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ZILCH CURRENTS, SUPERSYMMETRY AND KALUZA-KLEIN CONSISTENCY

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A B S T R A C T

We give a simple explanation for the fact that one can always perform a consistent truncation of a normal second-order Kaluza-Klein supergravity theory to the irreducible massless graviton supermultiplet. The basic reason for consistency is the highly restrictive requirement that massive fields of spin $s \geq 1$ couple at lowest order to conserved currents constructed from field strengths of the massless fields. In particular, one understands in this way the consistency of the truncation of $D = 11$ supergravity compactified on the seven-sphere to $d = 4$, $N = 8$ gauged supergravity.

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In many of the recent attempts to construct a unified quantum theory of gravity and particle interactions, one encounters the need to reduce a theory from some higher dimension D to a lower dimension d . The basic mechanism for this reduction is to use the Kaluza-Klein idea, in which a solution of the higher-dimensional theory is resolved into a tower of infinitely many massive states plus a finite number of massless states in the lower dimension. Typically, this decomposition is carried out by starting from a "vacuum" solution where the D -dimensional space-time has the form $M_d \times K_{D-d}$, where M_d is a maximally symmetric space-time of dimension d and K_{D-d} is a compact internal space of dimension $D-d$. One then expands the fluctuations of the D -dimensional fields around this ground state solution in terms of complete sets of harmonics on the internal space K_{D-d} . Thus, for a typical field $\Phi(x,y)$, we have

$$\Phi(x,y) = \sum_n \phi_n(x) Y_{(n)}(y) \quad (1)$$

where x^μ are co-ordinates on M_d and y^m are co-ordinates on K_{D-d} . The harmonics $Y_{(n)}(y)$ are chosen to be eigenfunctions of the mass operator on K_{D-d} , and the corresponding d -dimensional fields are the $\phi_{(n)}(x)$.

In practice, one is usually interested in restricting attention to a finite subset of the fields $\phi_{(n)}(x)$, typically the massless sector. In certain circumstances, one can make a consistent truncation of the D -dimensional theory down to a finite number of fields in d dimensions, simply by setting all the other fields to zero. When this can be done, the D -dimensional theory possesses superselection sectors and there exist solutions of the full D -dimensional field equations that are parametrized only by Cauchy data corresponding to the finite set of fields that are being retained. In other words, setting the infinite set of fields to zero is consistent with their equations of motion.

Sometimes the reasons for the existence of such a consistent truncation are clear from group theory. The problem that arises in attempting a truncation to a finite number of modes is that they may act as sources for the modes one wishes to set to zero. In that case, the truncation would be inconsistent. If group theory forbids the existence of source terms for the truncated modes, then consistency is guaranteed. This occurs, for example, if the full set of d -dimensional fields transforms under some group G , and we truncate to the set of singlets under any subgroup $K \subset G$ [see, e.g., Ref. 1)].

A much more interesting situation arises with consistent truncations that do not have any obvious group-theoretical explanation. There are few known examples of this type. The most notable is in the compactification of eleven-dimensional supergravity on a round seven-sphere. Truncation to the massless supermultiplet

in this compactification yields gauged $N = 8$, $d = 4$ supergravity²⁾. The consistency of this truncation has recently been shown³⁾ by a construction of the non-linear Kaluza-Klein ansatz.

In this paper, we present an intuitive analysis of why such non-trivial consistent truncations work. The main result is that one can always make a consistent truncation to the irreducible supermultiplet containing the graviton. [This was already conjectured in 1)]. The consistency of the truncation to the massless supermultiplet on the seven-sphere is the maximal case of this type.

The potential obstacle to performing a truncation to a finite set of d -dimensional modes results from the possible presence of terms in the d -dimensional Lagrangian that are linear in one of the modes to be truncated. Such dangerous terms may be schematically denoted $H(L)^n$, where H denotes one of the fields to be truncated and L denotes generically the fields to be retained. Typically, H will correspond to a heavy field and L will correspond to a massless field.

In the Kaluza-Klein theories that we are considering, all fields with spins $s \geq 1$ in d dimensions originate from massless gauge fields in the D -dimensional theory. Therefore, all heavy fields H with spins $s \geq 1$ acquire their masses by a Higgs mechanism. Consequently, the interactions of such heavy fields are governed by the D -dimensional gauge invariance. In particular, the lowest-order coupling for a given heavy field with $s \geq 1$ must be to a current that is conserved to lowest order.

