\frac{\partial M}{\partial L_i} = s_i + \bar{s}^i \quad (2.16)$$

and inverting this we get:

$$\hat{L}_i = \left(\frac{\partial M}{\partial L_i} \right)^{-1} (X, \bar{X}^V, s_i + \bar{s}^i) = L_i - a_i \Omega(V) . \quad (2.17)$$

Replacing this result back and making use again of Eq. (2.11a) we get again the theory in standard supergravity form, provided we perform the procedure for every L_i of course.

III. THE SUPERFIELD YANG-MILLS CHERN-SIMONS FORM

In this section we will show the validity of Eq. (2.11) for Abelian as well as non-Abelian gauge fields and obtain expressions for $\Omega(V)$ in both cases. Consider the superspace expression:

$$2 \operatorname{Re} \int d^6\mu S W^\alpha W_\alpha = \int d^6\mu S W^\alpha W_\alpha + \int d^6\bar{\mu} \bar{S} \bar{W}^\alpha \bar{W}_\alpha \quad (3.1)$$

where W_α is the Abelian field strength superfield:¹⁵

$$W_\alpha = (\bar{D}\bar{D} - 8R) D_\alpha V \quad (3.2)$$

and $\bar{W}_\alpha$ is the Hermitian conjugate. D_α and $\bar{D}_\alpha$ denote the covariant spinor derivatives in curved space. Since both S and W_α are chiral, we can write:

$$\int d^6\mu S W^\alpha W_\alpha = \int d^6\mu (\bar{D}\bar{D} - 8R) S D^\alpha V W_\alpha = \int d^8\mu S D^\alpha V W_\alpha \quad (3.3a)$$

and similarly:

$$\int d^6\bar{\mu} \bar{S} \bar{W}^\alpha \bar{W}_\alpha = \int d^6\bar{\mu} (\bar{D}\bar{D} - 8\bar{R}) \bar{S} \bar{D}^\alpha V \bar{W}_\alpha = \int d^8\bar{\mu} \bar{S} \bar{D}^\alpha V \bar{W}_\alpha \quad (3.3b)$$

Now adding and subtracting suitable terms, we can write:

$$\begin{aligned} & \int d^6\mu S W^\alpha W_\alpha + \int d^6\bar{\mu} \bar{S} \bar{W}^\alpha \bar{W}_\alpha \\ &= \int d^8\mu [(S+\bar{S}) (D^\alpha V) W_\alpha + (S+\bar{S}) (\bar{D}^\alpha V) \bar{W}_\alpha - \bar{S} (D^\alpha V) W_\alpha - S (\bar{D}^\alpha V) \bar{W}_\alpha] \quad (3.4) \end{aligned}$$

and then we can integrate by parts:

$$\int d^8\mu \bar{s} \mathcal{D}^\alpha_V W_\alpha = - \int d^8\mu \bar{s} V \mathcal{D}^\alpha_W W_\alpha \quad (3.5a)$$

$$\int d^8\mu s \bar{\mathcal{D}}_{\dot{\alpha}} V \bar{W}^{\dot{\alpha}} = - \int d^8\mu s V \bar{\mathcal{D}}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} . \quad (3.5b)$$

Finally, making use of the Bianchi identity:¹⁵

$$\mathcal{D}^\alpha_W W_\alpha = \bar{\mathcal{D}}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} , \quad (3.6)$$

we obtain:

$$\begin{aligned} 2 \operatorname{Re} \int d^6\mu s W^\alpha_W W_\alpha &= \int d^8\mu (S+\bar{S}) [\mathcal{D}^\alpha_V W_\alpha + \bar{\mathcal{D}}_{\dot{\alpha}} V \bar{W}^{\dot{\alpha}} + V \mathcal{D}^\alpha_W W_\alpha] \\ &= \int d^8\mu (S+\bar{S}) [\mathcal{D}^\alpha_V W_\alpha + \bar{\mathcal{D}}_{\dot{\alpha}} V \bar{W}^{\dot{\alpha}} + \frac{1}{2} V \mathcal{D}^\alpha_W W_\alpha + \frac{1}{2} V \bar{\mathcal{D}}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}] \end{aligned} \quad (3.7)$$

which has the desired form, as advanced in Eq. (2.11), which is suitable for the duality transformation.

So the Chern-Simons superfield for an Abelian gauge field V is:

$$\Omega^{(ab)}(V) = \mathcal{D}^\alpha_V W_\alpha + \bar{\mathcal{D}}_{\dot{\alpha}} V \bar{W}^{\dot{\alpha}} + V \mathcal{D}^\alpha_W W_\alpha . \quad (3.8)$$

Eq. (3.7) is a reflection of the fact that:

$$\Sigma[\Omega^{(ab)}(V)] = (\bar{\mathcal{D}}\mathcal{D} - 8R) \Omega^{(ab)}(V) = W^\alpha_W W_\alpha \quad (3.9a)$$

and

$$\bar{\Sigma}[\Omega^{(ab)}(V)] = (D\bar{D} - 8R) \Omega^{(ab)}(V) = \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}, \quad (3.9b)$$

since Eq. (3.8) is real by virtue of the Bianchi identity (3.6). A curious second property of this superfield is:

$$V \Omega^{(ab)}(V) = \frac{1}{2} D^{\alpha} (V^2 W_{\alpha}) + \frac{1}{2} \bar{D}_{\dot{\alpha}} (V^2 \bar{W}^{\dot{\alpha}}) \quad (3.10)$$

which implies

$$\int d^8\mu V \Omega^{(ab)}(V) = 0. \quad (3.11)$$

This can be thought of as the superfield extension of the obvious equation

$$v_{\mu} \epsilon^{\mu\nu\rho\sigma} v_{\nu} (\partial_{\rho} v_{\sigma} - \partial_{\sigma} v_{\rho}) = 0.$$

Under the gauge transformation

$$V' = V + i(\Lambda - \bar{\Lambda}) \quad (3.12)$$

the Chern-Simons form (3.8) transforms as:

$$\Omega'^{(ab)}(V') = \Omega^{(ab)}(V) + i D^{\alpha} (\Lambda W_{\alpha}) - i \bar{D}_{\dot{\alpha}} (\bar{\Lambda} \bar{W}^{\dot{\alpha}}). \quad (3.13)$$

Therefore if we require invariance of the superfield:

$$\hat{L} = L - a \Omega^{(ab)}(V), \quad (3.14)$$

we get the transformation rule for L_α :

$$L'_\alpha = L_\alpha + ai\Lambda W_\alpha . \quad (3.15)$$

The above procedure can be applied to the non-Abelian case with a few modifications. Now the starting point is:

$$2 \operatorname{Re} \int d^6\mu S \operatorname{Tr}(W^\alpha W_\alpha) = \int d^6\mu S \operatorname{Tr}(W^\alpha W_\alpha) + \int d^6\mu \bar{S} \operatorname{Tr}(\bar{W} \cdot \bar{W}^\alpha) \quad (3.16)$$

where W_α is here the non-Abelian field strength:¹⁵

$$W_\alpha = (\bar{D}\bar{D} - 8R) e^{-V} \mathcal{D}_\alpha e^V \quad (3.17a)$$

$$\bar{W}^\alpha = -(\mathcal{D}\mathcal{D} - 8\bar{R}) e^V \bar{\mathcal{D}}^\alpha e^{-V} . \quad (3.17b)$$

Let us introduce now:

$$W_\alpha(t) = (\bar{D}\bar{D} - 8R) e^{-tV} \mathcal{D}_\alpha e^{tV} , \quad 0 \leq t \leq 1 \quad (3.18)$$

$$W_\alpha(0) = 0 , \quad W_\alpha(1) = W_\alpha ,$$

such that

$$\operatorname{Tr}(W^\alpha W_\alpha) = \int_0^1 dt \frac{d}{dt} \operatorname{Tr}(W^\alpha(t) W_\alpha(t)) = 2 \int_0^1 dt \operatorname{Tr} \left(\frac{dW^\alpha(t)}{dt} W_\alpha(t) \right) . \quad (3.19)$$

Next let us define the t-dependent derivative

$$\nabla_{\alpha}(t) = e^{-tV} \mathfrak{D}_{\alpha} e^{tV} \quad (3.20)$$

which we can use to write:

$$\frac{dW^{\alpha}(t)}{dt} = (\bar{\mathfrak{D}}\bar{\mathfrak{D}} - 8R) [\nabla^{\alpha}(t), V] \quad (3.21)$$

and using the chirality of $W_{\alpha}(t)$ we get:

$$\begin{aligned} \text{Tr}(W^{\alpha} W_{\alpha}) &= 2 \int_0^1 dt \text{Tr}((\bar{\mathfrak{D}}\bar{\mathfrak{D}} - 8R) [\nabla^{\alpha}(t), V] W_{\alpha}(t)) \\ &= 2(\bar{\mathfrak{D}}\bar{\mathfrak{D}} - 8R) \int_0^1 dt \text{Tr}([\nabla^{\alpha}(t), V] W_{\alpha}(t)) . \end{aligned} \quad (3.22)$$

