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ON THE FUNDAMENTAL SPINOR FIELDS REPRESENTING
MASSLESS FERMIONS IN D-DIMENSIONS

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A B S T R A C T

We establish a relation between chirality and "helicity" in higher dimensions. This is a generalization of the well-known result that in four dimensions a spinor field of positive chirality can only be associated with a massless fermion of helicity $-\frac{1}{2}$. We give also the transformation properties of the fundamental spinor fields, representing massless fermions, in d-dimensions under the discrete symmetries of charge conjugation, parity, and time reversal. It is shown that self-charge-conjugate fundamental spinor fields can only exist in $d = 1, 2 \pmod{8}$ dimensions.

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It is well known that in four dimensions a spinor field of positive chirality can be associated with a massless fermion of helicity $-\frac{1}{2}$ but not with one of helicity $\frac{1}{2}$. This is indeed the case with a massless neutrino, where a positive chirality Weyl spinor is used to describe such a particle. This is an example of a more general result pointed out by Weinberg¹⁾ some time ago. Given that the general irreducible representation of the Lorentz group in four dimensions is labelled by (A,B) , where $2A$ and $2B$ are integers, Weinberg has shown that a massless particle annihilation operator, $a(\vec{p},\lambda)$, of helicity λ can only be used to construct fields which transform under the Lorentz group according to a representation (A,B) such that

$$B - A = \lambda \quad (1)$$

The limitation imposed by (1) is solely due to the non-semi-simple nature of the little group for massless particles. It is clear now that massless fermions with helicity $\lambda = \frac{1}{2}, -\frac{1}{2}$ can be associated with spinor fields transforming as $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$ respectively. These are Weyl spinors of chirality -1 and 1 . Consequently, there is a correlation between helicity and chirality in four dimensions.

In this note we ask what (if any) is the correlation between "helicity" and chirality in d -dimensions. First, however, we need to clarify what we mean by "helicity" in higher dimensions. In four dimensions, the helicity λ is a real number, which labels the one-dimensional physically permissible irreducible representations of the little group^{2),1)}. For global reasons the value of λ is restricted to be a positive or negative integer or half odd integer. In higher dimensions the "helicity" index λ describes the transformation properties of the massless particle states under the little group - for the massive case see Weinberg³⁾. In d -dimensions the physically relevant little group is $SO(d-2)$. When d is even the spinor representations of $SO(d-2)$ and those of the Lorentz group $SO(1,d-1)$ are doubled. If a massless particle rotates with one of the spinor representations of $SO(d-2)$, it is necessary to ask which spinor representation of $SO(1,d-1)$ may be associated with. We show that the non-semi-simple nature of the little group restricts this choice in a manner analogous to the four-dimensional case. Indeed we find that there is always a definite correlation between "helicity" [i.e. a spinor representation of $SO(d-2)$] and chirality in d -dimensions.

We may remark here that, if one is only interested in counting degrees of freedom, one does not need to know which spinor representation of $SO(d-2)$ is associated with a given spinor representation of the Lorentz group. On the other hand, if one is interested in finding how the spinor representation of the little group induces a spinor representation of the Lorentz group (as in the case of the work of Wigner²⁾), then one must be careful to take into account the constraints

imposed by the non-semi-simple nature of the little group as the analysis of Weinberg¹⁾ shows in four dimensions. This is also necessary, if one wants to study how massless spinor fields transform under the discrete symmetries of charge conjugation ($\mathcal{C}$), parity ($\mathcal{P}$), and time reversal ($\mathcal{T}$). In the past the analysis of $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$ transformations in d-dimensions has been carried out by making use of bispinors and the Dirac equation^{4),5)}. For a discussion of the $\mathcal{C}$ operation see also Ref. 6). In the case of massless fermions the fundamental spinor fields are not bispinors, and indeed in a parity violating theory bispinors are not always available. In such a case, one has to make use of fundamental spinor fields and the correct Weyl equations, which should embody the correct correlation between chirality and "helicity".