To illustrate this structure, consider a situation where a consistent truncation cannot be performed. For example, one might compactify D -dimensional Einstein gravity with a cosmological constant to a ground state that is the product (d-dimensional de Sitter space) $\times S^{D-d}$. In this case, the massless sector will contain vector fields $A_{\mu}^{ij} = -A_{\mu}^{ji}$ for the $SO(D-d+1)$ isometry group of the sphere S^{D-d} . Amongst the heavy fields will be a massive spin-two field $H_{\mu\nu}^{ij} = H_{\mu\nu}^{ji}$ that couples to the massless vectors through the interaction term $H_{\mu\nu}^{ij} T^{\mu\nu ij}$, where

$$T_{\mu\nu}^{ij} = F_{\mu\lambda}^{k(i} F_{\nu\sigma}^{j)k} g^{\lambda\sigma} - \frac{1}{4} g_{\mu\nu} g^{\lambda\sigma} g^{\rho\tau} F_{\lambda\rho}^{k(i} F_{\sigma\tau}^{j)k} \quad (2)$$

There are two important things to note about the structure of this current. Firstly, it is conserved to lowest order upon use of the lowest order-field equation for A_{μ}^{ij} (i.e., $\overset{\circ}{\nabla}^{\mu} F_{\mu\nu} = 0 + \dots$)

$$\overset{\circ}{\nabla}^{\mu} T_{\mu\nu}^{ij} = 0 + \dots, \quad (3)$$

where $\overset{\circ}{\nabla}_{\mu} = \partial_{\mu} + \frac{1}{2}\omega_m^{ab}\Sigma_{ab}$ is the covariant derivative in the background de Sitter space. In fact, the current $J_{\mu\nu}^{ij}$ is exactly gauge-covariantly conserved upon use of the gauge-covariant equation $D^{\mu} F_{\mu\nu}^{ij} = 0$, where $D_{\mu} = \nabla_{\mu} + A_{\mu}$ is the full gravitational and gauge-covariantized derivative. This full conservation property is related to the second point of note; that $T_{\mu\nu}^{ij}$ is a gauge-covariant object constructed from the Yang-Mills field strength $F_{\mu\nu}^{ij}$.

The two special properties of $T_{\mu\nu}^{ij}$ noted above are generic properties of the currents that couple to the heavy gauge fields in Kaluza-Klein theories.

The requirement of current conservation for the lowest-order coupling of a heavy gauge field H to other fields $\phi_1, \phi_2, \dots$ is a consequence of the gauge invariance associated to the heavy field. Consider for example a coupling such as $H^{\mu\nu A} T_{\mu\nu}^A(\phi_1, \phi_2, \dots)$ for some set of massive spin-two fields $H_{\mu\nu}^A$. The mass of $H_{\mu\nu}^A$ is acquired through a Higgs mechanism so the gauge symmetry associated to $H_{\mu\nu}^A$ is not lost, but merely becomes non-linearly realized. Since we are by assumption considering the lowest-order coupling of $H_{\mu\nu}^A$ to other fields, the cancellation of the inhomogeneous part of the gauge variation of $H_{\mu\nu}^A$ requires that $T_{\mu\nu}^A$ be conserved to lowest order subject to the field equations for $\phi_1, \phi_2, \dots$. Thus, for the spin-two field $H_{\mu\nu}^A$, the gauge transformation takes the form

$$\delta H_{\mu\nu}^A = \overset{\circ}{\nabla}_{\mu} \xi_{\nu}^A + \overset{\circ}{\nabla}_{\nu} \xi_{\mu}^A + \text{field dependent terms}, \quad (4)$$

and integration by parts in $\int \delta H^{\mu\nu A} T_{\mu\nu}^A$ yields the term

$$-2 \int \xi^{\mu A} \overset{\circ}{\nabla}^{\nu} T_{\mu\nu}^A(\phi_1, \phi_2, \dots). \quad (5)$$

Since the action is invariant under the gauge transformation parametrized by ξ_{μ}^A , this term must be cancelled. By the assumption that $H^{\mu\nu A} T_{\mu\nu}^A$ is the lowest-order coupling of $H_{\mu\nu}^A$ to other fields, the variation (5) can only be cancelled by a variation of the part of the action dependent only upon the $\phi_1, \phi_2, \dots$, i.e., it must vanish by virtue of the field equations for these fields.