Similarly, we can write:

$$\text{Tr}(\bar{W} \cdot \dot{\bar{W}}^{\dot{\alpha}}) = 2(\bar{\mathfrak{D}}\bar{\mathfrak{D}} - 8R) \int_0^1 dt \text{Tr}([\bar{\nabla}_{\dot{\alpha}}(t), V] \dot{\bar{W}}^{\dot{\alpha}}(t)) \quad (3.23)$$

with

$$\dot{\bar{W}}_{\alpha}(t) = -(\mathfrak{D}\mathfrak{D} - 8\bar{R}) e^{tV} \dot{\mathfrak{D}}_{\alpha} e^{-tV} \quad (3.24)$$

$$\dot{\bar{\nabla}}_{\dot{\alpha}}(t) = e^{tV} \dot{\mathfrak{D}}_{\dot{\alpha}} e^{-tV} .$$

Therefore:

$$\begin{aligned} \int d^6\mu S \text{Tr}(W^{\alpha} W_{\alpha}) &= 2 \int d^6\mu (\bar{\mathfrak{D}}\bar{\mathfrak{D}} - 8R) \int_0^1 dt S \text{Tr}([\nabla^{\alpha}(t), V] W_{\alpha}(t)) \\ &= 2 \int d^8\mu \int_0^1 dt S \text{Tr}([\nabla^{\alpha}(t), V] W_{\alpha}(t)) \end{aligned} \quad (3.25a)$$

and

$$\int d^6\mu \bar{S} \text{Tr}(\bar{W}_\alpha \dot{\bar{W}}^\alpha) = 2 \int d^8\mu \int_0^1 dt \bar{S} \text{Tr}([\bar{V}_\alpha(t), V] \dot{\bar{W}}^\alpha(t)) \quad (3.25b)$$

where use has been made of the fact that S ($\bar{S}$) is chiral (antichiral). Again we can add and subtract suitable terms:

$$\begin{aligned} & 2 \text{Re} \int d^6\mu S \text{Tr}(W^\alpha W_\alpha) \\ &= 2 \int d^8\mu \int_0^1 dt \left[(S+\bar{S}) \text{Tr}([\bar{V}^\alpha(t), V] W_\alpha(t) + (S+\bar{S}) \text{Tr}([\bar{V}_\alpha(t), V] \dot{\bar{W}}^\alpha(t) \right. \\ & \quad \left. - \bar{S} \text{Tr}([\bar{V}^\alpha(t), V] W_\alpha(t)) - S \text{Tr}([\bar{V}_\alpha(t), V] \dot{\bar{W}}^\alpha(t)) \right] \end{aligned} \quad (3.26)$$

and integrate by parts the last two terms:

$$\int d^8\mu \bar{S} \text{Tr}([\bar{V}^\alpha(t), V] W_\alpha(t)) = - \int d^8\mu \bar{S} \text{Tr}[V \mathcal{D}^\alpha (e^{tV} W_\alpha(t) e^{-tV})] \quad (3.27a)$$

$$\int d^8\mu S \text{Tr}([\bar{V}_\alpha(t), V] \dot{\bar{W}}^\alpha(t)) = - \int d^8\mu S \text{Tr}[V \bar{\mathcal{D}}_\alpha (e^{-tV} \dot{\bar{W}}^\alpha(t) e^{tV})] \quad (3.27b)$$

Finally, making use of the identity:

$$\mathcal{D}^\alpha (e^{tV} W_\alpha(t) e^{-tV}) = e^{tV} [\bar{\mathcal{D}}_\alpha (e^{-tV} \dot{\bar{W}}^\alpha(t) e^{tV})] e^{-tV} \quad (3.28)$$

which implies

$$\text{Tr}[V \mathcal{D}^\alpha (e^{tV} W_\alpha(t) e^{-tV})] = \text{Tr}(V \bar{\mathcal{D}}_\alpha [e^{-tV} \dot{\bar{W}}^\alpha(t) e^{tV}]) \quad (3.29)$$

we get the final result:

$$\begin{aligned}
& 2 \operatorname{Re} \int d^6 \mu \, S \operatorname{Tr}(W^\alpha W_\alpha) \\
&= 2 \int d^8 \mu \, (S + \bar{S}) \int_0^1 dt \operatorname{Tr} \left([\nabla^\alpha(t), V] W_\alpha(t) + [\bar{\nabla}_\alpha(t), V] \bar{W}^{\dot{\alpha}}(t) \right. \\
&\quad \left. + v \{ \nabla^\alpha(t), W_\alpha(t) \} \right) \\
&= 2 \int d^8 \mu \, (S + \bar{S}) \int_0^1 dt \operatorname{Tr} \left([\nabla^\alpha(t), V] W_\alpha(t) + [\bar{\nabla}_\alpha(t), V] \bar{W}^{\dot{\alpha}}(t) \right. \\
&\quad \left. + \frac{1}{2} v \{ \nabla^\alpha(t), W_\alpha(t) \} + \frac{1}{2} v \{ \bar{\nabla}_\alpha(t), \bar{W}^{\dot{\alpha}}(t) \} \right) , \tag{3.30}
\end{aligned}$$

where we have used:

$$\mathcal{D}^\alpha (e^{tV} W_\alpha(t) e^{-tV}) = e^{tV} \{ \nabla^\alpha(t), W_\alpha(t) \} e^{-tV} . \tag{3.31}$$

Eq. (3.30) has the desired form to make the duality transformation possible.

From Eq. (3.30) we have the expression for the Yang-Mills Chern-Simons form in superspace:

$$\begin{aligned}
\Omega^{(YM)}(V) &= 2 \int_0^1 dt \operatorname{Tr} \left([\nabla^\alpha(t), V] W_\alpha(t) + [\bar{\nabla}_\alpha(t), V] \bar{W}^{\dot{\alpha}}(t) \right. \\
&\quad \left. + v \{ \nabla^\alpha(t), W_\alpha(t) \} \right) . \tag{3.32}
\end{aligned}$$

Given that, by Eq. (2.11b), the chiral projection of $\Omega^{(YM)}(V)$ is gauge invariant, the gauge variation of $\Omega^{(YM)}(V)$ must be a linear multiplet just like in the Abelian case (see Eq. (3.13)). Thus we have

$$\delta_{YM} \Omega^{(YM)}_{(V)} = \mathfrak{D}^\alpha \delta_\alpha^{(YM)} + \bar{\mathfrak{D}}_{\dot{\alpha}} \bar{\delta}^{(YM)\dot{\alpha}} \quad (3.33)$$

which implies the transformation rule for the prepotential L_α in (2.5)

$$\delta_{YM} L_\alpha = a \delta_\alpha^{(YM)} \quad (3.34)$$

since again we must demand invariance of $\hat{L}$ in (2.13) if the theory is to be gauge invariant. This is entirely analogous to the Abelian case (see Eqs. (3.13)-(3.15)) aside from the fact that we have not given an expression for $\delta_\alpha^{(YM)}$. Also in the Yang-Mills case we do not have a property like Eq. (3.11) for the Abelian one, which is to be expected.

To finish this section we will just say that Eqs. (3.15) and (3.34) are just the superfield generalizations of the anomalous gauge transformations of the $b_{\mu\nu}$ tensor which will appear in Section VII.

IV. THE GREEN-SCHWARZ MECHANISM AND THE LORENTZ CHERN-SIMONS FORM

In this section we proceed to extend the analysis of Sections II and III in order to include the Lorentz group, which will give rise to the Green-Schwarz mechanism. Chiral superfields can also be coupled to the square of the gravitational (Weyl) curvature $W_{\alpha\beta\gamma}$ by a term in the action:

$$\text{Re} \int d^6\mu f^{(\text{grav})}(Z_I) W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} . \quad (4.1)$$

Again in this case, when $f^{(\text{grav})}(Z_I)$ is at most linear in the S_i fields:

$$f^{(\text{grav})}(Z_I) = b^i S_i + \hat{f}^{(\text{grav})}(X_j) , \quad b^i = \text{real coefficients} , \quad (4.2)$$

the duality transformation to a theory with linear superfields can still be performed, and the dual formulation will exhibit the Green-Schwarz mechanism. The basic fact now is that there is a real superfield $\Omega^{(\text{LOR})}$ which satisfies:

$$\Sigma[\Omega^{(\text{LOR})}] = (\bar{\mathcal{D}}\bar{\mathcal{D}} - 8R) \Omega^{(\text{LOR})} = W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} \quad (4.3)$$

in analogy with the gauge cases studied in the previous section. This immediately implies:

$$2 \text{Re} \int d^6\mu S W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} = \int d^8\mu (S + \bar{S}) \Omega^{(\text{LOR})} . \quad (4.4)$$

Unlike the cases studied in the previous section, we will not provide here a superfield derivation of Eq. (4.3) or Eq. (4.4) but will content ourselves with showing that it is true in component language in Section VII.

(We note that, unlike the Yang-Mills case, $\Omega^{(LOR)}$ is defined up to the addition of "covariant terms" of the form $a'E_{\alpha\alpha}E^{\alpha\alpha} + b'RR$ where a', b' are arbitrary coefficients. Here $E_{\alpha\alpha}$ and R are the Ricci and the scalar curvature superfields which together with $W_{\alpha\beta\gamma}$ completely specify the superspace geometry.¹⁵ In that case the Bianchi-Identity for $\Omega^{(LOR)}$ would be modified by adding a term $(\bar{D}\bar{D} - 8R)(a'E_{\alpha\alpha}E^{\alpha\alpha} + b'RR)$ on the right-hand side of Eq. (4.3). For special values of a' and b' , one obtains the super--Gauss-Bonnet combination.⁵⁾

When we have this addition to the action, the generalized linear superfields $\hat{L}_i$ will get an additional piece:

$$\hat{L}_i = L_i - a_i \Omega(V) - b_i \Omega^{(LOR)} \quad (4.5)$$

but the action will still be given by Eq. (2.14). Finally, for a linear superfield with non-vanishing b coefficient we get an anomalous Lorentz transformation for the prepotential L_α (on top of the one dictated by its spinorial index) when we demand invariance of $\hat{L}$.