To carry out the analysis we need a bit of a formalism about the spinor representations of (pseudo)-orthogonal groups. In an even dimensional space, $d = 2v$, let N and $\tilde{N}$ be the fundamental spinor representations of $SO(1, 2v-1)$ and let ψ^N and $\psi^{\tilde{N}}$ be the spinors transforming according to these representations. We denote also by r and $\tilde{r}$ the fundamental spinor representations of $SO(2v-2)$. Let $\gamma_j^{(v)}$ $j = 1, \dots, 2v$ be the $2^v \times 2^v$ matrices generating the Clifford algebra $C(R^{(1, 2v-1)})$. The set of matrices $Z_{jk} = 1/4[\gamma_j^{(v)}, \gamma_k^{(v)}]$ form a reducible representation of the algebra of $SO(1, 2v-1)$. In the reduced basis⁷⁾, where $\gamma_j^{(v)} = \gamma_j^{(v-1)} \otimes \sigma_2$ $j = 1, \dots, 2v-1$ and $\gamma_{2v}^{(v)} = 1^{(v-1)} \otimes \sigma_1$ with $\gamma_j^{(v-1)}$ being appropriate $2^{v-1} \times 2^{v-1}$ matrices, the generators Z_{jk} are diagonalized.

$$Z_{jk} = \begin{pmatrix} Z_{jk}^N & \\ & Z_{jk}^{\tilde{N}} \end{pmatrix}$$

with Ref. 8)

$$Z^N = \{Z_{jk}^N, Z_{j, 2v}^N\} = \left\{ \frac{1}{2} \gamma_j^{(v-1)} \gamma_k^{(v-1)}, -i \frac{1}{2} \gamma_j^{(v-1)} \right\} \quad (2)$$

$$Z^{\tilde{N}} = \{Z_{jk}^{\tilde{N}}, Z_{j, 2v}^{\tilde{N}}\} = \left\{ \frac{1}{2} \gamma_j^{(v-1)} \gamma_k^{(v-1)}, i \frac{1}{2} \gamma_j^{(v-1)} \right\} \quad (3)$$

$j, k = 1, 2, \dots, 2v-1$.

In this basis $\gamma_{2\nu+1}^{(\nu)} = \gamma_1^{(\nu)} \dots \gamma_{2\nu}^{(\nu)} \propto 1^{(\nu-1)} \otimes \sigma_3$ and the spinors ψ^N and $\psi^{\tilde{N}}$ have chiralities $\chi_N = 1$ and $\chi_{\tilde{N}} = -1$. The generators for the r and $\tilde{r}$ representations of $SO(2\nu-2) \subset SO(1,2\nu-1)$ are given by formulae analogous to (2) and (3) namely

$$Z^r = \left\{ \frac{1}{2} \gamma_j^{(\nu-2)} \gamma_k^{(\nu-2)}, \frac{1}{2} \gamma_j^{(\nu-2)} \right\} \quad (4)$$

$$Z^{\tilde{r}} = \left\{ \frac{1}{2} \gamma_j^{(\nu-2)} \gamma_k^{(\nu-2)}, -\frac{1}{2} \gamma_j^{(\nu-2)} \right\} \quad (5)$$

with $\gamma_j^{(\nu-2)}$ $j = 1, \dots, 2\nu-3$ appropriate $2^{\nu-2} \times 2^{\nu-2}$ matrices⁸⁾.

We are now ready to discuss the little group and its implications. For massless particles the algebra of the little group in d -dimensions is identified with that of all translations and rotations in $(d-2)$ dimensions. The generators corresponding to translations are given by Ref. 9)

$$L_a = Z_{a,2\nu-1} + Z_{a,2\nu} \quad , \quad a=1, \dots, 2\nu-2 \quad (6)$$

and it is easy to verify that they form an invariant Abelian subalgebra

$$[L_a, L_b] = 0 \quad (7)$$

In the N and $\tilde{N}$ representations, of course, we have

$$L_a^N = \frac{1}{2} \gamma_a^{(\nu-1)} \gamma_{2\nu-1}^{(\nu-1)} - i \frac{1}{2} \gamma_a^{(\nu-1)} \quad (8)$$

$$L_a^{\tilde{N}} = \frac{1}{2} \gamma_a^{(\nu-1)} \gamma_{2\nu-1}^{(\nu-1)} + i \frac{1}{2} \gamma_a^{(\nu-1)} \quad (9)$$