From the above discussion, we see that the lowest-order coupling between a single heavy field and other fields requires the existence of a conserved current such as $T_{\mu\nu}^A(\phi_1, \phi_2, \dots)$. Typically, the A index denotes a representation of some internal symmetry group, e.g., as in Eq. (2). Conserved charges carrying both space-time and internal symmetry indices exist for free fields on a maximally symmetric space-time background. For theories with finite numbers of interacting fields, the only conserved charges will correspond to a symmetry group that is the direct product of the space-time symmetry and an internal symmetry group. For interacting theories in flat space, this is the content of the Coleman-Mandula theorem⁵⁾.

Kaluza-Klein theories evade the restrictions of the Coleman-Mandula theorem by virtue of the infinite number of particle types. Nonetheless, in the order-by-order expansion that we are considering, the occurrence of lowest-order conserved charges carrying both space-time and internal symmetry indices will be limited to bilinears in the fields. These charges must be bilinear because they must correspond to the conserved charges for free fields discussed above, which are bilinear. Once the higher-order interactions are taken into account, conservation will only be regained after summing contributions at all orders, and including contributions from all Kaluza-Klein levels [for a discussion of the inevitability of summing to all orders to regain conservation in such an iterative construction, see Ref. 6)]. Thus, the couplings of the heavy gauge fields in Kaluza-Klein theories start out with currents that for a theory with a finite number of particle types could be conserved only for free fields. Such currents have been termed "Zilch" currents⁷⁾ in ordinary finite component particle theories, because they cannot be generalized to the interacting level⁸⁾. Owing to the infinite number of particle types in Kaluza-Klein theories, they can be generalized to include the interactions, and, indeed, Kaluza-Klein theory may be regarded as the general theory of Zilch couplings.

From the above, we have seen that the massive Kaluza-Klein gauge fields can appear linearly in the d-dimensional Lagrangian only if the lowest-order couplings of this type are to bilinear currents constructed from other fields. These bilinear currents must be conserved by virtue of the linearized field equations for the other fields. Conversely, the absence of couplings linear in a given massive gauge field may be established by demonstrating the absence of appropriate conserved bilinear currents.

In addition to the conservation requirement, the bilinear currents must be the lowest-order pieces of quantities that are covariant with respect to the unbroken gauge symmetries of the theory. For example, it was essential that the Zilch current (2) was constructed from field strengths $F_{\mu\nu}^{ij}$. Otherwise, ordinary

Yang-Mills covariance would have been broken by the coupling $H_{\mu\nu}^{ij} T^{\mu\nu ij}$ (where $H_{\mu\nu}^{ij}$ must transform as a tensor under ordinary Yang-Mills transformations). Similarly, any space-time symmetries, such as general covariance or local supersymmetry, must be respected by the couplings. Consequently, massless gauge fields can appear in the Zilch currents only through their linearized field strengths. In particular, the gravitational field can appear in Zilch currents only through the curvature tensor linearized with respect to fluctuations about the maximally symmetric background ($R_{\mu\nu\rho\sigma}^L$) and the gravitino field can appear only through the linearized Rarita-Schwinger field strength ($R_{\mu\nu}^{L\alpha}$).

In a Kaluza-Klein theory with at most second-order field equations, the above criteria are so restrictive as to preclude Zilch couplings involving the graviton and gravitino fields. This is because, once all the derivatives are required to form a linearized curvature or Rarita-Schwinger field strength, none are left over to act upon the other field in a bilinear current. With such a restricted structure (e.g., $R_{\mu\nu\rho\sigma}^L \phi$, $R_{\mu\nu}^L \lambda$, etc.) it is not possible to construct a current conserved by the free field equations. Thus, the conserved Zilch currents appropriate to a Kaluza-Klein theory, can involve massless fields of spin ≤ 1 only.

In a supersymmetric theory, the above considerations taken together with the constraints of supersymmetry rule out all couplings linear in a multiplet of massive fields to the irreducible graviton supermultiplet. This follows because in a theory with at most second-order field equations, the interaction terms are of too low dimension to satisfy simultaneously the requirements of supersymmetry, the massless multiplet's gauge invariances and the Zilch current conservation required for coupling to the massive multiplet.