In analogy to what happened to $\Omega(V)$ in the previous section, $\Omega^{(LOR)}$ is non-invariant under Lorentz transformations, but its chiral projection (4.3) is. So the change in $\Omega^{(LOR)}$ must be a linear superfield:

$$\delta \Omega^{(LOR)} = \bar{D}^\alpha \delta_\alpha^{(LOR)} + \bar{D}_\alpha \bar{\delta}^{(LOR)} \alpha \quad (4.6)$$

and therefore $\hat{L}$ can be made invariant with the anomalous transformation for L_α :

$$\delta^{anom} L_\alpha = b \delta_\alpha^{(LOR)} \quad (4.7)$$

which is the superfield generalization of the anomalous Lorentz transformation of the antisymmetric tensor $b_{\mu\nu}$ to be described in Section VII. With these provisions, the theory in terms of linear superfields is also Lorentz invariant just like the dual theory.

Again the analysis at the component level will be deferred until Section VII.

V. SUPERCONFORMAL APPROACH TO CHERN-SIMONS FORMS

In the superconformal tensor calculus (or superspace) formulation of matter coupling to $N = 1$ supergravity, superfields have definite superconformal properties. For a useful review of superconformal techniques, see Ref. 17.

In particular, the standard supergravity formulation corresponds to the so called "minimal formulation." This uses as a compensator a chiral multiplet S_0 whose last component $u = S_{\text{aux}} - iP$ corresponds to the scalar and pseudoscalar auxiliary fields of "minimal" supergravity. Each (scalar) conformal superfield has a definite Weyl and chiral weight. The linear multiplet L has Weyl weight 2 and being real, chiral weight zero. This means that the Chern-Simons forms Ω_3 have the same conformal properties of L . The covariant field strength $\hat{L} = L + \Omega_3$ satisfies the conformal invariant Bianchi-Identity

$$\Sigma(\hat{L}) = W^2, \quad (5.1)$$

where W^2 generally denotes a set of field strengths. In the dual formulation one introduces a chiral multiplet S (dual to L) through the Lagrange multiplier

$$-[L(S+\bar{S})]_D \quad (5.2)$$

(the subscript D means the D -density formula). Note that no compensator appears in Eq. (5.2) since Eq. (5.2) is a superconformal density.

The standard supergravity Lagrangian for the linear multiplet has the general form

$$\left[\Psi \left(\frac{\hat{L}}{s_o \bar{s}_o} \right) s_o \bar{s}_o \right]_D \quad (5.3)$$

where Ψ can depend on chiral matter fields X_j . By performing the Legendre transformation we get

$$\frac{\partial \Psi}{\partial \hat{L}/s_o \bar{s}_o} = s + \bar{s} \quad (5.4)$$

which can be solved to give $\Phi(s + \bar{s})$. So the dual Lagrangian is

$$[\Phi(s + \bar{s}) s_o \bar{s}_o]_D + [(s + \bar{s}) \Omega(V)]_D \quad (5.5)$$

which by use of Eq. (5.1) can be written as

$$[\Phi(s + \bar{s}) s_o \bar{s}_o]_D + 2 \operatorname{Re}(SW^2)_F. \quad (5.6)$$

We note that it is the expression in (5.3) and not the expression in (5.6) which directly comes from superstring compactification to four dimensions. The component expression of Eq. (5.1) contains the usual Bianchi identity

$$dH = -R \wedge R \quad (5.7)$$

where R is a gravitational or Yang-Mills curvature. In $D = 4$, Eq. (5.7) is actually a scalar equation (since a four-form is dual to a zero form) which is contained in the last (θ^2) component of the chiral equation in (5.1).

In the same way as for a given gravitational or Yang-Mills configuration, the solution of Eq. (5.7) is unique up to addition of a closed three-form, so the solution of Eq. (5.1) is unique up to addition of a linear multiplet which

has a component which is just a closed three-form.

To find the general solution to Eq. (5.1), it is best to study a more general problem. We shall prove the following theorem: Let Y be a superconformal chiral multiplet with Weyl $w = 3$ and chiral weight $n = 3$. Then there exists a local real multiplet Q_Y , with $w = 2$, $n = 0$ such that

$$Y = \Sigma(Q_Y) \quad (5.8)$$

if and only if the superconformal density

$$\text{Im}[Y]_F \quad (5.9)$$

is a total derivative, i.e. if

$$\int d^4x \text{Im}[Y]_F = \text{topological invariant} . \quad (5.10)$$

This must be so since we know that the imaginary part of the F-density formula for the chiral projection of a real superfield is in fact a total divergence. Moreover the same argument gives us an explicit formula for the components of Q_Y as a function of those of Y .

This argument applies to all topological invariants which have a supersymmetric completion (Euler characteristic, Hirzebruch signature, Pontryagin number, etc.). The proof of the above theorem consists of three steps. First we look for the most general solution of the equation

$$\Sigma(A) = Y \quad (5.11)$$

where Y is a given chiral multiplet with $w = n = 3$. Second, on the solution of Eq. (5.11) we shall impose the reality condition $A = \bar{A}$. Third we will require A to be local and this will require Eq. (5.10). Since, by definition, the most general solution to the homogeneous equation

$$\Sigma(\mathcal{L}) = 0 \quad (5.12)$$

is a complex linear multiplet, $\mathcal{L}$, then the most general solution to Eq. (5.11) is given by

$$A = A_0 + \mathcal{L} \quad (5.13)$$

where A_0 is any particular solution to Eq. (5.11). A particular solution is given by

$$A_0 = Y \bar{S}_0 (S_0 R)^{-1} \quad (5.14)$$

when R is the scalar curvature multiplet R defined as

$$S_0 R = \Sigma(\bar{S}_0) . \quad (5.15)$$

In fact

$$\Sigma(A_0) = Y (S_0 R)^{-1} \Sigma(S_0) = Y , \quad (5.16)$$

so the most general complex solution to Eq. (5.11) is given by Eq. (5.13) with A_0 given by Eq. (5.14). At this point we impose the reality condition $A = \bar{A}$.

The reality condition gives constraints on the components of A. χ must be a linear multiplet, moreover we have

$$A = \left(C, \bar{\zeta}, \frac{1}{2} (z+z^*), -\frac{i}{2} (z-z^*), V_m, i\gamma_5 \chi - \not{D}^C \zeta, \right. \\ \left. \frac{1}{2} (h+h^*) - \not{D}^C C \right), \quad (5.17)$$

where C, ζ are arbitrary real (Majorana) fields and V_m is real and subject to the constraint

$$D_m^C V^m = -\frac{i}{2} (h-h^*). \quad (5.18)$$

Here (z, χ_L, h) are the components of the chiral multiplet Y. To conclude the argument we have to find out whether (5.17) has a local solution. This amounts to seeing whether Eq. (5.18) can be solved with a local solution.

Since we know that if A is a solution, so is A-L and we can take L to be

$$L = (C, \zeta, 0, 0, V_m^L, -\not{D}^C \zeta, -\not{D}^C C), \quad (5.19)$$

where now

$$D_m^C V^{Lm} = 0. \quad (5.20)$$

Then we obtain

$$-\frac{i}{2} (h-h^*) = D_m^C V^m = e^{-1} \partial_\mu [e e^{m\mu} (V_m - V_m^L)] +$$

$$+ \frac{i}{2} z \bar{\Psi}_a \sigma^{ab} \Psi_{bR} - \frac{i}{2} z^* \bar{\Psi}_a \sigma^{ab} \Psi_{bL} + \frac{i}{2} \bar{\Psi} \cdot \gamma \gamma_5 \chi . \quad (5.21)$$

If we call $V_m^{\circ} = V_m - V_m^L$, Eq. (5.21) becomes

$$\begin{aligned} e^{-1} \partial_{\mu} (e e^{\mu} V_m^{\circ}) &= - \frac{i}{2} (h + z \bar{\Psi}_a \sigma^{ab} \Psi_{bR} + \bar{\Psi} \cdot \gamma \chi_L) + \text{h.c.} \\ &= e^{-1} \text{Im}[Y]_F \end{aligned} \quad (5.22)$$

where we used the component form of an F-density. Eq. (5.22) implies that for A to be real and local

$$\text{Im}[Y]_F = \partial_{\mu} (e^{\mu} K_m) \quad (5.23)$$

so that a particular solution to Eq. (5.22) is

$$V_m^{\circ} = K_m . \quad (5.24)$$

The general real solution is given by adding to the particular solution Q_Y° an arbitrary linear multiplet L. The components of Q_Y° are

$$Q_Y^{\circ} = \left[0, 0, -\frac{1}{2} (z+z^*), -\frac{i}{2} (z-z^*), K_m, i\gamma_5 \chi, \frac{1}{2} (h+h^*) \right] . \quad (5.25)$$

This concludes the conditions for the existence of the Chern-Simons form supermultiplets. The multiplet (5.19) has itself a local expression in terms of a chiral multiplet L_{α} with components

$$L_{\alpha} = \frac{1}{2} (1+\gamma_5) \left[\eta, -\sigma^{\mu\nu} (b_{\mu\nu} + \partial_{\mu} \xi_{\nu} - \partial_{\nu} \xi_{\mu}) - \frac{i}{4} (C+iD), \right. \\ \left. - \frac{1}{2} \zeta - \not{D} \eta \right] \quad (5.26)$$

where η , ξ_{μ} and D are undetermined and

$$v_m^L = e^{-1} e_{m\mu} \varepsilon^{\mu\nu\rho\sigma} \left(\partial_{\nu} b_{\rho\sigma} - \frac{i}{4} C \bar{\psi}_{\nu} \gamma_{\rho} \psi_{\sigma} \right) - \zeta_{mn} \psi^n. \quad (5.27)$$

This corresponds to Eq. (2.5) with the gauge invariance (2.6) in the superfield formalism. We would like to stress here that the particular solution Q_Y^0 is not a superconformal multiplet in the usual sense. It is a multiplet only under modified transformation rules designed to maintain the particular Wess-Zumino gauge condition which gives rise to (5.25).