On setting these generators to zero the little group algebra reduces to the algebra of $SO(2\nu-2)$. This procedure may appear rather ad hoc, but unless this is done, the massless particle states (the "helicity" states which form representations of the little group) will not form a finite set. From the above discussion we have $SO(2\nu-2) \subset \text{Little group} \subset SO(1,2\nu-1)$ and consequently the fundamental spinor representations N and $\tilde{N}$ of $SO(1,2\nu-1)$ decompose as

$$\begin{aligned} N &= r \oplus \tilde{r} \\ \tilde{N} &= r \oplus \tilde{r} \end{aligned} \quad (10)$$

For massless particles there is no rest frame so instead a "standard" light-like d-momentum $K^A = (k, 0, \dots, k)$ is introduced and the states, $u(k, \lambda)$, are labelled by k and the running "helicity" index λ . These states provide the numerical coefficients $u_n(k, \lambda)$, which satisfy

$$\left[\exp\left(\frac{1}{2}\omega^{jk}\gamma_j^{(\nu-1)}\gamma_k^{(\nu-1)}\right) \right]_{nm} u_n^{N,r}(k, \lambda) = u_n^{N,r}(k, \mu) \left[\exp\left(\frac{1}{2}\omega^{ab}\gamma_a^{(\nu-2)}\gamma_b^{(\nu-2)} + \frac{1}{2}\omega^{a,2\nu-2}\gamma_a^{(\nu-2)}\right) \right]_{\mu\lambda} \quad (11)$$

$$\left[\exp\left(\frac{1}{2}\omega^{jk}\gamma_j^{(\nu-1)}\gamma_k^{(\nu-1)}\right) \right]_{nm} u_n^{N,\tilde{r}}(k, \lambda) = u_n^{N,\tilde{r}}(k, \mu) \left[\exp\left(\frac{1}{2}\omega^{ab}\gamma_a^{(\nu-2)}\gamma_b^{(\nu-2)} - \frac{1}{2}\omega^{a,2\nu-2}\gamma_a^{(\nu-2)}\right) \right]_{\mu\lambda} \quad (12)$$

The coefficients $u^{N,r}$ and $u^{N,\tilde{r}}$ satisfy relations identical to (11) and (12).

As we have stated already in order that the states $u(k, \lambda)$ form a finite set, it is necessary to represent the translations of the little group by zero. In the N representation we must have

$$(L_a^N)_{nm} u_m^{N,r}(k, \lambda) = 0 \quad (13)$$

$$(L_a^N)_{nm} u_m^{N,\tilde{r}}(k, \lambda) = 0 \quad (14)$$

$$a = 1, \dots, 2\nu-2$$

A similar set of equations holds true with N replaced by $\tilde{N}$. Using the matrix representations for L^N and $L^{\tilde{N}}$ given by (8) and (9) respectively we immediately deduce that

$$(1 + i\gamma_{2\nu-1}^{(\nu-1)}) u^{N,r}(k, \lambda) = 0 \quad (15)$$

$$(1 + i\gamma_{2\nu-1}^{(\nu-1)}) u^{N,\tilde{r}}(k, \lambda) = 0 \quad (16)$$

$$(1 - i \gamma_{2\nu-1}^{(\nu-1)}) u^{\tilde{N},r}(k, \lambda) = 0 \quad (17)$$

$$(1 - i \gamma_{2\nu-1}^{(\nu-1)}) u^{\tilde{N},\tilde{r}}(k, \lambda) = 0 \quad (18)$$

We shall now obtain another set of equations for the numeric coefficients $u(k, \lambda)$. To do this we consider (11) and (12) and choose a special rotation made out of infinitesimal rotations in the planes $(1,2), (3,4), \dots, (2\nu-3, 2\nu-2)$. Comparing now equal powers of infinitesimal parameters on both sides of (11) and (12) we obtain, after some algebra, the following set of equations

$$(1 + i \gamma_{2\nu-1}^{(\nu-1)}) u^{N,r}(k, \lambda) = 0 \quad (19)$$

$$(1 + i \gamma_{2\nu-1}^{(\nu-1)}) u^{\tilde{N},r}(k, \lambda) = 0 \quad (20)$$

$$(1 - i \gamma_{2\nu-1}^{(\nu-1)}) u^{N,\tilde{r}}(k, \lambda) = 0 \quad (21)$$

$$(1 - i \gamma_{2\nu-1}^{(\nu-1)}) u^{\tilde{N},\tilde{r}}(k, \lambda) = 0 \quad (22)$$

