BOSONIC STRINGS AND COMPLEX STRUCTURES

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ABSTRACT

We write the action of the bosonic string in terms of two-dimensional complex structures rather than of two-dimensional metrics. We describe in some detail the behaviour under reparametrization of the world-sheet, and in particular we give an expression for the two-dimensional diffeomorphism anomaly. We describe a possible gauge-fixing procedure for the BRST quantization.

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The expression of the (bosonic) string action has evolved from the original Nambu-Goto form ^{1),2)} to a two-dimensional gravity-like form ^{3),4)} where the reparametrization invariance is made explicit with the help of a two-dimensional metric field. This form possesses a Weyl rescaling invariance as well, as emphasized by Polyakov ^{5),6)}.

Taking this form as a starting point we suggest another geometrically natural expression of the action of the bosonic string in terms of a two-dimensional complex structure rather than a metric. It turns out to be the integral of a two-form and, thus, is trivially invariant by reparametrization. The construction is along the following lines: for a two-dimensional surface Σ (oriented Riemann surface) with a metric g , embedded (by an embedding X) in a D -dimensional space-time T , equipped with a metric G , we first define

$$S(g, X, G) = \int_{\Sigma} \sqrt{g} g^{\alpha\beta} \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu} G_{\mu\nu} \quad (1)$$

On the one hand, S is invariant under the semi-direct product $\Gamma_{\Sigma} = D_{\Sigma} \times W_{\Sigma}$ of the group of diffeomorphisms D_{Σ} of Σ and the group W_{Σ} of conformal rescalings of the metric g , and consequently induces - for fixed embedding X and metric G - a function on the moduli space η_{Σ} ($\eta_{\Sigma} = \mathcal{M}/\Gamma_{\Sigma}$, where $\mathcal{M}$ is the space of metrics on Σ).

On the other hand η_{Σ} may also be considered as the quotient of the space $\mathcal{C}_{\Sigma}$ of complex structures on Σ by D_{Σ} , through the identification $\mathcal{C}_{\Sigma} \approx \mathcal{M}/W_{\Sigma}$ which we will detail later.

What we then do is to rewrite the action as a local functional $\mathcal{S}$ on $\mathcal{C}_{\Sigma}$, given by the integral over Σ of a certain two-form, of which we give the expression and identify the geometrical meaning. This two-form is constructed from a complex structure field J , an embedding X and a metric G . We take the functional $\mathcal{S}(J, X, G)$ [Eq. (6)] as our classical action. Of course, since J does not transform under Weyl rescalings, Weyl invariance is now an irrelevant concept.

The above construction was suggested to us by the observation that the conformal rescaling degrees of freedom look mathematically quite redundant [we have in mind especially Ref. 7)]. However, if we regard $\mathcal{S}(J, X, G)$ as a field theory action, two obvious but important remarks are in order. First, we do not expect $S(g, X, G)$ to lead, in general, to the same quantum action as $\mathcal{S}(J, X, G)$ due to the presence of the conformal anomaly in $D \neq 26$. Second, we must make sure that our theory based on $\mathcal{S}(J, X, G)$ is formulated in such a way as to guarantee locality.

Quantization of the theory means quantization of both embeddings and complex structures, i.e., in the framework of functional integration, a definition of

$$\int \mathcal{D}J \mathcal{D}X \ e^{-\mathcal{S}(J, X, G)}$$

Although the action $\mathcal{S}$ is invariant under D_Σ , quantization of the left- and right-mover part of the embeddings X separately, produces in general an anomaly. We give an expression of the cocycle to which this anomaly is proportional [Eq. (7)]. It is remarkable that this expression is local in J .

Finally, we briefly describe a possible gauge-fixing procedure for the quantization of the complex structure field, leading to the Teichmüller space (or the moduli space), and give the Faddeev-Popov determinant. The proposal presented here fits into the already extensive literature on the subject of string quantization⁸⁾.

