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EFFECTIVE POTENTIAL AND ASYMMETRIC INSTANTONS

M. Ciafaloni^{*)} and C. Destri^{**) .}
CERN - Geneva

ABSTRACT

Perturbing the double-well potential by a constant external source leads to instanton splitting and to unstable quantum modes. Nevertheless, we show that the effective potential is still dominated by instanton-like configurations which are only piecewise classical solutions, having a countable number of energy discontinuities and asymmetrical asymptotic behaviour. They are determined by maximizing the number of zero modes while minimizing the tadpole contributions which are shown to yield higher order corrections to the energy shift.

^{*)} On sabbatical leave from Dipartimento di Fisica, Università di Firenze and INFN, Sezione di Firenze, Italy.

^{**) On leave from Dipartimento di Fisica, Università di Parma, Parma, Italy.}

1. - INTRODUCTION

The quantum restoration of symmetries which may be classically broken is known to be induced by instantons, related to quantum tunnelling^{1),2)}. However, if the problem is analyzed via the effective potential $V_{\text{eff}}(\phi)$ (which should also embody non-perturbative effects), one meets a rather paradoxical situation, because the same classical symmetry which allows for instantons, is spoiled from the start by the constant external source $J = \partial V_{\text{eff}}/\partial \phi$.

Suppose, for instance, that we want to find $V_{\text{eff}}(\phi)$ for the simple double-well problem in quantum mechanics for which the Euclidean actions reads, in suitable units

$$S = \frac{i}{g} \int dt \left[\frac{1}{2} \dot{\phi}^2 + V(\phi) \right] \quad ; \quad V(\phi) = \frac{1}{2} (\phi^2 - 1)^2 \quad (1.1)$$

First of all, a straightforward perturbative approach is certainly inadequate. In fact, the naïve loop expansion

$$V_{\text{eff}}(\phi) = V(\phi) + \frac{g}{2T} \log \text{Det}_T S''(\phi) + \dots \quad , \quad T \rightarrow \infty \quad (1.2)$$

is plagued by instabilities - $V(\phi)$ being a non-convex function - and would anyway give small corrections, still allowing for symmetry breaking.

The first improvement of (1.2) is well known in statistical mechanics: the precise definition of V_{eff} as Legendre transform of $E_0(J)$ - the ground state energy in the external source J - leads to the so-called Maxwell construction for $V_{\text{eff}}(\phi)$ at tree level (Fig. 1a). However, loop corrections to it, although no longer plagued by instabilities, yield again a flat region around $\phi = 0$ and therefore cannot restore the symmetry. Only non-perturbative (tunnelling) effects can give rise to a global minimum at $\phi = 0$ (Fig. 1b) with curvature proportional to the (dimensionless) energy gap

$$V_{\text{eff}}''(0) \sim \Delta_0 = 4 \left(\frac{2g}{\pi} \right)^{1/2} \exp\left(-\frac{4}{3g}\right) \quad (1.3)$$

Here a problem comes out. In order to include such non-perturbative effects, we have to switch on the external source J , thus differentiating the two potential minima, i.e., breaking the symmetry explicitly. The instanton-like classical solutions of this new problem, defined by the shifted potential

$$V_J(\phi) = V(\phi) - J\phi \quad (1.4)$$

have several unwanted features. First of all, they are split (Fig. 2), with energy difference of order $2J$, each one of them spending a large time around only one of the two minima. Furthermore, the higher ranking one (type I), being unbounded, has an infinite action, while the other (type II) is a bounce which is known to yield unstable quantum modes²⁾. The situation is even more drastically changed at the level of multi-instantons which exist only for the bounce and have only half of the expected zero modes.

All this is not really surprising since it is appropriate for a situation of symmetry breaking, which is what happens also at a full quantum level for sufficiently large J . The clue to its understanding is, therefore, that the magnitude of the source J , which yields the effective potential for $|\phi| < 1$, is just of order $g \sim \langle (\delta\phi)^2 \rangle$, i.e., of the quantum fluctuations themselves. We are thus led to release the smoothness requirement for the action dominating trajectories, by allowing the velocity (and the energy) to change by an amount of order J in a countable number of points. Such trajectories will therefore be only piecewise classical solutions.

Let us remark that, even in the symmetric $J = 0$ case, the appropriate treatment of multi-instantons at finite time is difficult because there are not enough (real) classical solutions. One can try trajectory breaking by introducing ϕ -space completeness relations into the path integral³⁾ or resort to complex saddle points⁴⁾ with fairly satisfactory results. However, neither method works in the $J \neq 0$ case, the first one because ϕ integrals no longer reproduce the correct instanton measure and the second one because the pattern of complex solutions at $J \neq 0$ is spoiled too.

On the other hand, considering velocity discontinuities in the action dominating trajectories opens up a number of new possibilities. One could think, for instance, of taking type I trajectories and allowing velocity inversions (but not energy jumps) around the second minimum, in order to combine them into multi-instanton configurations. This choice does not work. It leads (Section 2) to a non-analytic behaviour in J (something ruled out by general quantum arguments as well as by an explicit WKB calculation - see Appendix A) and yields too small determinantal factors, due to the loss of half of the zero modes present at $J = 0$. Therefore, we will consider here the possibility of energy jumps, but, in order to

reduce ambiguities, we will introduce the essential requirement that all the zero modes of the $J = 0$ regime be there.

