CONCAVITY PROPERTIES OF ENERGY LEVELS AS FUNCTIONS OF ANGULAR
MOMENTUM AND LEVEL SPACING*

A. K. Common

Mathematical Institute,
University of Kent at Canterbury,
Canterbury, Kent, CT2 7NZ, England.

A. Martin

C.E.R.N., CH-1211 Geneva 23,
Switzerland.

ABSTRACT

We prove that the ground state energy in a central potential is a concave (convex) function of the angular momentum ℓ if the potential is a concave (convex) function of r^2 . We then deduce that the ground state energy of angular momentum $\ell+1$ is larger (smaller) than the average of the energies of the ground state and first radially excited state of angular momentum ℓ if the potential is concave (convex) in r^2 . Generalisations to higher radial excitations are discussed.

Submitted to Europhysics Letters

*Running head: Properties of Energy levels

If $E(n, \ell)$ is the energy of a state with n radial nodes and angular momentum ℓ in a central potential, we know that $E(0, \ell)$, the ground state energy with angular momentum ℓ , is a concave function of the variable $\ell(\ell+1)$. Indeed if a Hamiltonian has the form $H = A + \gamma B$, the ground state energy is a concave function of $\gamma^{(1)}$. Taking B to be the centrifugal term in the radial Schroedinger equation with coefficient $\ell(\ell+1)$ one obtains the above result. Hence, for an arbitrary potential:

$$E(0, 1) > \frac{2}{3} E(0, 0) + \frac{1}{3} E(0, 2). \quad (1)$$

However, there are physical situations where some extra information is known on the potential. For instance the potential between an infinitely heavy quark and anti-quark obtained from lattice Q.C.D. is monotonous increasing and concave with respect to $r^{(2)}$. Under these circumstances one can hope to improve on (1).

For the harmonic oscillator potential $V=r^2$, $E(n, \ell)$ and in particular $E(0, \ell)$ is a linear function of ℓ . This potential is such that

$$Y(r) = \frac{d}{dr} \left[\frac{1}{r} \frac{dV}{dr} \right] = 0, \quad \forall r > 0, \quad (2)$$

while $Y(r)$ is positive (negative) if V is a convex (concave) function of r^2 . Three years ago it was shown that the sign of $Y(r)$ is crucial to the ordering of energy levels. A theorem ⁽³⁾ was obtained saying that

$$E(n+1, \ell) \underset{<}{>} E(n, \ell+2) \quad (3)$$

if $Y(r) \underset{<}{>} 0, \forall r > 0$.

This theorem applied to the $c\bar{c}$ system allows one to "understand" why the D state called ψ'' is higher than the first S state radial excitation called ψ' since the potential is concave in r and monotonous and hence also concave with respect to r^2 .

We will show here that the concavity or convexity properties with respect to ℓ of the ground state energy $E(0, \ell)$ are controlled by the sign of Y . The proof makes use of two preliminary lemmas:

Lemma 1

If $Y(r) \underset{<}{>} 0, \forall r > 0$, then

$$-\frac{d}{dr} \left[\frac{u_\ell'}{ru_\ell} \right] \underset{<}{>} \frac{2(\ell+1)}{r^3}, \quad (4)$$

where u_ℓ is the reduced ground state wave function with angular momentum ℓ assumed normalised to unity.

Proof

This property could be proved directly, but we will instead use the change of variables and functions ⁽³⁾:

$$z = r^2, \quad L = \frac{(2\ell-1)}{4}, \quad w(z) = r^{\frac{1}{2}} u_\ell(r). \quad (5)$$

The transformed potential is then

$$U(z, E) = \frac{[V(r) - E]}{4z} \quad (6)$$

$$J = \int_0^{\infty} \frac{dr}{r} \int_0^r \left[\frac{2\ell+1}{r'^2} - \frac{\partial E(0, \ell)}{\partial \ell} \right] u_{\ell}^2(r') dr'.$$

Again, using integration by parts on the first term on the R.H.S. of (13),

$$\int_0^{\infty} u_{\ell}' \frac{\partial u_{\ell}}{\partial \ell} / r dr = - \int_0^{\infty} dr \frac{d}{dr} \left[\frac{u_{\ell}'}{ru_{\ell}} \right] \left[\int_0^r u_{\ell}(r') \frac{\partial u_{\ell}(r')}{\partial \ell} dr' \right]. \quad (14)$$