VI. LOOP EFFECTS IN SUPERSTRING THEORIES AND THE 4D-ANOMALY CANCELLING TERM

In this section we describe the general form of the Lagrangian for the superstring effective action, taking into account loop effects. It is well known that the classical string effective action has a global scale invariance under a shift of the metric and of the dilaton field⁹

$$g_{\mu\nu} \rightarrow \lambda g_{\mu\nu} , \quad \text{Re } s \rightarrow \lambda \text{ Re } s \quad (6.1)$$

given by

$$\mathcal{L} \rightarrow \lambda \mathcal{L} . \quad (6.2)$$

The n-loop counterterm to the Lagrangian scales as

$$\mathcal{L}_{\text{n-loop}} \rightarrow \lambda^{(1-n)} \mathcal{L}_{\text{n-loop}} \quad (6.3)$$

under the same transformation.¹⁰ In effective theories coming from Calabi-Yau compactification, there is an extra global symmetry associated to the breathing mode of the Calabi-Yau manifold⁹

$$s \rightarrow r^{-1/2} s , \quad T \rightarrow r^{1/2} T , \quad C_i \rightarrow r^{1/4} C_i , \quad g_{\mu\nu} \rightarrow g_{\mu\nu} \quad (6.4)$$

where C_i are the charged matter fields and T is the chiral superfield associated to the breathing mode of the manifold. The classical Lagrangian under (6.4) scales as

$$\mathcal{L} \rightarrow r^{-1/2} \mathcal{L} \quad (6.5)$$

while the n-loop counterterm¹⁰ scales as

$$\mathcal{L}_{n\text{-loop}} \rightarrow r^{n-1/2} \mathcal{L}_{n\text{-loop}} . \quad (6.6)$$

In the superconformal formulation, the linear multiplet is introduced through the term

$$\int d^8\Omega \, LU \quad (6.7)$$

which shows that L is inert under both λ - and r -scale transformations. The compensator on the other hand scales as $S_0 \rightarrow S_0 \lambda^{1/3}$, $S_0 \rightarrow S_0 r^{-5/12}$. Using the λ and r symmetries as well as the superconformal invariance of the overall Lagrangian, one finds the general n-loop correction to the standard supergravity Lagrangian written in terms of the linear multiplet L , the chiral multiplets T and C_i and the compensator S_0

$$\left(\frac{L}{S_0 \bar{S}_0} \right)^{1/2 (3n-1)} S_0 \bar{S}_0 (\tilde{\phi})^{(3-n)/2} f \left(\frac{C_i}{(T+T^*)^{1/2}}, \frac{C^{*i}}{(T+T^*)^{1/2}} \right) \quad (6.8)$$

where $\tilde{\phi} = T + T^* - C_i C^{*i}$. Gauge invariance requires the following replacements:

$$L \rightarrow \hat{L}, \quad C^{*i} \rightarrow (e^V C^*)^i . \quad (6.9)$$

There is a further one-loop modification for the gauge part¹¹

$$\alpha T W^2 \tag{6.10}$$

as well as a one-loop anomaly cancelling term

$$c \hat{L} V \tag{6.11}$$

where V is a $U(1)$ gauge field of some gauge group with anomalous fermion representations in four dimensions. At one loop, Eq. (6.8) is particularly simple since it becomes independent of the compensator

$$\hat{L} \tilde{\phi} f . \tag{6.12}$$

For the case of a simple dimensional reduction to $D = 4$ or in the case of orbifold compactifications, the f function is a constant and we get

$$k \hat{L} (T + \bar{T} - c_i C^{*i}) . \tag{6.13}$$

Eqs. (6.10), (6.11), and (6.13) give all possible one-loop modifications to the standard supergravity forms. By performing the duality transformation, one gets a modified Kahler potential for the S field

$$-\log[S + \bar{S} + k(T + \bar{T} - c e^V C^* + cV)] . \tag{6.14}$$

One can in fact check that the bosonic components of the dimensionally reduced Wess-Zumino term do indeed reproduce Eqs. (6.10), (6.11), and (6.13). From Eq. (6.14) we see that the presence of an anomalous $U(1)$ group, which means $c \neq 0$ gives a mass to the vector multiplet. The massive vector field is given

by the superfield $cV + S + \bar{S} + k(T + \bar{T})$ which implies that the Goldstone degree of freedom is the combination $\partial_\mu \text{Im } s + k \partial_\mu \text{Im } t$. In the dual formulation this is a linear combination of the $b_{\mu\nu}$ and b_{ij} fields. We note that the relative coefficient between the $T + \bar{T}$ and $C_i C^{*i}$ term in Eq. (6.13) need not be equal (this is equivalent to choosing f in Eq. (6.8) to be)

$$1 + \delta \frac{CC^*}{T + T^* - CC^*}$$

so the Goldstone mode depends on a detailed calculation and cannot be inferred by the terms

$$C_i C^{*i} \Omega(V) \tag{6.15}$$

whose bosonic component $C_i \partial_\mu C^{*i} \Omega_3^\mu$ has been explicitly calculated by dimensional reduction of the 10-D anomaly terms. Eq. (6.8) also shows that the Kahler potential of the tree level gets in principle modified at each order in string perturbation theory. This implies a different restriction from Kahler geometry on higher curvature terms, in a way which depends on the loop expansion. The restrictions derived in Ref. 18 are the restrictions on higher curvature forms only for the case of the classical effective action.

VII. COMPONENT EXPRESSION FOR LINEAR MULTIPLIETS WITH GREEN-SCHWARZ MECHANISM

This section is devoted to expressing all the results of Sections II, III, and IV in component language as well as to give the Chern-Simons multiplets of Section V explicitly. In particular, we will continue the program of Ref. 7 to include the full contribution of the term

$$\text{Re} \int d^6 \mu \, b \, S \, W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} \quad (7.1)$$

in the action. From now on we will consider only the case of only one superfield of the type S and one linear superfield L dual to it.

First, we will show that the chiral multiplet $W_{\alpha\beta\gamma} W^{\alpha\beta\gamma}$ satisfies the conditions of the theorem of Section V. Since it has Weyl weight $w = 3$ we need only to show that the imaginary part of its F-type action is a total divergence. Its components are¹⁹

$$\begin{aligned} W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} = & \left[\bar{R}_{ab}(Q) R_L^{ab}(Q), \frac{1}{2} (\Gamma^{cd} W_{abcd} - 3i R_{ab}(A)) R_L^{ab}(Q), \right. \\ & 4\bar{R}_L^{ab}(Q) R_{ab}^{(cov)}(S) + \frac{1}{2} W_{abcd} W^{abcd} - \frac{1}{4} \epsilon^{abcd} W_{\mu\nu ab} W^{\mu\nu cd} \\ & \left. - \frac{3}{4} R_{ab}(A) R^{ab}(A) + \frac{3}{4} R_{ab}(A) \tilde{R}^{ab}(A) \right] \quad (7.2) \end{aligned}$$

in conformal tensor calculus. To obtain the multiplet in Poincaré tensor calculus we have to add to the last component:

$$-u \bar{R}_{ab}(Q) R_L^{ab}(Q) \quad (7.3)$$

where

$$u = S_{\text{aux}} - iP$$

in terms of the auxiliary fields in minimal supergravity. The generalized Weyl tensor and conformal curvature appearing in Eq. (7.2) are the ones appearing in Ref. 20. The conventions we will use here are the ones in Ref. 13, and the Γ -tensors are defined as usual.