In Section 2, we show how this criterion leads indeed to an essentially unique choice, and we also show how the tadpoles, which are introduced by the velocity discontinuities, only give rise to a small correction in the single instanton case. The multi-instanton case, leading to the correct effective potential, is analyzed in Section 3 where we also discuss how tadpoles affect the result in the general case. The improved WKB analysis and the determinant calculations are given in the Appendices.

2. - PIECEWISE CLASSICAL TRAJECTORIES

The general problem that we approach here is to determine the field configurations $\phi(t)$ that dominate the path integral for the effective potential.

Usually, only classical solutions are considered, for which the expansion of the action in quantum fluctuation $\eta(t)$ does not contain linear terms $\sim \sigma\eta$ (tadpoles). The reason for this requirement is that tadpoles will generate large contributions $O(-\sigma^2/g)$ to the effective action, signalling that the original field was not really dominating. In our case, this conventional procedure would imply solving the classical equations of motions in the presence of the source J , with all the problems mentioned in the Introduction. We shall therefore look for field configurations which in a sense minimize the tadpoles [which will be $O(J)$ for small J], while retaining all zero modes of the $J = 0$ solutions, and thus maximizing their weight (determinantal factor).

Hence we analyze continuous trajectories $\phi(t)$ which are only piecewise classical solutions having a countable number of localized tadpoles. While $\phi(t)$ is kept continuous (as required by the kinetic term in the action), $\dot{\phi}(t)$ is allowed to have a finite number of discontinuities at $t = t_i$ ($i = 1, \dots, n$).

The quadratic expansion of the action around such a trajectory $\phi_{\tilde{t}}(t)$ [$\tilde{t} \equiv (t_1, \dots, t_n)$] reads

$$gS[\phi_{\tilde{t}} + \eta] = \frac{1}{2} \sum_{i=0}^n \int_{t_i}^{t_{i+1}} dt \left[\dot{\phi}_{\tilde{t}}^2 + (\phi_{\tilde{t}}^2 - 1)^2 \right] - \sum_{i=1}^n \Delta v_i \eta(t_i) +$$

$$+ \frac{1}{2} \int_{t_0}^{t_{n+1}} dt \eta(t) (D\eta)(t) + O(\eta^3) \quad , \quad (2.1)$$

where $t_0 = -T/2$, $t_{n+1} = T/2$ label the initial and final times, $\Delta v_i = \dot{\phi}(t_i+0) - \dot{\phi}(t_i-0)$ are the velocity discontinuities and D is the inverse propagator for the quantum modes:

$$D = -\partial_t^2 + V_J''(\phi_t) = -\partial_t^2 + 6\phi_t^2 - 2 \quad . \quad (2.2)$$

We have now to fix the trajectory $\phi_{\tilde{t}}$ so as to meet the boundary conditions at $t = \pm T/2$ and maximize the number of zero modes. To be definite, we shall consider the calculation of the kernel:

$$K(T) = \langle \phi_+ | e^{-HT/g} | \phi_- \rangle \quad , \quad (2.3)$$

where $\phi_{\pm} = \pm 1 + J/4$ are the two potential minima. For T large enough, trajectories $\phi_{\tilde{t}}(t)$ with boundary conditions $\phi_{\tilde{t}}(\pm T/2) = \phi_{\pm}$ exist which go through $\phi = 0$ any odd number, $n = 1, 3, 5, \dots$, of times, corresponding to one instanton and multi-instanton configurations of the symmetric $J = 0$ case. In this section we analyze the $n = 1$ case to start with.

As discussed before, the $n = 1$ trajectory $\phi_{t_1}(t) = (\phi_1(t), \phi_2(t))$ should satisfy two criteria:

a) Presence of a zero mode of D when $T \rightarrow \infty$. This means that $\phi_{t_1}(t)$ should spend a large time around both minima ϕ_+ and ϕ_- . Therefore, the two classical components ϕ_1 and ϕ_2 of ϕ_{t_1}

$$\phi_{t_1}(t) = \begin{cases} \phi_1(t) & , -T/2 < t < t_1 \\ \phi_2(t) & , t_1 < t < T/2 \end{cases} \quad (2.4)$$

must have different energies (see Fig. 3)

$$\begin{aligned} E_1 &= \frac{1}{2} \dot{\phi}_1^2 - V_J(\phi_1) \simeq -J + O(\bar{e}^{T_1}) , \\ E_2 &= \frac{1}{2} \dot{\phi}_2^2 - V_J(\phi_2) \simeq +J + O(\bar{e}^{T_2}) , \end{aligned} \quad (2.5)$$

where $T_1 = T/2 + t_1$ and $T_2 = T/2 - t_1$ are assumed to be both of order T as $T \rightarrow \infty$.

b) The tadpoles should be as small as possible, consistently with condition a). Since, by (2.5)

$$\Delta v \simeq [2(J + V_J(\phi))]^{1/2} - [2(-J + V_J(\phi))]^{1/2}, \quad (2.6)$$

where $\phi = \phi_1(t_1) = \phi_2(t_1)$, this singles out the local maximum ϕ_0 of $V_J(\phi) = V(\phi) - J\phi$, i.e.,

$$V'(\phi_0) = J, \quad \phi_0 \simeq -J/2, \quad V_J(\phi_0) \simeq \frac{1}{2} + O(J^2). \quad (2.7)$$

Thus, the discontinuity in $\dot{\phi}_{t_1}$ should occur close to $\phi = 0$ and is of order $\Delta v \simeq 2J$.