Solving equations (16) to (22) we obtain

$$\begin{aligned} u^{N,\tilde{r}}(k, \lambda) &= 0 \\ u^{\tilde{N},r}(k, \lambda) &= 0 \end{aligned} \quad (23)$$

with $u^{N,r}(k, \lambda)$ and $u^{\tilde{N},\tilde{r}}(k, \lambda)$ in general different from zero.

The same analysis goes through with numeric coefficients $v^{N,r}(k, \lambda)$, $v^{\tilde{N},r}(k, \lambda)$, $v^{N,\tilde{r}}(k, \lambda)$ and $v^{\tilde{N},\tilde{r}}(k, \lambda)$ which satisfy equations similar to (11) and (12) but with the right-hand side transforming according to the complex conjugate representations r^* and $\tilde{r}^*$ respectively. The result is that all the numeric coefficients of the type $v^{N,\tilde{r}}$ and $v^{\tilde{N},r}$ vanish, and only $v^{N,r}$ and $v^{\tilde{N},\tilde{r}}$ are in general different from zero.

The vanishing of all the numeric coefficients $u_n^{N,\tilde{r}}$, $u_n^{\tilde{N},r}$, $v_n^{N,\tilde{r}}$, $v_n^{\tilde{N},r}$ in d -dimensions is the analogue of (1), the constraint satisfied by the helicity λ , pointed out by Weinberg in four dimensions. We can now follow Weinberg¹⁾ and boost

the non-vanishing numerical coefficients $u^{N,r}(k,\lambda)$, $u^{\tilde{N},\tilde{r}}(k,\lambda)$, $v^{N,r}(k,\lambda)$ and $v^{\tilde{N},\tilde{r}}(k,\lambda)$ of the "standard" reference frame to a frame defined by an arbitrary light-like momentum $p^A = \{|\vec{p}|, \vec{p}\}$. First we consider the boost $B(|\vec{p}|)$ along the (d-1)-direction given by

$$B^A_B(|\vec{p}|) = \begin{pmatrix} \cosh \Theta & 0 & \dots & \sinh \Theta \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sinh \Theta & 0 & \dots & \cosh \Theta \end{pmatrix}$$

where $\Theta = \ln[|\vec{p}|/k]$, which takes $k^A = \{k, 0, \dots, k\}$ into $\{|\vec{p}|, 0, \dots, |\vec{p}|\}$. Let $R(\vec{p})$ be the rotation which takes the (d-1)-axis into the direction of the unit vector $\hat{p} = \vec{p}/|\vec{p}|$. Then the boosted coefficients are given by

$$u_n^{N,r}(\vec{p}, \lambda) = (D^N[R(\vec{p})] D^N[B(|\vec{p}|)])_{nm} u_m^{N,r}(k, \lambda) \quad (24)$$

The numeric coefficients $u_n^{\tilde{N},\tilde{r}}(\vec{p}, \lambda)$, $v_n^{N,r}(\vec{p}, \lambda)$ and $v_n^{\tilde{N},\tilde{r}}(\vec{p}, \lambda)$ are constructed in a similar manner.

The general relativistic fields in d-dimensions, $\{\phi^M\}$, transform under various irreducible representations, M , of the Lorentz group $SO(1, d-1)$. It is necessary, however, to label the fields with both M and a label s which specifies the transformation properties under the "little" group $SO(d-2)$; there is in general no one-to-one connection between M and s ³⁾. In the case of spinors ϕ^N and $\psi^{\tilde{N}}$ our analysis shows that the only fields that can be constructed are

$$\psi_n^{N,r}(x) = \int d\mu(\vec{p}) \sum_{\lambda} \left[e^{-i\vec{p} \cdot x} u_n^{N,r}(\vec{p}, \lambda) a(\vec{p}, \lambda, r) + e^{i\vec{p} \cdot x} v_n^{N,r}(\vec{p}, \lambda) a^*(\vec{p}, \lambda, \bar{r}) \right] \quad (25)$$