The F-type conformal Lagrangian density for a chiral multiplet $[z, \chi_L, h]$ of weight $w = 3$ is given by:

$$e^{-1} \mathcal{L} = h + \bar{\Psi}_R \cdot \gamma \chi_L + \frac{1}{2} \bar{\Psi}_{\mu R} \Gamma^{\mu\nu} \Psi_{\nu R} z. \quad (7.4)$$

Applying this formula to the multiplet in (7.2), we get

$$\begin{aligned} e^{-1} \mathcal{L}(W_{\alpha\beta\gamma} W^{\alpha\beta\gamma}) &= \frac{1}{2} W_{\mu\nu ab} W^{\mu\nu ab} - \frac{3}{4} R_{\mu\nu}(A) R^{\mu\nu}(A) + 4 R_L^{\mu\nu}(Q) R_{\mu\nu}^{(\text{cov})}(S) \\ &\quad - \frac{1}{4} \epsilon^{abcd} W_{cd}^{\mu\nu} + \frac{3}{4} R_{\mu\nu}(A) \tilde{R}^{\mu\nu}(A) \\ &\quad + \frac{1}{2} \bar{\Psi}_{\rho R} \gamma^\rho (\Gamma^{ab} W_{\mu\nu ab} - 3i R_{\mu\nu}(A)) R_L^{\mu\nu}(Q) + \frac{1}{2} \bar{\Psi}_{\rho R} \Gamma^{\rho\sigma} \Psi_{\sigma R} \bar{R}_{\mu\nu}(Q) R_L^{\mu\nu}(Q). \end{aligned} \quad (7.5)$$

If one works within the framework of Poincaré tensor calculus, one obtains again the Lagrangian (7.5), since the terms containing u cancel. The imaginary part of (7.5) just gives:

$$i \text{Im}[W_{\alpha\beta\gamma} W^{\alpha\beta\gamma}]_F =$$

$$= e^{-1} \epsilon^{\mu\nu\rho\sigma} \partial_\mu \left[-\Omega_{\nu\rho\sigma}^{\text{LOR}} + \frac{2}{3} \Omega_{\nu\rho\sigma}^{(A)} - 8 \bar{\phi}_\nu \mathcal{D}_\rho \psi_\sigma + \bar{\phi}_\nu \gamma_\rho \psi_\sigma \right] \quad (7.6)$$

in terms of the Lorentz and A_μ Chern-Simons forms:

$$\Omega_{\nu\rho\sigma}^{\text{LOR}} = \omega_{[\nu}^{\text{ab}} R_{\rho\sigma]ab} + \frac{2}{3} \omega_{\nu ab} \omega_\rho^{bc} \omega_{\sigma c}^a \quad (7.7)$$

$$\Omega_{\nu\rho\sigma}^{(A)} = A_{[\nu} F_{\rho\sigma]} \quad (7.8)$$

with

$$F_{\rho\sigma} = \partial_\rho A_\sigma - \partial_\sigma A_\rho .$$

The remaining objects in Eq. (7.6) are¹⁷

$$\phi_\nu = \frac{1}{3} \gamma^\rho \left(s_{\nu\rho} + \frac{1}{2} \gamma_5 \tilde{s}_{\nu\rho} \right)$$

$$s_{\nu\rho} = -(\mathcal{D}_\nu \psi_\rho - \mathcal{D}_\rho \psi_\nu) \quad (7.9)$$

$$\mathcal{D}_\mu \psi_\nu = \left(\partial_\mu + \frac{1}{4} \omega_\mu^{\text{ab}} \Gamma_{ab} + \frac{i}{2} \gamma_5 A_\mu \right) \psi_\nu$$

and the spin connection contains ψ_μ torsion but no b_μ terms since the dilation field b_μ cancels from the Weyl multiplet.²⁰ The Lorentz Chern-Simons form satisfies:

$$\partial_{[\mu} \Omega_{\nu\rho\sigma]}^{\text{LOR}} = \frac{1}{2} R_{[\mu\nu}^{\text{ab}} R_{\rho\sigma]ab} \quad (7.10)$$

and the A_μ -form obeys:

$$\partial_{[\mu} \Omega_{\nu\rho\sigma]}^{(A)} = \frac{1}{2} F_{[\mu\nu} F_{\rho\sigma]} \quad (7.11)$$

which is the same as for an Abelian gauge field.

Next let us construct the chiral multiplet $SW_{\alpha\beta\gamma} w^{\alpha\beta\gamma}$ by multiplying (7.2) with the multiplet:

$$S = [z_s, \chi_{Ls}, h_s] \quad (7.12)$$

using the rules of tensor calculus. The result is:

$$\begin{aligned} SW_{\alpha\beta\gamma} w^{\alpha\beta\gamma} = & \left[z_s \bar{R}_{\mu\nu}^{(Q)} R_L^{\mu\nu}(Q), \right. \\ & \frac{1}{2} z_s (\Gamma^{ab} w_{\mu\nu ab} - 3i R_{\mu\nu}(A)) R_L^{\mu\nu}(Q) + \bar{R}_{\mu\nu}^{(Q)} R_L^{\mu\nu}(Q) \chi_{Ls}, \\ & z_s \left(\frac{1}{2} w_{\mu\nu ab} w^{\mu\nu ab} - \frac{3}{4} R_{\mu\nu}(A) R^{\mu\nu}(A) + 4 \bar{R}_L^{\mu\nu}(Q) R_{\mu\nu}^{(cov)}(S) \right. \\ & \left. - \frac{1}{4} \varepsilon^{abcd} w_{\mu\nu ab} w^{\mu\nu}_{cd} + \frac{3}{4} R_{\mu\nu}(A) \tilde{R}^{\mu\nu}(A) \right) + h_s \bar{R}_{\mu\nu}^{(Q)} R_L^{\mu\nu}(Q) \\ & \left. - \bar{\chi}_{Ls} (\Gamma^{ab} w_{\mu\nu ab} - 3i R_{\mu\nu}(A)) R_L^{\mu\nu}(Q) \right]. \quad (7.13) \end{aligned}$$

Now in order to write the Lagrangian corresponding to (7.1) we take the real part of the F-formula (7.4) applied to the multiplet (7.13):

$$\begin{aligned} e^{-1} \mathcal{L}(SW_{\alpha\beta\gamma} w^{\alpha\beta\gamma}) = & \frac{1}{2} z_s \left[\frac{1}{2} w_{\mu\nu ab} w^{\mu\nu ab} - \frac{3}{4} R_{\mu\nu}(A) R^{\mu\nu}(A) \right. \\ & \left. + 4 \bar{R}_L^{\mu\nu}(Q) R_{\mu\nu}^{(cov)}(S) - \frac{1}{4} \varepsilon^{abcd} w_{\mu\nu ab} w^{\mu\nu}_{cd} + \frac{3}{4} R_{\mu\nu}(A) \tilde{R}^{\mu\nu}(A) \right] + \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} h_s \bar{R}_{\mu\nu}(Q) R_L^{\mu\nu}(Q) - \frac{1}{2} \bar{\chi}_{Ls} (\Gamma^{ab} W_{\mu\nu ab} - 3iR_{\mu\nu}(A)) R^{\mu\nu}(Q) \\
& + \frac{1}{4} z_s \bar{\psi}_{R\rho} \gamma^\rho (\Gamma^{ab} W_{\mu\nu ab} - 3iR_{\mu\nu}(A)) R^{\mu\nu}(Q) + \frac{1}{2} \bar{R}_{\mu\nu}(Q) R_L^{\mu\nu}(Q) \bar{\psi}_{\rho R} \gamma^\rho \chi_{Ls} \\
& + \frac{1}{2} \bar{\psi}_{\rho R} \Gamma^{\rho\sigma} \psi_{\sigma R} z_s \bar{R}_{\mu\nu}(Q) R_L^{\mu\nu}(Q) + \text{h.c.} \tag{7.14}
\end{aligned}$$

Now we are ready to perform the duality transformation in component language to the density corresponding to the superfield action:

$$\int d^8\mu \phi(X, \bar{X}e^V, S+\bar{S}) + a \text{Re} \int d^6\mu S W_A W^A + b \text{Re} \int d^6\mu S W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} . \tag{7.15}$$

To this end we will collect the terms containing the auxiliary field h_s and the terms containing the complex scalar field z_s explicitly. (The other terms in the Lagrangian (7.15) only contain $\text{Re } z_s$ implicitly through the ϕ function and its derivatives.) The ones corresponding to the first two terms in Eq. (7.15) can be read off from Ref. 13 and the rest from Eq. (7.14). The relevant terms are:

$$\begin{aligned}
e^{-1}\mathcal{L}(z_s, h_s) = & -\frac{1}{2} \phi_{ss} \partial_\mu z_s \partial^\mu z^{*s} - \frac{1}{2} \phi_s^i D_\mu z_i \partial^\mu z^{*s} - \frac{1}{2} \phi_{is} \partial_\mu z_s D^\mu z^{*i} \\
& + \frac{1}{2} \phi_{ss} |h_s|^2 + \frac{1}{2} \phi_s^i h_i h^{*s} + \frac{1}{2} \phi_{si} h_s h^{*i} - \phi_s^{IJ-} \chi_{LI} \chi_{LJ} h^{*s} \\
& + \phi_J^{Is-} \chi_{LI} \partial z_s \chi_R^J + \frac{1}{3} u^* \phi_s h_s + \frac{i}{3} A^\mu \phi_s \partial_\mu z_s \\
& + \phi_s^{I-} \bar{\psi}_{\mu L} \partial z^{*s} \gamma^\mu \chi_{LI} - \frac{1}{8} e^{-1} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_\nu \psi_\rho \phi_s \partial_\sigma z_s \\
& + \frac{a}{2} z_s \left(-\frac{1}{4} F_{\mu\nu}^A F_A^{\mu\nu} - \frac{1}{2} \bar{\lambda}^A \tilde{\not{D}} \lambda_A + \frac{1}{2} D^A D_A + \frac{1}{4} F_{\mu\nu}^A \bar{\psi}_\rho \sigma^{\mu\nu} \gamma^\rho \lambda_A \right) +
\end{aligned}$$