The trajectory (2.4) constructed according to (2.6) and (2.7) has a zero-mode roughly corresponding to translations in the matching time t_1 , when T_1 and T_2 both tend to infinity. Its explicit form is

$$f_0(t) = \begin{cases} \bar{\rho}^1 \dot{\phi}_1(t) & , t < t_1 \\ \rho \dot{\phi}_2(t) & , t > t_1 \end{cases} \quad (2.8)$$

where $\rho^2 = \dot{\phi}_1(t_1)/\dot{\phi}_2(t_1) = 1 - \Delta v/\dot{\phi}_2(t_1) \simeq 1 - 2J$. Let us note that, since $\ddot{\phi}_{t_1}(t_1) = 0$ [Eq. (2.7)], both f_0 and $\dot{f}_0$ are continuous at $t = t_1$, so that f_0 is a bona fide zero-mode of the second order operator D . Furthermore, $f_0 \simeq -(\partial\phi_{t_1}/\partial t_1)(1 + O(J))$, and f_0 is therefore connected with the translational invariance of the $T_{1,2} \rightarrow \infty$ solutions.

In contrast, the fully classical solution interpolating between ϕ_- and ϕ_+ has no zero modes. In fact, it can be identified with the (uniquely determined) trajectory $\phi_{t_1}^-(t)$ where $\bar{E}_1$ [and $E_1 = E_1(\bar{E}_1)$] are chosen so as to have $\Delta v(\bar{E}_1) = 0$.

It is easy to see that such a trajectory has $E_1 = E_2 \sim J$ and therefore it spends most of its time around ϕ_+ , with $T_1 \sim |\log J|$ staying finite as $T \rightarrow \infty$. The one-loop contribution of this classical solution $\phi_{t_1}^-$ can be evaluated by standard methods (Appendix B) and is

$$K_{\mathcal{A}}^{(1)}(T) = \left(\frac{\pi}{2gJ}\right)^{1/2} \Delta_0 \exp\left(JT/g + O(J \log J)\right), \quad (2.9)$$

where g (related to more conventional parameters by $g = 2\hbar/m\omega a^2$, $2a$ being the wall separation) is the effective coupling [see Eq (1.1)] and the $+JT/g$ term in the exponent (2.9) signals the fact that $\phi_{t_1}^-$ is a type I "instanton". Note that (2.9) suffers from two unwanted singularities in the J variable: one coming from the $\sim \log J$ time spent by $\phi_{t_1}^-$ near ϕ_- , and the other ($\sim 1/\sqrt{J}$) from the lowest eigenvalue of the operator D in the determinant. Both will disappear by taking into account all trajectories of type $\phi_{t_1}(t)$.

In order to evaluate the contribution of the family ϕ_{t_1} to the kernel (2.3) we should calculate: (a) the classical action and the zero mode factors à la Polyakov, and (b) the non-zero mode contributions (including tadpoles) and their determinant. We shall therefore write:

$$K^{(1)}(T) = N \int \left[\frac{(f_0, f_0)}{2\pi g} \right]^{1/2} dt_1 \mathcal{D}\eta_{\perp} \exp(-S[\phi_{t_1} + \eta_{\perp}]) , \quad (2.10)$$

where we have used the expansion:

$$\phi(t) = \phi_{t_1}(t) + \eta_{\perp}(t) \quad ; \quad (f_0, \eta_{\perp}) \equiv \int dt f_0 \eta_{\perp} = 0 , \quad (2.11)$$

and by (2.1),

$$gS \simeq \frac{4}{3} + 2JT_1 - \Delta v \eta_{\perp}(t_1) + \frac{1}{2} \int dt \eta_{\perp} D_{\perp} \eta_{\perp} + O(J) . \quad (2.12)$$

In the $O(J)$ terms we have lumped all those not proportional to the (large) parameter t_1 and $D_{\perp}$ denotes the restriction of (2.2) to the space orthogonal to the zero mode f_0 . Note that the $J \log J$ term in (2.9) has disappeared in (2.12) in favour of the t_1 dependent exponent ($|\log J| \rightarrow T/2 + t_1$), with t_1 to be integrated over. Therefore, non-analytic terms will not appear in the final result.

By choosing the normalizing factor N in the standard way, the Gaussian integration over transverse modes $\eta_{\perp}$ in (2.10) provides the factor

$$\left[\frac{\text{Det } D_{\perp}}{\text{Det } D_0} \right]^{-1/2} \exp \left\{ \frac{\Delta v^2}{2g} D_{\perp}^{-1}(t_1, t_1) \right\}, \quad (2.13)$$

where $D_0 = -\partial_t^2 + 4$ is the inverse harmonic propagator. The determinant ratio is evaluated in Appendix B,

$$\frac{\text{Det } D_{\perp}}{\text{Det } D_0} = \frac{(f_0, f_0)}{64} [1 + O(J)], \quad (2.14)$$

and it does not depend on t_1 for large $T_{1,2}$. Since $D_{\perp}^{-1}(t_1, t_1)$ is order 1, the tadpole term in (2.13) gives a negligible contribution of order J^2/g . Let us recall that we are actually interested in sources $J = O(g)$, for which the classical energy shift due to J is smaller than the ground state quantum gap.