The supersymmetry that is relevant for this analysis is the linearized, rigid, unbroken supersymmetry that organizes the Kaluza-Klein states into supermultiplets. Although the full higher-dimensional supersymmetry transformation will be non-linear, one may use just the linearized transformations for analysis of the lowest-order coupling terms linear in a supermultiplet of massive fields. These lowest-order couplings must be invariant under the linearized supersymmetry transformations when the free-field equations for the fields are imposed. This is because the supersymmetry transformations of the lowest-order coupling terms must cancel among themselves when the variations of the quadratic terms in the action are annihilated by imposing the free-field equations.

Consequently, one may analyze the structure of the lowest-order couplings using superfields for the linearized rigid supersymmetry. For a Kaluza-Klein

theory compactified on an internal manifold of finite volume, this rigid supersymmetry will generally be an orthosymplectic supersymmetry. However, since the particular size of the internal manifold will play no role in the analysis of the structure of the coupling terms, we may consider the limit in which this size tends to infinity, and all the Kaluza-Klein massive levels telescope down to zero mass. Thus, it is sufficient to analyze the possibilities for coupling using Poincaré rigid supersymmetry. In order to satisfy the gauge invariance requirements for the graviton supermultiplet, this multiplet must appear only through its superfield strength. From this, we immediately see that truncation to $d = 4$, $N = 1$ supergravity must be consistent, since the couplings would have at best the form

$$\int d^4x d^2\theta W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} \phi_h, \quad (6)$$

where $W_{\alpha\beta\gamma}$ is the chiral $N = 1$ superfield strength, and ϕ_h could be any chiral "massive" multiplet. But (6) involves coupling of ϕ_h to an operator of dimension 4 (equivalent to four bosonic derivatives), and can never occur in a second-order theory. The coupling (6) is of the same dimension as $(R_{\mu\nu\alpha\beta})^2 \phi_h$, which is the form that would be required for graviton coupling in a purely bosonic theory.

Similarly, in a theory that compactifies to preserve an $N = 2$, $d = 4$ supersymmetry, there are no possible couplings of the correct dimension built from the $N = 2$ supergravity field strength $W_{\alpha\beta}$. Although the dimension of $W_{\alpha\beta}$ is smaller by $\frac{1}{2}$ than that of $W_{\alpha\beta\gamma}$ in the $N = 1$ case, the dimension of the measure required to make a supersymmetric invariant grows to compensate: for a chiral heavy multiplet ϕ_h , the lowest coupling one could build is

$$\int d^4x d^4\theta W'_{\alpha\beta} W'^{\alpha\beta} \phi_h, \quad (7)$$

which could couple the massive multiplet ϕ_h to a dimension-four operator, just as happened in the $N = 1$ case.

For the various cases with an unbroken d -dimensional N -extended supersymmetry, there are a number of different ways in which supersymmetric invariants can be formed, corresponding to superspace measures with differing numbers of $d\theta$ integrations⁹⁾. However, for a decrease in the number of $d\theta$ integrations, there is a corresponding strengthening of the constraint that the integrand must satisfy in order for the integral to be supersymmetric. For example, in the $N = 3$, $d = 4$ case, one might think to construct an invariant quadratic in the $N = 3$ supergravity field strength W_α (of dimension $\frac{1}{2}$) and an integration over two $d\theta$'s only. There is an $N = 3$ "superaction"⁹⁾ with a measure $d^4x d\theta^{\alpha(i} d\theta^{\alpha(j)}$, requiring for supersymmetry that the integrand K_{ij} satisfy

$$D_\alpha{}^\kappa K_{ij} = \frac{1}{4} (\delta_i^\kappa D_\alpha{}^\ell K_{\ell j} + \delta_j^\kappa D_\alpha{}^\ell K_{i\ell}),$$

$$\bar{D}_\alpha{}^\kappa K_{ij} = 0. \quad (8)$$

Similarly, using the methods of Ref. 9), one can show that there is another superaction with measure $d^4x d^{2\theta}{}^{[ij]}{}_{(\alpha\beta)}$, requiring for supersymmetry that the integrand $K_{\alpha\beta ij}$ satisfy

$$D_\gamma{}^\kappa K_{\alpha\beta ij} = 0,$$

$$D_\gamma{}^\kappa K_{\alpha\beta ij} = \frac{1}{4} (\delta_i^\kappa D_\gamma{}^\ell K_{\alpha\beta \ell j} + \delta_j^\kappa D_\gamma{}^\ell K_{\alpha\beta i\ell}),$$

$$\bar{D}_\gamma{}^\kappa K_{\alpha\beta ij} = 0. \quad (9)$$

However, there is no way to construct an integrand using only $W_\alpha W_\beta$ and ϕ_h that satisfies the constraints (8) or (9) for any matter superfield ϕ_h . Once again, the lowest-order allowed coupling for ϕ_h turns out to be to an operator of dimension four, which is not possible in a second-order theory. Similarly, one may show that no couplings linear in ϕ_h to the graviton supermultiplet can be made for $N = 4$, $d = 4$ supersymmetry.