$$\begin{aligned}
& - \frac{a}{8} e^{-1} \varepsilon^{\mu\nu\rho\sigma} \Omega_{\nu\rho\sigma}^{(YM)} \partial_{\mu} z_s + \frac{a}{4} \partial_{\mu} z_s \bar{\lambda}^A \gamma_5 \gamma^{\mu} \lambda_A - \frac{a}{4} h_s \bar{\lambda}_L^A \lambda_{LA} \\
& - \frac{b}{4} e^{-1} \varepsilon^{\mu\nu\rho\sigma} \left[- \Omega_{\nu\rho\sigma}^{LOR} + \frac{2}{3} \Omega_{\nu\rho\sigma}^{(A)} - 8 \bar{\phi}_v \mathcal{D}_{\rho} \psi_{\sigma} + \bar{\phi}_v \gamma_{\rho} \psi_{\sigma} \right] \partial_{\mu} z_s \\
& + \frac{b}{2} h_s \bar{R}_{\mu\nu}^{(Q)} R_L^{\mu\nu}(Q) + \text{h.c.} \tag{7.16}
\end{aligned}$$

where some integrations by parts have been performed. A few words of explanation are due. The indices I,J run over all the chiral multiplets including S, while the index i runs over all the chiral multiplets excluding S, just like in Section II. The index A is the Yang-Mills index corresponding to the adjoint representation of the gauge group. The fields $u = S_{\text{aux}} - iP$ and A_{μ} are the auxiliary fields of minimal supergravity. D^A is the auxiliary field which is the scalar partner of the gaugino λ^A and the gauge vector field A_{μ}^A in the gauge multiplet. Next, for the terms that survive when we put $b = 0$ in (7.16) we take the following explanation from Ref. 13: (a) $\tilde{D}_{\mu} \lambda^A$ is a covariant derivative with respect to supergravity and gauge transformations, but without the $\psi_{\mu} D^A$ term, while the other derivatives are Yang-Mills and general coordinate covariant but have no ψ_{μ} torsion; (b) the indices attached to ϕ mean differentiation:

$$\phi^I = \frac{\partial \phi}{\partial z_I}, \quad \phi_J = \frac{\partial \phi}{\partial z^{*J}}$$

and so on. Finally $\Omega_{\mu\nu\rho}^{(YM)}$ is the Yang-Mills Chern-Simons form:

$$\Omega_{\mu\nu\rho}^{(YM)} = A_{[\mu}^A F_{\nu\rho]A} - \frac{1}{3} \tilde{g} C_{ABC} A_{\mu}^A A_{\nu}^B A_{\rho}^C \tag{7.17}$$

where $\tilde{g}$ is the gauge coupling constant, C_{ABC} are the structure constants of

the Yang-Mills group and of course $F_{\mu\nu}^A$ is the non-Abelian field strength. $\Omega_{\mu\nu\rho}^{(YM)}$ satisfies:

$$\epsilon^{\mu\nu\rho\sigma} \partial_{\mu} \Omega_{\nu\rho\sigma}^{(YM)} = e F_{\mu\nu}^A \tilde{F}_A^{\mu\nu}. \quad (7.18)$$

The imaginary part $\text{Im } z_s$ of the component field z_s only appears in the Lagrangian through its derivative $\partial_{\mu} \text{Im } z_s$ (let us remember that ϕ and therefore its derivatives depend only on $\text{Re } z_s$). As a consequence we can perform a Legendre transformation to replace $\text{Im } z_s$ by a massless antisymmetric tensor $b_{\mu\nu}$ by a well known procedure explained below.²¹ However, additional steps must be taken to make sure that the theory in the dual formulation is supersymmetric.

The steps are the following:⁷

1. eliminate the auxiliary field of the S multiplet h_s from the Lagrangian through its equation of motion;
2. replace $\partial_{\mu} \text{Im } z_s$ by L_{μ} which from then on is assumed unconstrained;
3. add to the Lagrangian a term with the Lagrange multiplier $b_{\mu\nu}$

$$\mathcal{L}_c = \frac{i}{4} \epsilon^{\mu\nu\rho\sigma} b_{\mu\nu} \partial_{\rho} L_{\sigma} \quad (7.19)$$

(at this point if one varies with respect to $b_{\mu\nu}$ and replaces back the result, one obtains the initial theory, of course); and

4. eliminate L_{μ} by using its equation of motion as derived from the Lagrangian:

$$\mathcal{L} + \mathcal{L}_c. \quad (7.20)$$

This procedure requires some comment. Step 1 is necessary because $b_{\mu\nu}$ belongs to a linear multiplet which, as we mentioned in Section II, contains no auxiliary fields and therefore h_s has no place in the final theory. The fields h_s and L_μ as given by their equations of motion have to be replaced everywhere: in the Lagrangian as well as the supersymmetry transformation rules of all the other fields. This fixes the transformation rules of all the fields in the new theory except $b_{\mu\nu}$. The transformation rule of the $b_{\mu\nu}$ field is obtained by demanding invariance of the Lagrangian (7.20) with the transformation law of L_μ given by one of the $\partial_\mu \text{Im } z_s$.²¹ The point is that now $\partial_{[\mu} L_{\nu]}$ cannot be put equal to zero, and $\delta b_{\mu\nu}$ has to be chosen such that the terms of this type cancel.

Let us apply the procedure to the case at hand. First we write the Lagrangian (7.16) replacing $\partial_\mu \text{Im } z_s$ by L_μ and add $\mathcal{L}_c$ in (7.19):

$$\begin{aligned}
 e^{-1} \mathcal{L} = & -\phi_{ss} (\partial_\mu \text{Re } z_s \partial^\mu \text{Re } z_s + L_\mu L^\mu) \\
 & - \partial_\mu \text{Re } z_s (\phi_{si} D^\mu z^{*i} + \phi_s^i D^\mu z_i) - i L_\mu (\phi_{si} D^\mu z^{*i} - \phi_s^i D^\mu z_i) \\
 & + \phi_{ss} |h_s|^2 + \phi_s^i h_i h^{*s} + \phi_{si} h_s h^{*i} - \phi_s^{IJ-} \chi_{LI} \chi_{LJ} h^{*s} \\
 & - \phi_{sIJ} \bar{\chi}_R^I \chi_R^J h_s + 2i \phi_{sJ} \bar{\chi}_{LI}^I \gamma^\mu \chi_R^J L_\mu + \frac{1}{3} \phi_s (u^* h_s + u h^{*s}) \\
 & - \frac{2}{3} \phi_s^A L_\mu + \partial_\mu \text{Re } z_s \bar{\psi}_\rho \gamma^\mu \gamma^\rho (\phi_s^I \chi_{LI}^I + \phi_{sI} \chi_R^I) \\
 & + i L_\mu \bar{\psi}_\rho \gamma^\mu \gamma^\rho (\phi_{sI} \chi_R^I - \phi_s^I \chi_{LI}^I) \\
 & - \frac{i}{4} e^{-1} \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_\nu \psi_\rho \phi_s^L \sigma +
 \end{aligned}$$

$$\begin{aligned}
& + a \operatorname{Re} z_s \left(-\frac{1}{4} F_{\mu\nu}^A F_A^{\mu\nu} - \frac{1}{2} \bar{\lambda}^A \not{D} \lambda_A + \frac{1}{2} D^A D_A + \frac{1}{4} F_{\mu\nu}^A \bar{\psi}_\rho \sigma^{\mu\nu} \gamma^\rho \lambda_A \right) \\
& - \frac{ia}{4} e^{-1} \varepsilon^{\mu\nu\rho\sigma} L_\mu \left[\Omega_{\nu\rho\sigma}^{(YM)} + \frac{1}{3!} \bar{\lambda}^A \Gamma_{\nu\rho\sigma} \lambda_A \right] - \frac{a}{4} h_s \bar{\lambda}_L^A \lambda_{LA} - \frac{a}{4} h^{*s} \bar{\lambda}_R^A \lambda_{RA} \\
& + \frac{ib}{2} e^{-1} \varepsilon^{\mu\nu\rho\sigma} L_\mu \left[-\Omega_{\nu\rho\sigma}^{LOR} + \frac{2}{3} \Omega_{\nu\rho\sigma}^{(A)} - 8\bar{\phi}_\nu \not{D}_\rho \psi_\sigma + \bar{\phi}_\nu \gamma_\rho \psi_\sigma \right] \\
& + \frac{b}{2} h_s \bar{R}_{\mu\nu}(Q) R_L^{\mu\nu}(Q) + \frac{b}{2} h^{*s} \bar{R}_{\mu\nu}(Q) R_R^{\mu\nu}(Q) \\
& - \frac{i}{4} \varepsilon^{\mu\nu\rho\sigma} b_{\mu\nu} \partial_\rho L_\sigma .
\end{aligned} \tag{7.21}$$