By substituting (2.13), (2.15) and (2.12) in (2.10), the t_1 integration is easily performed with the result

$$K^{(1)}(T) = [1 + O(J)] \frac{\Delta_0}{g} \int_{-T/2}^{T/2} dt_1 e^{-2Jt_1/g} = [1 + O(J)] \frac{\Delta_0}{J} \text{sh} \frac{JT}{g} \quad (2.15)$$

where [cf., Eq. (1.3)]

$$\Delta_0 = 4 \left(\frac{2g}{\pi} \right)^{1/2} e^{-4/3g} \quad (2.16)$$

is the well-known energy splitting induced by instantons at $J = 0$. Notice that the $J^{-1/2}$ factor in Eq. (2.9) has been replaced by $1/J$. This is essential for (2.15) to reduce to the expected value $\Delta_0 T/g$ in the $J \rightarrow 0$ limit.

Let us remark that, by integrating t_1 over the whole T interval, we have neglected the finite instanton size, $\tau_J = \tau_0 + O(J)$. Keeping track of it would amount to replace T in (2.15) by $T - \tau_J$ and would eventually give rise to small corrections to the energy gap, of relative order Δ_0/g (Section 3).

3. - MULTI-INSTANTON CONTRIBUTION

The generalization of the approach based on piecewise classical trajectories to multi-instanton-like configurations is rather straightforward. We consider fluctuations around trajectories $\phi_{\tilde{t}}(t)$ where $\tilde{t} = (t_1, \dots, t_n)$ parametrizes the velocity jumps, which are located in ϕ space at the local maxima (2.7) of the potential $V_J(\phi)$, i.e., $\phi_{\tilde{t}}(t_i) = 0(J)$.

We again require that such trajectories spend a large time around both minima by tuning the energies E_i in each continuity interval (t_{i-1}, t_i) , with the obvious constraint that the total time be T (Fig. 4). In this way, all matching times t_i become genuine translational parameters for this class of trajectories.

Note that, to be rigorous, this procedure should be used for multi-instantons also in the $J = 0$ case, as a finite time alternative to the dilute gas approximation valid at infinite time. The difference is that Δv_i would in this case be $O(e^{-T/n})$ instead of being $O(J)$, and therefore the tadpole contributions would be completely negligible (see below).

The evaluation of the quantum fluctuations around $\phi_{\tilde{t}}$ parallels the one performed for Eq. (2.10). By choosing the same boundary conditions $\phi_{\tilde{t}}(\pm T/2) = \phi_{\pm}$, the action (2.1) becomes

$$\begin{aligned} gS = & \frac{4}{3}n + J \sum_{i=0}^n (-)^i (t_{i+1} - t_i) - \sum_{i=1}^n \Delta v_i \eta_{\perp}(t_i) + \\ & + \frac{1}{2} \int dt \eta_{\perp} D_{\perp}^{(n)} \eta_{\perp} + O(J) \end{aligned} \quad (3.1)$$

where, we recall, $n = 2k+1$ is odd, $t_0 = -T/2$, $t_{n+1} = T/2$ and the $O(J)$ corrections essentially do not depend on $t_1, \dots, t_n$. The Gaussian integration over $\eta_{\perp}$ now provides the factor

$$\left[\frac{\text{Det } D_{\perp}^{(n)}}{\text{Det } D_0} \right]^{-1/2} \exp \left\{ \frac{1}{2g} \sum_{i,j=1}^n \Delta v_i D_{\perp}^{(n)-1}(t_i, t_j) \Delta v_j \right\} \quad (3.2)$$

where the determinant ratio takes on the factorized form (see Appendix B):

$$\frac{\text{Det } D_{\perp}^{(n)}}{\text{Det } D_0} = \prod_{i=1}^n \frac{(f_0^i, f_0^i)}{64} , \quad (3.3)$$

and the tadpole contribution in the exponent can be shown to be negligible.

It is, in fact, important to realize that this contribution is not of order n^2 , as it would naïvely appear from the double sum in (3.2). Indeed, due to the complete subtraction of the n zero modes, we can prove in Appendix B that each tadpole disturbance only propagates over an instanton length and not over all times. The total tadpole contribution to the effective action is therefore of type

$$-\text{const} \cdot \frac{J^2}{g} \cdot n , \quad (3.4)$$

where the constant is an upper bound on the $O(1)$ quantities $D_{\perp}^{(n)-1}(t_i, t_i)$. Replacing n by its average value in the instanton gas, which is of order $\Delta_0 T/g$, shows that the tadpoles induce overall corrections of order $\Delta_0 J^2/g$ to the energy gap (see below).