and

$$\psi_n^{\tilde{N},\tilde{r}}(x) = \int d\mu(\vec{p}) \sum_{\lambda} \left[e^{-i\vec{p} \cdot x} u_n^{\tilde{N},\tilde{r}}(\vec{p}, \lambda) a(\vec{p}, \lambda, \tilde{r}) + e^{i\vec{p} \cdot x} v_n^{\tilde{N},\tilde{r}}(\vec{p}, \lambda) a^*(\vec{p}, \lambda, \tilde{r}) \right] \quad (26)$$

and no other ones. Thus, the annihilation operator $a(\vec{p}, \lambda, r)$ is associated with the spinor field $\psi^{N,r}$ of chirality $\chi_N = +1$, and $a(\vec{p}, \lambda, r)$ is associated with the spinor

field $\psi^{\tilde{N}, \tilde{r}}$ of chirality $\chi_{\tilde{N}} = -1$. For example, it is inconsistent to use $a(\vec{p}, \lambda, r)$ to construct a spinor field $\psi^{\tilde{N}, r}$ which has negative chirality. This is the generalization of Weinberg's result in higher dimensions.

The previous analysis was confined to even dimensions. In odd dimensions, $d = 2v+1$, there is a unique spinor representation, M , of $SO(1, 2v)$, which decomposes under the "little" group $SO(1, 2v-1)$ as $M = r \oplus r$, where r is the unique spinor representation of $SO(2v-1)$. In this case we can write uniquely

$$\psi_n^{M, r}(x) = \int d\mu(\vec{p}) \sum_{\lambda} \left[e^{-ip \cdot x} u_n^{M, r}(\vec{p}, \lambda) a(\vec{p}, \lambda, r) + e^{ip \cdot x} v_n^{M, r}(\vec{p}, \lambda) a^*(\vec{p}, \lambda, \tilde{r}) \right]^{(27)}$$

To complete this note we give also the transformations of the spinor fields $\psi^{N, r}(x)$, $\psi^{\tilde{N}, \tilde{r}}(x)$ and $\psi^{M, r}(x)$ under $\mathcal{L}$, $\mathcal{P}$, and $\mathcal{J}$. This is of interest in even dimensional spaces, because one can see directly whether the chirality is maintained or not after $\mathcal{L}$, $\mathcal{P}$, and $\mathcal{J}$. Moreover, once the transformation under $\mathcal{L}$ is known, it is easy to find out what values of d can accommodate massless self-charge-conjugate fundamental spinor fields. We summarize our results in Tables 1, 2, and 3^{*}). In Table 1 $z_{N, r}$, $\alpha_{N, r}$ and $\beta_{N, r}$ are generally complex phases, although it is possible to take $\alpha_{N, r} = \chi_N \alpha_r$ with α_r real, and $\lambda_s^v = (-1)^{vs+s(s-1)/2+v(v+1)/2}$. The transformation properties for $\psi^{\tilde{N}, \tilde{r}}(x)$ are obtained from Table 1 by substitution $N \rightarrow \tilde{N}$, $r \rightarrow \tilde{r}$ and taking into account the relations $z_{\tilde{N}, \tilde{r}} = z_{N, r}$, $\beta_{\tilde{N}, \tilde{r}} = -\beta_{N, r}$ and $\alpha_{\tilde{N}, \tilde{r}} = -\alpha_{N, r}^*$. In Table 2 the properties of $\psi^{\tilde{N}, \tilde{r}}(x)$ are obtained again by substitution and taking into account the relation $\hat{\beta}_{\tilde{N}, \tilde{r}} = \hat{\beta}_{N, r} (\alpha_{N, r}^*)^2$. With the choice $\alpha_{N, r} = \chi_N \alpha_r$ and α_r real the latter gives $\hat{\beta}_{\tilde{N}, \tilde{r}} = \hat{\beta}_{N, r}$.