The L_μ part of the Lagrangian is just:

$$e^{-1} \mathcal{L}(L_\mu) = -\phi_{ss} L_\mu L^\mu + L_\mu N^\mu \tag{7.22}$$

with

$$\begin{aligned}
N^\mu &= -i(\phi_{sI} D^\mu z^{*I} - \phi_s^I D^\mu z_I) + 2i\phi_{sJ}^I \bar{\chi}_{LI} \gamma^\mu \chi_R^J - \frac{2}{3} \phi_s^A A^\mu \\
&+ i\bar{\psi}_\rho \gamma^\mu \gamma^\rho (\phi_{sI} \chi_R^I - \phi_s^I \chi_{LI}) + \frac{i}{4} e^{-1} \varepsilon^{\mu\nu\rho\sigma} \phi_s \bar{\psi}_\nu \gamma_\rho \psi_\sigma + e^{-1} H^\mu
\end{aligned} \tag{7.23}$$

where H^μ is the generalized field strength tensor for $b_{\mu\nu}$:

$$\begin{aligned}
H^\mu &= \frac{i}{4} \varepsilon^{\mu\nu\rho\sigma} \left[\partial_\nu b_{\rho\sigma} - a \left(\Omega_{\nu\rho\sigma}^{(YM)} + \frac{1}{3!} \bar{\lambda}^A \Gamma_{\nu\rho\sigma} \lambda_A \right) + 2b \left(-\Omega_{\nu\rho\sigma}^{LOR} + \frac{2}{3} \Omega_{\nu\rho\sigma}^{(A)} \right. \right. \\
&\quad \left. \left. + 8\bar{\phi}_\nu \not{D}_\rho \psi_\sigma + \bar{\phi}_\nu \gamma_\rho \psi_\sigma \right) \right] .
\end{aligned} \tag{7.24}$$

(The value of the coefficient of $\Omega_{\nu\rho\sigma}^{(A)}$ here and therefore in the rest of the

paper would be modified if one added the covariant terms mentioned in the remark under Eq. (4.4). For the particular values of a', b' which correspond to the super--Gauss-Bonnet combination, see Ref. 7.) The L_μ equation of motion implies:

$$L^\mu = \frac{1}{2\phi_{ss}} N^\mu \quad (7.25)$$

and replacing in (7.22) we get:

$$\mathcal{L}(L_\mu) = \frac{e}{4\phi_{ss}} N_\mu N^\mu. \quad (7.26)$$

The h_s part of the Lagrangian is:

$$e^{-1} \mathcal{L}(h_s) = \phi_{ss} |h_s|^2 + A^* h_s + A h^{*s} \quad (7.27)$$

with

$$A = \phi_s^i h_i - \phi_s^{IJ-} \chi_{LI} \chi_{LJ} + \frac{1}{3} \phi_s u - \frac{a}{4} \bar{\lambda}_R^A \lambda_{RA} + \frac{b}{2} \bar{R}_{\mu\nu}(Q) R_R^{\mu\nu}(Q). \quad (7.28)$$

The h_s equation of motion gives:

$$h_s = - \frac{A}{\phi_{ss}} \quad (7.29)$$

and hence:

$$\mathcal{L}(h_s) = -e \frac{|A|^2}{\phi_{ss}}. \quad (7.30)$$

Next we will derive the supersymmetry transformation law of $b_{\mu\nu}$. As mentioned earlier, we only need to look at the terms containing $\partial_{[\rho} L_{\sigma]}$ in the variation of the Lagrangian (7.20). The relevant terms only come from the variation of (7.21) and they are:

$$\begin{aligned}
 & i e L_{\mu} (\delta \bar{\psi}_{\rho}) \gamma^{\mu} \gamma^{\rho} (\phi_{sI} \chi_R^I - \phi_s^I \chi_{LI}) + \frac{i}{2} \phi_s \varepsilon^{\mu\nu\rho\sigma} (\delta \bar{\psi}_{\nu}) \gamma_{\rho} \psi_{\sigma} L_{\mu} \\
 & - \frac{i}{4} \varepsilon^{\mu\nu\rho\sigma} (\delta b_{\mu\nu}) \partial_{\rho} L_{\sigma} - \frac{i}{4} \varepsilon^{\mu\nu\rho\sigma} \left[a \delta \Omega_{\nu\rho\sigma}^{(YM)} \right. \\
 & \left. + 2b \left(-\delta \Omega_{\nu\rho\sigma}^{LOR} + \frac{2}{3} \delta \Omega_{\nu\rho\sigma}^{(A)} \right) \right] L_{\mu}
 \end{aligned} \tag{7.31}$$

since the variations of the Rarita-Schwinger field and the Chern-Simons forms contain derivatives. Indeed:

$$\delta \Omega_{\nu\rho\sigma}^{(YM)} = 2A_{[v}^A \partial_{\rho} \delta A_{\sigma]A} + \text{other terms} \tag{7.32a}$$

$$\delta \Omega_{\nu\rho\sigma}^{LOR} = 2\omega_{[v}^{ab} \partial_{\rho} \delta \omega_{\sigma]ab} + \text{other terms} \tag{7.32b}$$

$$\delta \Omega_{\nu\rho\sigma}^{(A)} = 2A_{[v} \partial_{\rho} \delta A_{\sigma]} + \text{other terms} . \tag{7.32c}$$

From Eq. (7.31) we can finally obtain the terms containing $\partial_{[\rho} L_{\sigma]}$ in the supersymmetry variation of the Lagrangian:

$$\begin{aligned}
 \delta \mathcal{L}(\partial_{[\rho} L_{\sigma]}) &= -\frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\Gamma}^{\mu\nu\rho\sigma} (\phi_s^I \chi_{LI} + \phi_{sI} \chi_R^I) \\
 &- \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \phi_s \varepsilon_{\rho}^{\mu\nu\rho\sigma} \gamma_{\rho} \psi_{\sigma} \partial_{\nu} L_{\mu} - \frac{i}{4} \varepsilon^{\mu\nu\rho\sigma} (\delta b_{\mu\nu}) \partial_{\rho} L_{\sigma} +
 \end{aligned}$$

$$- \frac{ia}{2} \varepsilon^{\mu\nu\rho\sigma} A_\rho^A (\delta A_{\sigma A}) \partial_\nu L_\mu + ib \varepsilon^{\mu\nu\rho\sigma} (\omega_\rho^{ab} \delta \omega_{\sigma ab} - \frac{2}{3} A_\rho \delta A_\sigma) \partial_\nu L_\mu . \quad (7.33)$$

The variations of the spin connection and the axiliary field A_μ are given²² by

$$\delta \omega_{\mu ab} = \frac{1}{4} \bar{\varepsilon} (2\gamma_{[a} \psi_{b]\mu}^{(\text{cov})} - \gamma_\mu \psi_{ab}^{(\text{cov})} + \{\Gamma_{ab}, \eta\} \psi_\mu) \quad (7.34a)$$

$$\delta A_\mu = \frac{3i}{4} \bar{\varepsilon} \gamma_5 \left(R_\mu^{(\text{cov})} - \frac{1}{3} \gamma_\mu \gamma \cdot R^{(\text{cov})} \right) \quad (7.34b)$$

where $R_\mu^{(\text{cov})}$ is given by:

$$R^\mu^{(\text{cov})} = \frac{1}{2} e^{-1} \varepsilon^{\mu\nu\rho\sigma} \gamma_5 \gamma_\nu \psi_{\rho\sigma}^{(\text{cov})}$$

$$\psi_{\rho\sigma}^{(\text{cov})} = 2 \left(D_{[\rho} + \frac{i}{2} A_{[\rho} \gamma_5 - \frac{1}{2} \gamma_{[\rho} \eta \right] } \psi_{\sigma]} \right) \quad (7.35)$$

$$\eta = - \frac{1}{3} (S_{\text{aux}} - i\gamma_5 P - i\lambda \gamma_5) .$$

The transformation rule of the gauge field is just:

$$\delta A_\mu^A = - \frac{1}{2} \bar{\varepsilon} \gamma_\mu \lambda^A . \quad (7.36)$$

Therefore, demanding that the expression (7.33) vanishes we obtain the transformation law for $b_{\mu\nu}$:

$$\delta b_{\mu\nu} = 2\bar{\varepsilon} \Gamma_{\mu\nu} (\phi_s^I \chi_{LI}^I + \phi_{sI} \chi_R^I) + 2\phi_s \bar{\varepsilon} \gamma_{[\mu} \psi_{\nu]} \\ - a A_{[\mu}^A \bar{\varepsilon} \gamma_{\nu]} \lambda_A - b \left[\bar{\varepsilon} (2\gamma^a \psi_{[\nu}^{(\text{cov})b} - \gamma_{[\nu} \psi^{(\text{cov})ab} + \{\Gamma^{ab}, \eta\} \psi_{[\nu]} \omega_{\mu]ab} + \right.$$

$$- 2i A_{[\mu} \bar{\epsilon} \gamma_5 \left(R^{(\text{cov})}_{\nu]} - \frac{1}{3} \gamma_{\nu]} \gamma \cdot R^{(\text{cov})} \right) \Big] . \quad (7.37)$$

We have seen in Section II that $b_{\mu\nu}$ has 2 partners: a real scalar C^L and a Majorana fermion χ^L . The scalar partner⁷ is given by

$$C^L = \phi_s \quad (7.38)$$

and the fermion partner is obtained from the transformation rule $\delta C^L = \bar{\epsilon} \chi^L$ ⁷

$$\chi^L = \phi_s^I \chi_{LI}^I + \phi_{sI} \chi_R^I . \quad (7.39)$$

The transformation rule of χ^L can be read off from Ref. 13 replacing h_s by the expression (7.29) and $\partial_\mu \text{Im } z_s$ by (7.25)

$$\begin{aligned} \delta \chi^L = & \frac{1}{4} \hat{\partial} C^L \epsilon + \frac{1}{2} C^L \eta \epsilon - \frac{i}{4} e^{-1} H^{\mu} \gamma_{\mu} \gamma_5 \epsilon \\ & + \frac{1}{16} e^{-1} \epsilon^{\mu\nu\rho\sigma} (C^L \bar{\psi}_{\nu} \gamma_{\rho} \psi_{\sigma} + 2 \bar{\psi}_{\nu} \Gamma_{\rho\sigma} \chi^L) \gamma_{\mu} \gamma_5 \epsilon \\ & + \frac{a}{8} (\bar{\lambda}_R^A \lambda_{RA} \epsilon_L + \bar{\lambda}_L^A \lambda_{LA} \epsilon_R) \\ & - \frac{b}{4} (\bar{R}_{\mu\nu}(Q) R^{\mu\nu}_R(Q) \epsilon_L + \bar{R}_{\mu\nu}(Q) R^{\mu\nu}_L(Q) \epsilon_R) \end{aligned} \quad (7.40)$$

where the Poincaré supercovariant derivative of C^L is:

$$\hat{D}_{\mu} C^L = \partial_{\mu} C^L - \bar{\psi}_{\mu} \chi^L . \quad (7.41)$$