Thus, to leading order, the class $\phi_{\tilde{t}}$ of piecewise classical trajectories contributes to the kernel $K(T)$ of Eq. (2.3) the quantity:

$$\begin{aligned} K^{(n)}(T) &\simeq \left(\frac{\Delta_0}{g}\right)^n \int_{-T/2}^{T/2} d^n t \, \theta(t_1 < t_2 \dots < t_n) \exp \left\{ \frac{J}{g} \sum_{i=0}^n (-)^i (t_{i+1} - t_i) \right\} \\ &= \left(\frac{\Delta_0}{g}\right)^{2k+1} \int_0^T d\tau \, e^{J(T-2\tau)/g} \frac{\tau^k}{k!} \cdot \frac{(T-\tau)^k}{k!} , \end{aligned} \quad (3.5)$$

where $n = 2k+1$ and τ is the total lapse of time spent by $\phi_{\tilde{t}}$ near ϕ_- .

Expression (3.5) is easily diagonalized and resummed over k by a Laplace transform, yielding

$$\tilde{K}(E) = \int_0^\infty dT \, K(T) e^{-ET/g} = \frac{1}{\Delta_0} \sum_{k=0}^{\infty} \left(\frac{\Delta_0^2}{E^2 - J^2} \right)^{k+1} =$$

$$= \frac{\Delta_0}{E^2 - J^2 - \Delta_0^2} \quad (3.6)$$

We recognize here the leading energy eigenvalues

$$\begin{aligned} E_{\pm} &= \pm \Delta (1 + O(J^2/g)) + O(\Delta_0^2/g) \\ \Delta &\equiv +\sqrt{J^2 + \Delta_0^2} \end{aligned} \quad (3.7)$$

which, by Legendre transform in J yield the desired effective potential

$$V_{\text{eff}}(\phi) = \Delta_0 \sqrt{1 - \phi^2} \quad , \quad |\phi| < 1 \quad (3.8)$$

The same expression (3.7) also arises from a conventional semi-classical treatment of the Schrödinger equation (Appendix A) and indeed exhibits the expected analyticity in J and, for $V_{\text{eff}}(\phi)$ in (3.8), the curvature proportional to Δ_0 [Eq. (1.3)]. From (3.6) we can also recover the form of the kernel $K(T)$:

$$K(T) = \frac{\Delta_0}{\Delta} \text{sh} \frac{\Delta T}{g} \quad (3.9)$$

which is the one expected on quantum mechanical grounds from the reduced "effective" Hamiltonian

$$h \sim \begin{pmatrix} J & \Delta_0 \\ \Delta_0 & -J \end{pmatrix} \quad (3.10)$$

The corrections explicitly written out in (3.7) come from two sources:

(a) the tadpoles mentioned above which add terms $\sim nJ^2/g$ to the action, and therefore correspond to a shift $\Delta_0 \rightarrow \Delta_0 e^{0(J^2/g)} \simeq \Delta_0 (1 + O(J^2/g))$, yielding the correction to $E_{\pm}$ of relative order J^2/g , and

(b) finite instanton size effects which can also be taken into account by the replacements $\tau \rightarrow \tau - n\tau_J$, $T - \tau \rightarrow T - \tau - n\tau_J$ in Eq. (3.5). The resummation (3.6) can still be performed by the Laplace transform method and an E-dependent shift $\Delta_0 \rightarrow \Delta_0 e^{-E\tau_J/g}$ ensues that, when expanded in E/g , leads to the corrections $O(\Delta_0^2/g)$ in (3.7). Both types of corrections fit nicely with the WKB estimates of Appendix A.

Let us remark that our final results eventually justify the naïve application of the dilute gas approximation to computing external source effects. In the naïve approach, one would simply saturate the expectation value of $\exp\{\int dt J(t)\phi(t)\}$ with the one-loop instanton measure, thus getting an action quite similar to (3.1) except for the tadpoles. This procedure is justifiable, however, only for localized sources such that $J(t) \rightarrow 0$, as $|t| \rightarrow \infty$, fast enough not to change the stationary states of the system. On the contrary, we have seen that a J constant in time modifies in a drastic way the asymptotic behaviour of the instanton system, thus requiring the introduction of piecewise classical trajectories and the careful evaluation of the corresponding tadpole terms.

We have found here that tadpoles are small, for small enough J, in our simple quantum mechanical system. However, their counting is not trivial (as mentioned above) and could drastically increase in a field theoretic situation. Therefore, we think that our approach to the non-perturbative analysis of the effective potential sets precise limits of applicability (whose validity should be checked also in more complicated cases) to instanton based approximations.

From another point of view, let us remark that we have seen how some classical solutions leading to unstable quantum modes (like the bounce) have been stabilized by nearby piecewise classical trajectories for sufficiently small J. Thus, the existence of some unstable directions around classical solutions does not necessarily imply the need for a change of ground state; it may be that, especially in field theoretical cases, one has to allow for piecewise classical solutions in order to recognize truly unstable vacua from stable ones.

APPENDIX A

THE UNIFORM WKB APPROXIMATION

In this appendix we derive a uniform WKB approximation for the ground state energy $E_0(J)$ of the forced double-well oscillator [see Eq. (1.1)]:

$$H = -\frac{g^2}{2} \frac{d^2}{d\phi^2} + V(\phi) - J\phi \quad ; \quad V(\phi) = \frac{1}{2}(\phi^2 - 1)^2 \quad . \quad (A.1)$$

First of all, we shall construct improved WKB solutions of the Schrödinger equation $H\psi = E\psi$ by means of the so-called "comparison equation" method⁵⁾ and a suitable matching procedure. This will allow us to obtain a uniform semi-classical approximation for the exact eigenfunctions, that is an approximation valid also at the classical turning points.