Now the R.H.S. of (12) is positive so that $\left[\frac{\partial u_{\ell}(r)}{\partial \ell} / u_{\ell}(r) \right]$ is an increasing function of r and has one zero at $r=r_0$ since $\int_0^{\infty} u_{\ell} \frac{\partial u_{\ell}}{\partial \ell} dr = 0$ from the normalisation of u_{ℓ} to unity. Therefore

$$\int_0^r u_{\ell}(r') \frac{\partial u_{\ell}(r')}{\partial \ell} dr' < 0,$$

so that we can use (4) in (14) to give

$$\begin{aligned} - \int_0^{\infty} u_{\ell}' \frac{\partial u_{\ell}}{\partial \ell} / r dr &\geq - \int_0^{\infty} \frac{2(\ell+1)}{r^3} \left[\int_0^r u_{\ell} \frac{\partial u_{\ell}}{\partial \ell} dr' \right] dr \\ &= - \int_0^{\infty} (\ell+1) u_{\ell} \frac{\partial u_{\ell}}{\partial \ell} / r^2 dr. \end{aligned} \quad (15)$$

The L.H.S. of (13) re-appears and we have

$$- (2\ell+1) \int_0^{\infty} u_{\ell} \frac{\partial u_{\ell}}{\partial \ell} / r^2 dr \leq J \quad (16)$$

if $Y(r) \geq 0, \forall r \geq 0$.

However J can be rewritten exchanging integrations as

$$\begin{aligned} J &= - \int_0^{\infty} u_{\ell}^2 \left[\frac{2\ell+1}{r^2} - \frac{\partial E(0, \ell)}{\partial \ell} \right] \log r dr \\ &= - (2\ell+1) \left[\langle r^{-2} \log r \rangle < 1 \rangle - \langle r^{-2} \rangle \langle \log r \rangle \right]. \end{aligned}$$

Applying inequality (9) with $\beta=-2$, we get

$$J \leq \langle r^{-2} \rangle \text{ if } Y(r) \geq 0, \forall r \geq 0. \quad (17)$$

Combining (17) with (16) and inserting in (11),

$$\frac{d^2 E(0, \ell)}{d\ell^2} \geq 0 \text{ if } Y(r) \geq 0, \forall r \geq 0. \quad (18)$$

We have therefore proved the main result of this paper that:

The ground state energy $E(0, \ell)$ is a concave (convex) function of ℓ when the potential $V(r)$ is a concave (convex) function of r^2 .

A different proof limited to the concavity case was presented in Ref. 7.

A consequence of (18) is that

$$E(0, \ell+1) \leq \frac{1}{2} [E(0, \ell) + E(0, \ell+2)] \quad (19)$$

Acknowledgements

We would like to express our gratitude to Harald Grosse who has allowed us to present his unpublished result on perturbation around the harmonic oscillator (equation (21)). One of us (A.M.) would also like to thank A. De Rujula for raising the question of the spacing of the $c\bar{c}$ levels many years ago.

REFERENCES

- [1] See for instance W. Thirring, Quantum Mechanics of Atoms and Molecules, Springer-Verlag (New York, Wien) (1981).
- [2] E. Seiler, Phys. Rev. D18 (1978) 482. C. Bachas, Phys. Rev. D33 (1986) 2723.
- [3] B. Baumgartner, H. Grosse and A. Martin, Nuclear Physics B254 (1985) 528.
- [4] B. Baumgartner, H. Grosse and A. Martin, Phys. Letters 146B (1984) 363.
- [5] A. K. Common, J. Phys. A18 (1985) 2219.
- [6] A. K. Common and A. Martin, CERN preprint Th 4488/86 (1986), to be published in J. Phys. A.
- [7] A. Martin, lectures at the XXVI Internationale Universitätswochen für Kernphysik, Schladming 1987, to be published by Springer-Verlag.
- [8] See for instance, Review of Particle Properties, Particle Physics Group, Phys. Letters 170B (1986).
- [9] H. Grosse and A. Martin, Phys. Letters 134B (1984) 368.
- [10] H. Grosse, unpublished. For a sketch of the proof see reference 7 .