In terms of C^L and χ^L we can rewrite (7.37):

$$\begin{aligned}
 \delta b_{\mu\nu} = & 2\bar{\epsilon}\Gamma_{\mu\nu}\chi^L + 2C^L\bar{\epsilon}\gamma_{[\mu}\psi_{\nu]} - aA^A_{[\mu}\bar{\epsilon}\gamma_{\nu]}\lambda_A \\
 & - b\left[\bar{\epsilon}(2\gamma^a\psi^{(\text{cov})b}_{[\nu} - \gamma_{[\nu}\psi^{(\text{cov})ab} + \{\Gamma^{ab}, \eta\}\psi_{\nu]}\omega_{\mu]ab} \right. \\
 & \left. - 2iA_{[\mu}\bar{\epsilon}\gamma_5\left(R^{(\text{cov})}_{\nu]} - \frac{1}{3}\gamma_{\nu]}\gamma\cdot R^{(\text{cov})}\right)\right] .
 \end{aligned} \tag{7.42}$$

Finally we need to replace the fields $\text{Re } z_s$ and χ_s of the old theory by C^L and χ^L of the new one. From Eq. (7.38) we get $\text{Re } z_s$:

$$\text{Re } z_s = \frac{1}{2} \phi_s^{-1}(C^L) \tag{7.43}$$

while from Eq. (7.39) we can obtain χ_s :

$$\chi_s = \frac{1}{\phi_{ss}} \chi^L . \tag{7.44}$$

The field $b_{\mu\nu}$ must transform under the gauge group as well as the Lorentz one if the final Lagrangian is to be invariant under these two groups of transformations. The way $b_{\mu\nu}$ transforms is derived by imposing invariance of its generalized field strength H^μ in (7.24). From the transformations:

$$\delta_{\text{YM}}(\Lambda^A) \Omega_{\mu\nu\rho}^{\text{YM}} = 2\partial_{[\mu}(\Lambda^A\partial_{\nu}A_{\rho]}A) \tag{7.45}$$

and

$$\delta_{\text{LOR}}(\Lambda^{ab}) \Omega_{\mu\nu\rho}^{\text{LOR}} = 2\partial_{[\mu}(\Lambda^{ab}\partial_{\nu}\omega_{\rho]}ab) , \tag{7.46}$$

the conditions of invariance of the generalized field strength:

$$\delta_{YM}(\Lambda^A) H^\mu = 0 \quad (7.47a)$$

$$\delta_{LOR}(\Lambda^{ab}) H^\mu = 0 \quad (7.47b)$$

imply the anomalous transformations of $b_{\mu\nu}$:

$$\delta_{YM}(\Lambda^A) b_{\mu\nu} = 2a\Lambda^A \partial_{[\mu} A_{\nu]A} \quad (7.48a)$$

$$\delta_{LOR}(\Lambda^{ab}) b_{\mu\nu} = -4b\Lambda^{ab} \partial_{[\mu} \omega_{\nu]ab} . \quad (7.48b)$$

In conformal tensor calculus $b_{\mu\nu}$ has also anomalous chiral gauge transformations due to the presence of the Chern-Simons A_μ form in (7.24). Under these transformations, the change in the chiral field is

$$A_\mu^{\text{chiral}} = A_\mu^{\text{chiral}} + \partial_\mu \theta \quad (7.49)$$

implying:

$$\delta_{\text{chiral}} \Omega_{\nu\rho\sigma}(A) = -\frac{3}{2} \partial_{[\nu} (\theta F_{\rho\sigma]} \quad (7.50)$$

and thus change in $b_{\mu\nu}$ is

$$\delta_{\text{chiral}} b_{\mu\nu} = \theta F_{\mu\nu} . \quad (7.51)$$

As mentioned in Section II, the fields C^L , χ^L and $b_{\mu\nu}$ form the prepotential multiplet L_α . This multiplet can be embedded into the vector multiplet $\hat{L}$:

$$\hat{L} = (C^L, Z^L, H^L, K^L, B_\mu^L, \Lambda^L, D^L) \quad (7.52)$$

where

$$Z^L = -2i\gamma_5 \chi^L \quad (7.53)$$

and we can read off the H^L , K^L and B_μ^L components from (7.40) and the transformation rule of Z^L which is:

$$\delta Z^L = \frac{1}{2} (i\gamma_5 H^L - K^L - B^L + i\hat{\psi} C^L \gamma_5) \varepsilon. \quad (7.54)$$

So:

$$H^L = \frac{2}{3} C^L S_{aux} - \frac{a}{4} \bar{\lambda}^A \lambda_A + \frac{b}{2} \bar{R}_{\mu\nu}(Q) R^{\mu\nu}(Q) \quad (7.55a)$$

$$K^L = -\frac{2}{3} C^L P - \frac{a}{4} i\bar{\lambda}^A \gamma_5 \lambda_A + \frac{b}{2} i \bar{R}_{\mu\nu}(Q) \gamma_5 R^{\mu\nu}(Q) \quad (7.55b)$$

$$B^{L\mu} = \frac{2}{3} C^L A^\mu - \frac{i}{4} e^{-1} \varepsilon^{\mu\nu\rho\sigma} (C^L \bar{\psi}_\nu \gamma_\rho \psi_\sigma + 2 \bar{\psi}_\nu \Gamma_{\rho\sigma} \chi^L) - e^{-1} H^\mu \quad (7.55c)$$

where H^μ is the generalized field strength given in (7.24). The terms involving C^L and χ^L and the $b_{\mu\nu}$ part in H^μ correspond to a pure linear multiplet. The components Λ^L and D^L will be given below. The chiral projection of the multiplet $\hat{L}$ in (7.52) has as its first component:²³

$$\begin{aligned} & \frac{1}{2} \left[H^L - \frac{2}{3} C^L S_{aux} - i \left(K^L + \frac{2}{3} C^L P \right) \right] \\ &= -\frac{a}{2} \bar{\lambda}_L^A \lambda_{LA} + b \bar{R}_{\mu\nu}(Q) R_L^{\mu\nu}(Q) \end{aligned} \quad (7.56)$$

which is precisely the first component of the chiral multiplet:

$$a W^A W_A + b W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} .$$

Therefore we conclude

$$\Sigma(\hat{L}) = a W^A W_A + b W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} . \quad (7.57)$$

Note that even though in this section we have worked up to now within the framework of Poincaré tensor calculus, the auxiliary fields S_{aux} and P disappear at the end as they should since $W^A W_A$ and $W_{\alpha\beta\gamma} W^{\alpha\beta\gamma}$ are also conformal multiplets.

Finally, the Λ^L and D^L components in conformal tensor calculus can be obtained from Eqs. (5.17) in terms of the components of the multiplet in (7.57):

$$a W^A W_A + b W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} = [z, \chi_L^{(W^2)}, h] \quad (7.58)$$

(the components of $W^A W_A$ can be extracted from Ref. 13, for instance). The result is:

$$\begin{aligned} \Lambda^L &= i\gamma_5 \chi^{(W^2)} - \not{D}^C Z^L \\ D^L &= \frac{1}{2} (h+h^*) - \not{D}^C C^L . \end{aligned} \quad (7.59)$$

Of course, we can set C^L , χ^L and $b_{\mu\nu}$ all equal to zero and still obtain a multiplet satisfying (7.57) since this amounts to subtracting a linear

multiplet from the result. The components Λ^L and D^L of the Poincaré multiplet can be obtained using standard rules, while the H^L , K^L and B_μ^L components of the conformal $\hat{L}$ multiplet are obtained from Eq. (7.55) by simply dropping the first term in each of these equations.

VIII. CONCLUSION

In the present paper, we have derived super Chern-Simons forms both in superspace and in tensor calculus language. They appear naturally in supergravity theories containing linear multiplets. The general structure of these theories has been derived through duality transformations from a theory in the so-called standard form. We have also emphasized the relevance of our results in connection with the effective Lagrangian of supersymmetric strings.

The "minimal" Chern-Simons multiplets are related through a Bianchi Identity to the square of the supersymmetric Yang-Mills field strength and the square of the Weyl multiplet. The corresponding effective Lagrangian contains (supersymmetric) ghosts. The ghost-free combination related to the super Gauss-Bonnet invariant is more complicated to derive in component form. Also different quadratic invariants in the gravitational curvatures, other than the Weyl square multiplet, depend on the supergravity (compensator) auxiliary fields. Depending on the particular set of auxiliary fields, these terms will generally lead to different choices of effective Lagrangians.

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