Let E and J be such that there are four real classical turning points $\phi_1 < \phi_2 < \phi_3 < \phi_4$; we then introduce the following quantities

$$\begin{aligned} k(z) &= \sqrt{2(E - V(z) + Jz)} \quad ; \quad W(\phi, \phi') = \frac{1}{g} \int_{\phi}^{\phi'} dz k(z) \\ \tau_- &= \frac{1}{\pi} W(\phi_1, \phi_2) \quad ; \quad \tau_+ = \frac{1}{\pi} W(\phi_3, \phi_4) \\ S &= |W(\phi_2, \phi_3)| = \frac{1}{g} \int_{\phi_2}^{\phi_3} dz \sqrt{2(V(z) - Jz - E)} \end{aligned} \quad (A.2)$$

In this regime, the semi-classical approximation is valid whenever $S \gg \tau_{\pm}$, that is for all the states with energy E such that gS stays non-zero as $g \rightarrow 0$. These states are the ground states and first few excited states, namely those for which τ_+ and τ_- are both of order 1 or less and the pair (ϕ_1, ϕ_2) is well separated from the pair (ϕ_3, ϕ_4) [this is perfectly consistent with the assumption $J \sim O(g)$, which is the relevant source strength for the effective potential in the interval $|\phi| < 1$]. We therefore treat each well by itself and then impose a matching condition on the wave functions deep into the interval $[\phi_2, \phi_3]$.

The absence of other (real or complex) turning points besides $\phi_1, \dots, \phi_4$ makes this problem ideal for the application of the parabolic comparison method. The wave function $\psi(\phi)$ having the usual WKB form as $\phi \rightarrow \pm\infty$ is thus given by:

$$\psi(\phi) = \begin{cases} A_+ \left(\frac{d\sigma_+}{d\phi} \right)^{-1/2} D_{\tau_+ - 1/2}(\sigma_+(\phi)) & , \phi > 0 , \\ A_- \left(\frac{d\sigma_-}{d\phi} \right)^{-1/2} D_{\tau_- - 1/2}(\sigma_-(\phi)) & , \phi < 0 , \end{cases} \quad (\text{A.3})$$

where $D_\nu(z)$ is Weber's parabolic cylinder function⁶⁾ and the two so-called "mapping functions" $\sigma_\pm(\phi)$ are defined by:

$$g \frac{d\sigma_\pm}{d\phi} = \pm k(\phi) \left[\tau_\pm - \frac{1}{4} \sigma_\pm^2 \right]^{-1/2} ; \quad \sigma_+(\phi_4) = \sigma_-(\phi_1) = 0 . \quad (\text{A.4})$$

Notice that, by definition, $d\sigma_\pm/d\phi$ is non-zero at the classical turning points where $K(\phi)$ instead vanishes, either linearly, for finite $\tau_\pm$, or quadratically when τ_+ and/or τ_- vanishes.

Using the asymptotic form of $D_{\tau_\pm - 1/2}(\sigma_\pm(\phi))$ in the matching region around $\phi = 0$ and the obvious relation

$$|W(\phi_3, \phi)| = S - |W(\phi_2, \phi)| \quad ; \quad \phi_2 < \phi < \phi_3 \quad (\text{A.5})$$

one determines the ratio A_+/A_- and obtains the quantization condition for E in the form of an implicit relation among τ_+ , τ_- and S :

$$\cot(\pi\tau_+) \cot(\pi\tau_-) = \frac{\pi}{2} \frac{e^{(\tau_+ + \tau_-)} \tau_+^{\tau_+} \tau_-^{\tau_-}}{\Gamma(\frac{1}{2} + \tau_+) \Gamma(\frac{1}{2} + \tau_-)} e^{-2S} . \quad (\text{A.6})$$

Next, we exploit the symmetry of the problem under simultaneous reflection $\phi \rightarrow -\phi$, $J \rightarrow -J$, and the analyticity of $V(\phi)$ (allowing contour deformation of phase integrals) to establish the useful relations:

$$\tau_+(J) = \tau_-(-J) \quad ; \quad S(J) = S(-J) \quad ; \quad \tau_+ - \tau_- = J/g \quad (\text{A.7})$$

It follows that Eq. (A.4) determines the allowed values of E as a real analytic function of J^2 . Hence the WKB approximation succeeds in yielding a symmetric, real analytic ground state energy $E_0(J)$ and correspondingly a real analytic effective potential $V_{\text{eff}}(\phi)$, drastically changing the tree level approx-

imation. Notice that the ground state quantization corresponds for $J > 0$ ($J < 0$) to $\tau_+(\tau_-)$ very close to $\frac{1}{2}$. Thus the last of Eqs. (A.7) shows that semi-classical tunnelling occurs at the ground state energy only for $|J| < g/2$. For larger values of $|J|$ such ground state tunnelling becomes impossible and the tree level approximations for $E_0(J)$, $|J| > g/2$ and $V_{\text{eff}}(\phi)$, $|\phi| > 1$ are already qualitatively correct.