In Tables 1 and 2 $P_{2^{v-1}}^{(v-1)}$ is a $2^{v-1} \times 2^{v-1}$ matrix⁸⁾ satisfying

$$\gamma_j^{(v-1)*} = (-1)^v P_{2^{v-1}}^{(v-1)} \gamma_j^{(v-1)} P_{2^{v-1}}^{(v-1)-1}, \quad j=1, \dots, 2^{v-1} \quad (28)$$

^{*}) These $\mathcal{L}$, $\mathcal{P}$, and $\mathcal{J}$ transformations were also derived independently by S. Chadha.

Moreover, $P_{2\nu-1}^{(\nu-1)} P_{2\nu-1}^{(\nu-1)*} = P_{2\nu-1}^{(\nu-1)*} P_{2\nu-1}^{(\nu-1)} = \lambda_{2\nu-1}^{\nu-1}$. It is clear also that $P_{2\nu-1}^{(\nu-1)}$ is the matrix that brings about the equivalence

$$N^* \sim \tilde{N} \quad , \quad \text{for } \nu \text{ even}$$

(29)

$$N^* \sim N \quad , \quad \text{for } \nu \text{ odd}$$

It is of interest to consider the case $d = 4k+2$ and define a new field $\psi^{\bar{N}, \bar{r}}(x) := \mathcal{L} \psi^{N, r}(x) \mathcal{L}^{-1}$. In this case one can show that $\mathcal{L} a(\vec{p}, \lambda, r) \mathcal{L}^{-1} = a(\vec{p}, \lambda, r)$ and $\mathcal{L} a^*(\vec{p}, \lambda, r) \mathcal{L}^{-1} = \lambda_{2\nu-2}^{\nu-1} a^*(\vec{p}, \lambda, r)$. Hence, the field $\psi^{\bar{N}, \bar{r}}(x)$ can be constructed in terms of creation and annihilation operators as follows

$$\psi_n^{\bar{N}, \bar{r}}(x) = \int d\mu(\vec{p}) \sum_{\lambda} \left[e^{-i\vec{p} \cdot x} u_n^{N, r}(\vec{p}, \lambda) a(\vec{p}, \lambda, \bar{r}) + \lambda_{2\nu-2}^{\nu-1} e^{i\vec{p} \cdot x} v_n^{N, r}(\vec{p}, \lambda) a^*(\vec{p}, \lambda, r) \right]$$

Now,

$$\lambda_{2\nu-2}^{\nu-1} \Big|_{\nu=2k+1} = (-1)^k = \begin{cases} 1, & k=2q, \quad d=8q+2 \\ -1, & k=2q+1, \quad d=8q+6 \end{cases}$$

With $\lambda_{2\nu-2}^{\nu-1} = 1$, $\psi^{\bar{N}, \bar{r}}(x)$ can be identified with $\psi^{N, r}$ and it is, therefore, clear that self-charge-conjugate massless fermions can exist in $d = 2 \pmod{8}$.

We now turn our attention to the odd dimensional case. It is well known that, if parity in odd dimensions is defined in the usual way of the reflection of all the spatial co-ordinates, it is simply equivalent to a proper Lorentz transformation. So one can arbitrarily define parity as the reflection of all spatial co-ordinates $\vec{x}_1$ except $x_{2\nu}^{(4)}$. One could equally define parity as the improper transformation obtained by reflecting any odd number of spatial co-ordinates, but all these reflections should be equivalent to the above choice by proper Lorentz transformations.

In Table 3 $P_{2\nu}^{(\nu)}$ is a $2^\nu \times 2^\nu$ matrix satisfying

$$\gamma_j^{(\nu)*} = (-1)^\nu P_{2\nu}^{(\nu)} \gamma_j^{(\nu)} P_{2\nu}^{(\nu)-1}$$

Following the procedure outlined for the even dimensional case one can show in the odd dimensional case that it is possible to have self-charge-conjugate massless spinor fields in $d = 1 \pmod{8}$.

Finally, we would like to remark that, if we combine $\psi^{N,r}(x)$ and $\psi^{\tilde{N},\tilde{r}}(x)$ to form a bispinor field

$$\psi(x) = \begin{pmatrix} \psi^{N,r}(x) \\ \psi^{\tilde{N},\tilde{r}}(x) \end{pmatrix}$$

then it is easy to check that the results of Tables 1 and 2 are in agreement with Ref. 4). The results for self-charge-conjugate spinors also agree with those of Wetterich and van Nieuwenhuizen⁶⁾.