1. - TWO-DIMENSIONAL COMPLEX STRUCTURES

A complex structure⁹⁾ is the data at each point x of Σ of an orientation preserving linear endomorphism J of the tangent space such that $J^2 = -1$. An explicit parametrization may be realized locally in co-ordinates $z, \bar{z}$ in terms of a complex-valued function μ ($|\mu|^2 < 1$) by

$$J^\alpha_\beta = \frac{i}{1-\mu\bar{\mu}} \begin{pmatrix} 1+\mu\bar{\mu} & 2\mu \\ -2\bar{\mu} & -(1+\mu\bar{\mu}) \end{pmatrix} \quad (2)$$

The "function" μ is actually a Beltrami differential¹⁰⁾ and the solution of the Beltrami equation $\mu = u_{\bar{z}}/u_z$ yields co-ordinates $u, \bar{u}$ which trivialize the complex structure into the form $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. The group of diffeomorphism of Σ acts naturally on Beltrami differentials through composition with the solution of the Beltrami equation⁷⁾. The infinitesimal action takes the form:

$$\delta\mu = \xi^\alpha \partial_\alpha \mu + \mu (\partial_{\bar{z}} \xi^{\bar{z}} - \partial_z \xi^z) - \mu^2 \partial_z \xi^{\bar{z}} + \partial_{\bar{z}} \xi^z$$

which shows that μ is far from being a tensor for non-holomorphic diffeomorphisms. One has to notice that for this transformation of μ , the complex structure J is a tensor, and its transformation coincides with the one induced by pull-back of the action of D_Σ on the tangent space. The infinitesimal transformation reads in co-ordinates:

$$\delta J^\alpha_\beta = \xi^\gamma \partial_\gamma J^\alpha_\beta + \partial_\beta \xi^\gamma J^\alpha_\gamma - \partial_\gamma \xi^\alpha J^\gamma_\beta$$

2. - METRIC VERSUS COMPLEX STRUCTURE

Among the various descriptions of the isomorphism $\mathcal{C}_\Sigma \approx \mathcal{H}/D_\Sigma$, we will give two useful ones:

(a) one way to parametrize locally metrics up to a real positive factor is by using Beltrami differentials and setting:

$$ds^2 = \lambda^2 |dz + \mu d\bar{z}|^2 \quad (3)$$

The metric is then proportional to

$$g_{\alpha\beta} = \frac{1}{2} \begin{pmatrix} \frac{2\bar{\mu}}{1-\mu\bar{\mu}} & \frac{1+\mu\bar{\mu}}{1-\mu\bar{\mu}} \\ \frac{1+\mu\bar{\mu}}{1-\mu\bar{\mu}} & \frac{2\mu}{1-\mu\bar{\mu}} \end{pmatrix} \quad (4)$$

where $g_{\alpha\beta}$ is chosen to have determinant 1, and the correspondence is the association of g and J constructed with the same μ .

(b) Another way is to ask for the compatibility of the metric and the complex structure, i.e., demanding $g(J\tau, J\eta) = g(\tau, \eta)$ for any two tangent vectors τ and η . The condition fixes the complex structure once g is given, and fixes the metric up to a Weyl transformation once J is given.

The correspondences (a) and (b) are the same and they read:

$$\sqrt{g} g^{\alpha\beta} = -2 K^\alpha_\gamma \epsilon^{\gamma\beta} \quad (5)$$

where $K = -iJ$ (so $K^2 = +1$) and $\epsilon^{\alpha\beta}$ is the antisymmetric symbol with $\epsilon^{z\bar{z}} = +1$.

Remark: the metric g of determinant 1 given by (4) is just one choice of a representative of the class of metrics above the complex structure given by (2). It has the advantage of being local in μ but the disadvantage of not being a tensor with the prescribed transformation of μ (because the condition $\det = 1$ is not invariant by diffeomorphism).

Another natural choice of a representative is to adjust the factor λ in (3) in such a way as to get a metric of constant negative curvature $R=-1$. This condition has the advantage of being invariant by diffeomorphisms, but the inconvenience that λ is non-local in μ [actually λ is constructed from the isothermal co-ordinate u introduced above and equals $\lambda = -u_z \bar{u}_{\bar{z}} \cdot (1-\mu\bar{\mu}) \cdot (u-\bar{u})^2$].