Due to the complex implicit structure of Eq. (A.6), it is not easy to read out the explicit functional form of $E_0(J)$. For $|J| \ll g$, however, one can develop an expansion in $J/g = \tau_+ - \tau_-$ whose coefficients are to be estimated in the semi-classical limit $g \rightarrow 0$, which also forces $J \rightarrow 0$. In the crudest approximation we have $\tau_+ + \tau_- \approx E/g + O(J^2)$ and $S \approx S_0 + O(J^2/Eg)$ where

$$S_0 = \frac{2}{g} \int_0^{\phi_E} dz \sqrt{2(V(z) - E)} \quad , \quad \phi_E = (1 - \sqrt{2E})^{1/2} . \quad (\text{A.8})$$

Then, since at $E = 0$ we have $gS_0 = \text{one-instanton action} = 4/3$, we see that for the ground state (and first excited state) $\tau_{\pm} \rightarrow \frac{1}{2}$ as $g \rightarrow 0$ which implies $E \rightarrow g(1 + O(J/g))$ (the one-loop result neglecting tunnelling). Setting $E = g + \Delta$ and expanding Eq. (A.6) in Δ/g and J/g yields to leading order

$$\left(\frac{\Delta}{g}\right)^2 + \left(\frac{J}{g}\right)^2 = \frac{1}{\pi e} e^{-2S_0} [1 + O\left(\frac{\Delta}{g}\right) + O\left(\frac{J^2}{g}\right)] \quad (\text{A.9})$$

where S_0 is evaluated at $E = g$. Keeping track of the leading logarithms of g in S_0 we have

$$e^{-S_0} \simeq 4 \left(\frac{2e}{g}\right)^{1/2} e^{-4/3g} [1 + O(g)] ,$$

so that (A.9) can be written as

$$\Delta^2 = J^2 + \Delta_0^2 [1 + O\left(\frac{\Delta}{g}\right) + O\left(\frac{J^2}{g}\right)] ,$$

where

$$\Delta_0 = 4 \left(\frac{2g}{\pi}\right)^{1/2} e^{-4/3g} [1 + O(g)]$$

is the usual instanton induced tunnelling gap [see Eq. (1.3)].

In conclusion, the energies of the ground state and first excited states are

$$E_0 = g + \Delta_- \quad ; \quad E_1 = g + \Delta_+$$

with

$$\Delta_{\pm} = \pm \sqrt{J^2 + \Delta_0^2} \cdot [1 + O(\frac{J^2}{g})] + O(\frac{\Delta_0^2}{g}) , \quad (A.10)$$

to be compared with Eq. (3.7) of the text.

APPENDIX B

DETERMINANTS AND TADPOLES

Here we want to compute the determinant and the propagator of the non-zero modes and to discuss the magnitude of the tadpole terms.

Let us start with the single instanton-like case of Section 2. By working at finite time T , we will factorize out the $O(e^{-2T})$ "zero mode" eigenvalue from the complete determinant. Let us therefore define the solution $\psi(t)$ of $D\psi = 0$ such that $\psi(-T/2) = 0$, $\dot{\psi}(-T/2) = 1$. It is given by

$$\psi(t) = \begin{cases} g_1(-T/2)f_1(t) - f_1(-T/2)g_1(t) & , \quad -T/2 < t < t_1 \\ \rho^2 g_1(T/2)f_2(t) - \bar{\rho}^2 f_1(-T/2)g_2(t) & , \quad t_1 < t < T/2 \end{cases} \quad (B.1)$$

where

$$\rho^2 = f_1(t_1)/f_2(t_1) ,$$

and

$$f_i(t) = \dot{\phi}_i(t) \quad , \quad g_i(t) = f_i(t) \int_{t_1}^t \frac{dt'}{f_i(t')^2} \quad , \quad (B.2)$$

$$W[f_i, g_i] \equiv f_i \dot{g}_i - \dot{f}_i g_i = 1 \quad ; \quad (i=1,2)$$

are the usual regular and irregular solutions, respectively with $g_1(t_1) = g_2(t_1) = 0$. The complete determinant is then given by²⁾

$$N^{-2} \text{Det } D = \psi(T/2) = -f_1(-T/2)f_2(T/2) \left[\rho^2 T_1'(E_1) + \bar{\rho}^2 T_2'(E_2) \right] \quad (B.3)$$

where the normalization factor N is the same as for the reference oscillator.

Let us now find the zero mode eigenvalue and eigenfunction. We seek a solution of $D\phi = \lambda_0 \phi$ of the form

$$\varphi_0(t) = \begin{cases} \bar{p}' [f_1(t) + \delta_1 g_1(t)] & , \quad t < t_1 \\ p [f_2(t) + \delta_2 g_2(t)] & , \quad t > t_1 \end{cases} \quad (\text{B.4})$$

where δ_1, δ_2 are slowly varying functions of t , to be determined from the boundary conditions $\phi_0(-T/2) = \phi_0(T/2) = 0$. Therefore,

$$\delta_1(-T/2) = -[T_1'(E_1)]^{-1}, \quad \delta_2(T/2) = -[T_2'(E_2)]^{-1}, \quad (\text{B.5})$$

where

$$T_i(E_i) = \left| \int_{\phi_0}^{\phi_i} \frac{d\phi}{\sqrt{2(E_i + V_J(\phi))}} \right| \simeq \frac{1}{2} \log \frac{8}{|\dot{\phi}_i + 2\phi_i|}. \quad (\text{B.6})$$

Since ϕ_0 is continuous with its derivative and the δ_i 's are small, $O(\lambda_0)$, the usual Wronskian theorem yields

$$\lambda_0(f_0, f_0) = -\frac{1}{T_1' T_2'} [p^2 T_1'(E_1) + \bar{p}^2 T_2'(E_2)] + O(\lambda_0^2) \quad (\text{B.7})$$

where f_0 , defined in Eq. (2.8), is the $\delta_i = 0$ form of (B.4).