For $N > 4$, $d = 4$ supersymmetry it can be seen immediately that no couplings of the form $W \dots W \dots \phi_h$ can be made, for all currents bilinear in $W \dots$ contain spins higher than 2 for these N ¹⁰⁾. In a Kaluza-Klein theory, ϕ_h can contain massive fields of spin only up to two.

The above analysis shows that it is always possible to make a consistent truncation of a normal second-order supersymmetric Kaluza-Klein theory to the massless supermultiplet containing the graviton. In particular, this gives a simple explanation for the consistency ³⁾ of truncation to the $N = 8$, $d = 4$ gauged supergravity theory from $D = 11$ supergravity compactified on a seven-sphere. Evidence for the consistency of this truncation was given in Refs. 4), 11)-13). In this case, the rigid unbroken supersymmetry is $OSp(8/4)$, under which the higher Kaluza-Klein levels form irreducible short massive multiplets.

Another non-trivial instance of consistent truncation without an obvious group-theoretical explanation is provided by the compactification of type IIB, $D = 10$ supergravity on the five-sphere. Evidence for the consistency of truncation in this case to five-dimensional $N = 4$ supergravity has been given in Ref. 14), where it was shown that the trilinear coupling between the massless gauge bosons and a massive spin-two mode is absent. This is again an instance of

truncation to the irreducible graviton supermultiplet, and is consequently explained by the analysis given in this paper.

Although we have concentrated on consistent truncations in supersymmetric theories in this paper, the essential constraints on coupling between the massive and massless levels also operate in purely bosonic cases. Again, the couplings of the heavy fields of spin ≥ 1 will be constrained by the requirement that they couple to conserved currents, and again the massless gauge fields can appear in the dangerous couplings only through their field strengths. For theories with at most second-order field equations, these requirements rule out couplings linear in the massive gauge fields to the massless graviton. If there are other unbroken symmetries present that link the other dangerous couplings to the above excluded ones, then a consistent truncation may be possible. An important open problem of this type is to explain the apparent consistency of truncation to the full massless sector of the bosonic string compactified on a group manifold¹⁵⁾.

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REFERENCES

- 1) M.J. Duff and C.N. Pope - Nucl.Phys. B255 (1985) 355.
- 2) M.J. Duff and C.N. Pope - in Supersymmetry and Supergravity 82,
Eds. S. Ferrara, J.G. Taylor and P. van Nieuwenhuizen (World Scientific,
Singapore, 1983).
- 3) B. de Wit and H. Nicolai - Nucl.Phys. B281 (1987) 211.
- 4) M.J. Duff, B.E.W. Nilsson, C.N. Pope and N.P. Warner - Phys.Lett. 149B
(1984) 90.
- 5) S. Coleman and J. Mandula - Phys.Rev. 159 (1967) 1251.
- 6) S. Deser - J.Gen.Rel.Grav. 1 (1970) 9.
- 7) D.M. Lipkin - J.Math.Phys. 5 (1964) 646.
- 8) S. Deser and H. Nicolai - Phys.Lett. 98B (1981) 45.
- 9) P.S. Howe, K.S. Stelle and P.K. Townsend - Nucl.Phys. B191 (1981) 445.
- 10) P.S. Howe, K.S. Stelle and P.K. Townsend - Nucl.Phys. B192 (1981) 332.
- 11) B.E.W. Nilsson - Phys.Lett. 115B (1985) 54.
- 12) B. de Wit, H. Nicolai and N.P. Warner - Nucl.Phys. B255 (1985) 29.
- 13) C.N. Pope - Class. Quantum Grav. 2 (1985) L77.
- 14) T.T. Tsikas - Class. Quantum Grav. 3 (1986) 773.
- 15) M.J. Duff, C.N. Pope, B.E.W. Nilsson and N.P. Warner - Phys.Lett. 171B
(1986) 70.