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$d = 2v = 4k, v \text{ even}$	
ℓ	$\ell^{\phi^{N,r}(x)} \ell^{-1} = -\lambda_{2^{v-2}}^{v-1} z_{N,r} \beta_{N,r}^{P_{2^{v-1}}^{(v-1)}^{-1}} (\phi^{\tilde{N}, \tilde{r}}(x))^*$
$\wp$	$\wp^{\phi^{N,r}(t, \vec{x})} \wp^{-1} = \phi^{\tilde{N}, \tilde{r}}(t, -\vec{x})$
$\mathcal{J}$	$\mathcal{J}^{\phi^{N,r}(t, \vec{x})} \mathcal{J}^{-1} = z_{N,r}^* \alpha_{N,r}^* \beta_{N,r}^* P_{2^{v-1}}^{(v-1)} \phi^{N,r}(-t, \vec{x})$

TABLE 1

$d = 2v = 4k+2, v \text{ odd}$	
ℓ	$\ell^{\phi^{N,r}(x)} \ell^{-1} = \lambda_{2^{v-2}}^{v-1} z_{N,r} \hat{\beta}_{N,r}^{P_{2^{v-1}}^{(v-1)}^{-1}} (\phi^{N,r}(x))^*$
$\wp$	$\wp^{\phi^{N,r}(t, \vec{x})} \wp^{-1} = \phi^{\tilde{N}, \tilde{r}}(t, -\vec{x})$
$\mathcal{J}$	$\mathcal{J}^{\phi^{N,r}(t, \vec{x})} \mathcal{J}^{-1} = z_{N,r}^* \alpha_{N,r}^* \hat{\beta}_{N,r}^* P_{2^{v-1}}^{(v-1)} \phi^{\tilde{N}, \tilde{r}}(-t, \vec{x})$

TABLE 2

$d = 2v+1$	
ℓ	$\ell^{\phi^{M,r}(x)} \ell^{-1} = \lambda_{2^{v-1}}^{v-1} z_{M,r} \beta_{M,r}^{P_{2^v}^{(v)}^{-1}} (\phi^{M,r}(x))^*$
$\wp$	$\wp^{\phi^{M,r}(t, \vec{x}_1, x_{2v})} \wp^{-1} = \gamma_{2^v}^{(v)} \gamma_0^{(v)} \phi^{M,r}(t, -\vec{x}_1, x_{2v})$
$\mathcal{J}$	$\mathcal{J}^{\phi^{M,r}(t, \vec{x})} \mathcal{J}^{-1} = \lambda_{2^v}^v z_{M,r}^* \beta_{M,r}^* \alpha_{M,r}^* P_{2^v}^{(v)} \phi^{M,r}(-t, \vec{x})$

TABLE 3

REFERENCES

- 1) S. Weinberg, Phys. Rev. 134 (1964) 882.
- 2) E.P. Wigner, Ann. Math. 40 (1939) 149.
- 3) S. Weinberg, Phys. Lett. 143B (1984) 97.
- 4) M. Belen Gavela and R.I. Nepomechie, Class. Quantum Grav. 1 (1984) 21.
- 5) C. Wetterich, Nucl. Phys. B234 (1984) 413.
- 6) F. Gliozzi, J. Scherk and D. Olive, Nucl. Phys. B122 (1977) 253;
P. van Nieuwenhuizen, in Supergravity 1981, S. Ferrara and J.G. Taylor eds.,
(Cambridge, CUP, 1982);
C. Wetterich, Nucl. Phys. B211 (1983) 177;
T. Kugo and P. Townsend, Nucl. Phys. B221 (1983) 357;
P. van Nieuwenhuizen, in Groups and Topology II, Les Houches 1983, B. De Witt
and R. Stora eds., (North Holland, 1984).
- 7) M. Gourdin, "Basics of Lie Groups", Editions Frontières, Gif-sur-Yvette,
France (1982).
- 8) S. Chadha and M. Daniel, Rutherford Appleton preprint RAL-85-040 (1985).
- 9) N. Ohta, Phys. Rev. D31 (1985) 442.