Notice that the eigenvalue λ_0 in (B.7) is, by (B.6), of order e^{-2T} [and not $O(J)!]$, so that (B.4) is a bona-fide zero mode. By dividing out λ_0 in (B.3), the transverse mode determinant becomes

$$N^{-2} \text{Det } D_{\perp} = (f_0, f_0) f_1(-T/2) f_2(T/2) T_1'(E_1) T_2'(E_2). \quad (\text{B.8})$$

This expression is actually independent, for $T \rightarrow \infty$, of both the initial and final velocities and of the intermediate time. In fact, by using (B.6) and by recalling that in our units the oscillator frequency is $\omega = 2$, we get

$$-f_1(-T/2) T_1' = \frac{1}{16} e^{2T_1} \quad ; \quad -f_2(T/2) T_2' = \frac{1}{16} e^{2T_2}. \quad (\text{B.9})$$

Therefore, by dividing out $\text{Det } D_0 \sim 1/4 e^{2T}$, we obtain

$$\frac{\text{Det } D_{\perp}}{\text{Det } D_0} = \frac{1}{64} (f_0, f_0), \quad (\text{B.10})$$

which is the value used in the text.

The n-instanton case (Fig. 4) is treated by noticing that the zero modes f_0^i ($i = 1, \dots, n$) are essentially non-overlapping so that (B.8) is replaced by

$$\begin{aligned} \bar{N}^2 \text{Det } D_{\perp}^{(n)} &= \prod_{i=1}^n (f_0^i, f_0^i) \cdot (-T_1' f_1(-T/2)) \left[\prod_{i=2}^n (-T_i') \right] (-T_{n+1}' f_{n+1}(T/2)) \\ &= \left[\prod_{i=1}^n \frac{1}{64} (f_0^i, f_0^i) \right] \cdot \frac{1}{4} \exp \left(2 \sum_{i=1}^{n+1} T_i \right) \end{aligned} \quad (\text{B.11})$$

where we have used (B.9) and the analogous relation for internal times, i.e.,

$$-T_i'(E_i) \approx \frac{1}{64} e^{2T_i}, \quad (i = 2, \dots, n) \quad (\text{B.12})$$

By dividing out the reference determinant D_0 we finally get

$$\frac{\text{Det } D_{\perp}^{(n)}}{\text{Det } D_0} = \prod_{i=1}^n \frac{1}{64} (f_0^i, f_0^i) \quad (\text{B.13})$$

The transverse Green's function can be computed, using again non-overlapping zero modes, by the projection

$$G_{\perp}(t, t') = P G P, \quad P = 1 - \sum_{i=1}^n \frac{f_0^i \otimes f_0^i}{(f_0^i, f_0^i)}, \quad (\text{B.14})$$

where

$$\begin{aligned} G(t, t') &= \bar{D}^{-1}(t, t') = \sum_i G_i(t, t'), \\ G_i(t, t') &= -f_i(t) g_i(t') \theta(t - t') + (t \leftrightarrow t'). \end{aligned} \quad (\text{B.15})$$

Note that the g_i 's in (B.15) are defined with $g_i(t_i) = 0$, by exploiting the subtraction of the $f_i \otimes f_i$ part implied by the transverse projection (B.14). If we then compute the tadpole contributions to the action, the mixed terms vanish, i.e.,

$$-g\Delta S_t = \frac{1}{2} \sum_{ij} \Delta v_i \Delta v_j G_{\perp}(t_i, t_j) \simeq \frac{1}{2} (2J)^2 \sum_i G_{\perp}(t_i, t_i) \quad (\text{B.16})$$

and the diagonal terms, because of $g_i(t_i) = 0$, are of order 1, i.e.,

$$G_{\perp}(t_i, t_i) = f(t_i)^2 \int_{t_{i-1}}^{t_i} dt f(t)^2 \int_{t_{i-1}}^t dt' f(t')^{-2} \cdot (f, f)^{-1} = O(1) \quad (\text{B.17})$$

Therefore, $\Delta S_t \simeq -\text{const. } J^2 n/g$ only increases linearly with n , giving rise to the estimates (3.4) and (3.7).

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FIGURE CAPTIONS

- Fig. 1 : (a) Maxwell construction (dotted line) and (b) expected instanton contribution to the effective potential for the double-well problem.
- Fig. 2 : Instanton splitting for $J = 0$, leading to (I) the infinite action solution and to (II) the bounce with unstable quantum modes.
- Fig. 3 : Instanton-like piecewise classical trajectory with (a) velocity discontinuity at $t = t_1$ and (b) corresponding energy diagram.
- Fig. 4 : Multi-instanton-like trajectory with three velocity discontinuities and corresponding partial times.

(a)

(b)

Fig. 1

(I)

(III)

Fig. 2

(a)

(b)

Fig. 3

Fig. 4