3. - THE ACTION FUNCTIONAL

Using the correspondence described above, and in particular its form (5), we see that the action (1) can be rewritten in terms of the complex structure as:

$$\mathcal{J} = 2 \int_{\Sigma} \varepsilon^{\alpha\beta} \partial_{\alpha} X^{\mu} K^{\gamma}_{\beta} \partial_{\gamma} X^{\nu} G_{\mu\nu} \quad (6)$$

or

$$\mathcal{J} = \int_{\Sigma} \omega$$

where ω is a two-form with local components $\phi_{\alpha\beta} = 2\partial_{[\alpha} X^{\mu} K^{\gamma}_{\beta]} \partial_{\gamma} X^{\nu} G_{\mu\nu}$. We may add to the action a constraint term $\mathcal{J}_c$ to ensure $K^2 = 1$

$$\mathcal{J}_c = \int_{\Sigma} (\phi \text{Tr} K + \psi (\det K - 1)) \quad (7)$$

where ϕ and ψ are scalar Lagrange multipliers with weight one [we may replace them with scalar fields with weight zero provided that we multiply them by $\sqrt{g(J_0)}$ where $g(J_0)$ is a metric compatible with a background complex structure J_0]. The geometrical meaning of ω is very simple. Let $\gamma_{\alpha\beta} = \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu} G_{\mu\nu}$. We do not expect γ to be compatible with the complex structure $J = iK$, but we may extract its compatible part $h(\tau, \eta) = \gamma(\tau, \eta) + \gamma(J\tau, J\eta)$.

The two-form ω is nothing but the fundamental two-form of h ⁹⁾ (up to a factor i). In other words

$$\omega(\tau, \eta) = h(\tau, K\eta) = \gamma(\tau, K\eta) - \gamma(K\tau, \eta)$$

The equations of motion obtained by varying $\mathcal{J} + \mathcal{J}_c$ express the fact that J is compatible with γ .

4. - ANOMALY COCYCLE FOR TWO-DIMENSIONAL DIFFEOMORPHISMS

The classical action $\mathcal{J} + \mathcal{J}_c$ is invariant by diffeomorphisms. Functionally integrating over X^{μ} (for flat background metric) produces an effective action which is expected to be invariant in any space-time dimension. However, if we consider the left- or the right-moving sector separately, we are bound to find an anomaly. It is an important test of the locality of our action to verify that the corresponding cocycle can be written as a polynomial of J and its derivatives. We proceed as follows.

Two BRS operators acted on $\mathcal{H}$: δ_D for the diffeomorphisms (ξ will denote the corresponding ghost) and δ_W for Weyl transformations¹¹⁾. The anomaly cocycle expressed in terms of metrics and connections is a priori a cocycle for the total

operator $\delta = \delta_D + \delta_W$. Fortunately, it is possible to find as a representative a cochain Δ which is a δ_D -1-cocycle and a δ_W -0-cocycle, that is to say a Weyl invariant form of the anomaly of diffeomorphisms¹¹⁾:

$$\Delta = c \int_{\Sigma} \varepsilon^{mn} \left(\Gamma_{mi}^j \partial_n \partial_j \xi^i - \frac{1}{2} \partial_m \partial_j \xi^i g^{il} g_{ni} \Gamma_{lp}^p - \frac{1}{2} \Gamma_{mj}^i g^{jl} g_{ni} \partial_l \partial_p \xi^p \right).$$

Δ has to be the pull back of some δ_D cocycle A on D_{Σ} by the projection $\Gamma_{\Sigma} \rightarrow D_{\Sigma}$ where A is a local function of the complex structure and not of the metric. We find

$$A = \frac{c}{2} \int_{\Sigma} \varepsilon^{\alpha\beta} [K_{\beta}^{\gamma} \partial_{\alpha} K_{\gamma}^{\delta} \partial_{\beta} \partial_{\gamma} \xi^{\delta} - K_{\alpha}^{\gamma} (\partial_{\beta} K_{\gamma}^{\delta} - \partial_{\beta} K_{\delta}^{\gamma}) \partial_{\gamma} \partial_{\beta} \xi^{\delta} - K_{\alpha}^{\gamma} \partial_{\beta} K_{\gamma}^{\delta} \partial_{\beta} \partial_{\gamma} \xi^{\delta}] \quad (8)$$

which is indeed δ_D -closed for $K^2 = 1$, as can be checked directly. The numerical coefficient c can be computed by standard techniques^{5), 12)} and, including also the ghost contribution, it turns out to be proportional to $D-26(26-D)$ in the right(left) moving sector.

5. - QUANTIZATION OF THE COMPLEX STRUCTURE

The invariance of $\mathcal{J} + \mathcal{J}_c$ under D_{Σ} leads us to add a gauge fixing term complemented with the adequate Faddeev-Popov term to the action, or equivalently to write the functional integral over the Teichmüller space if we quotient only by the connected component of the identity of D_{Σ} . If we want our functional integral to descend to the moduli space $\eta_{\Sigma} = \mathcal{C}_{\Sigma}/D_{\Sigma}$, we are bound to find the anomaly pointed out in Ref. 13) which is absent only in $D = 26$. We remark that unless we consider separately the left- or the right-moving sector, this seems to be the only place in our treatment where a critical dimension comes into play.

To fix the gauge we may use the D_{Σ} invariant scalar product defined on the tangent space $T_J(\mathcal{C}_{\Sigma})$ to $\mathcal{C}_{\Sigma}$ at each point as follows¹⁴⁾: let $g(J)$ be the metric over Σ of constant curvature $R = -1$ compatible with J . If X and Y are vectors in $T_J(\mathcal{C}_{\Sigma})$ define:

$$(X, Y)_J = \int \sqrt{g(J)} X_{\beta}^{\alpha} Y_{\alpha}^{\beta} \quad (9)$$

Then, if $\nabla_J \xi$ denotes the variation of the complex structure J under the infinitesimal diffeomorphism ξ , we may define the adjoint ∇_J^* of ∇_J with respect to the scalar product (9) (or rather ∇_K and ∇_K^* with obvious notations). As gauge conditions associated to a background complex structure J_0 we choose

$$\nabla_0^* (K - K_0) = 0 \quad (10)$$

where $\nabla_0 = \nabla_{K_0}$. Although in connection with Eq. (10) a slice theorem is not available in the literature, it is reasonable to assume that Eq. (10) represents a good gauge. Condition (10) can be implemented with auxiliary fields B^α and complemented with the adequate Faddeev-Popov term $\mathcal{Y}_{FP}$:

$$\mathcal{Y}_{GF} + \mathcal{Y}_{FP} = \int_{\Sigma} \sqrt{g(I_0)} B^\alpha \left(\nabla_0^* (K - K_0) \right)_\alpha + \int_{\Sigma} \sqrt{g(I_0)} \bar{\xi}^\alpha \left(\nabla_0^* \nabla_K \xi \right)_\alpha \quad (11)$$

where $\xi^\alpha(\bar{\xi}^\alpha)$ are the (anti)ghost fields. The BRS transformations of B^α , ξ^α and $\bar{\xi}^\alpha$ are defined by

$$\delta B^\alpha = 0, \quad \delta \xi^\alpha = \xi^\beta \partial_\beta \xi^\alpha, \quad \delta \bar{\xi}^\alpha = B^\alpha$$

Now following Refs. 14) and 15), the correct Faddeev-Popov determinant associated to the functional integration of (11) over the ghost system $(\xi^\alpha$ and $\bar{\xi}^\alpha)$ is given by

$$\Delta_{FP} = \det \sqrt{\nabla_K^* \nabla_K} \cdot \omega_{WP}$$

where ω_{WP} is the Weil-Petersson measure. Notice that for genus $g \geq 2$ the Faddeev-Popov operator $\nabla_K^* \nabla_K$ has trivial kernel and cokernel. In particular, the antighosts $\bar{\xi}^\alpha$ do not have zero modes. Hence we are not obliged to find new gauge fixing conditions for the Faddeev-Popov action, as is the case for the approaches of Ref. 8).

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