CERN-TH.4820/87

THE SELF-DUAL CLOSED BOSONIC MEMBRANES

B. Biran, E.G.F. Floratos^{*)} and G.K. Savvidy^{**)}

CERN - Geneva

A B S T R A C T

The free string cannot change by itself its own topology. For membranes nonlinearities cannot be gauged away. Therefore "free" membranes contain interactions and can change their own topology. This property is analogous to non-Abelian gauge field theories and gives the opportunity to introduce the notion of self-duality for the closed bosonic membrane.

---

^{*)} Permanent address: Physics Dept., Univ. of Crete, Iraklion, Crete

^{**)} Permanent address: Yerevan Physics Institute, Markarian St. 2, Yerevan 375 036, Armenia, U.S.S.R.

Presently there is some intensive research in the physics of interacting strings, considered as fundamental objects for both particle and statistical physics¹⁾.

In this work we consider some physical aspects of the two-dimensional extended object, the closed bosonic membrane. There are some reasons for doing that. First of all why nature would choose one-dimensional objects rather than higher-dimensional ones, if there exist consistent quantum theories for the latter ones. Indeed higher-dimensional objects seem to have richer dynamical properties and less ambiguities in the definition of the interaction than the string. Secondly, strings can be considered as a limiting case of higher-dimensional objects and may improve our understanding of the string phenomena. Thirdly we hope that there are deep analogies between membrane dynamics and collective excitations of non-Abelian gauge fields.

In the case of strings, the main motivation for particle physics is the unification of gravity with the other forces in a predictive theory which is finite²⁾. The replacement of the known gauge principles by the geometric principle of minimal area propagation in conformally invariant backgrounds gives rise to an infinity of string excitations which can be regarded as independent particles, dynamically correlated through the duality symmetry²⁾. This symmetry makes the (super) string theory finite. What is lacking from the string theory is a principle which singles out a unique vacuum. Maybe this is related to the arbitrariness for the choice of string interactions in a string field theory³⁾.

The free string, as a one-dimensional object ($n=1$), cannot change by itself its own topology because the non-linearities of free string can be gauged away and the interaction term must be put in by hand. This is a reason for the study of higher-dimensional extended objects like membranes ($n=2,3,\dots$) whose non-linearities cannot be gauged away, contrary to the string case.

In fact, this difference has a deep origin. Indeed any two closed compact extended objects considered as n -dimensional manifolds - for $n = 1,2,3$ but not for $n = 0,4$ - can be joined in a smooth way by interpolating between them an $n+1$ dimensional compact manifold having those two as boundaries. Then these two manifolds are called cobordant and this is an equivalence relation which is a topological invariant. So the n -dimensional closed and compact manifolds are separated into equivalence classes, the set of which carries the structure of a ring with, in addition, the set union and multiplication of the Cartesian product. This ring is called a Thom ring Ω_n . There is also a $\mathbb{Z}_2$ structure of this ring if one takes into account orientation. Now the statement is that $\Omega_1 = \Omega_2 = \Omega_3 = 0$ ⁴⁾. So the

lowest dimensional object for which there exists the possibility that the equation of motion, being non-linear, connects at two different times different topologies is the two-dimensional membrane ($\Omega_2 = 0$).

The action of the membrane is proportional to the swept out volume⁵⁾

$$L = T g^{1/2}$$

$$g = \det \{ \partial_a x^i \partial_b x^i \} \quad (1)$$

where $x^i(\sigma_1, \sigma_2, \tau)$, $i = 1, \dots, D$ is the embedding of the closed membrane in the D -dimensional Euclidean space-time, σ_1, σ_2 parametrize the surface and τ is the "time" variable. In (1) $\partial_a x^i$ means the derivative with respect to parameters ($\xi^a = \tau, \sigma_1, \sigma_2$). If we define the momenta

$$P_i^a = \frac{\delta L}{\delta x_a^i} = T g^{1/2} g^{ab} \partial_b x_i \quad (2)$$

then the equation of motion reads

$$\partial_a P_i^a = 0 \quad i = 1, \dots, D ; a = 1, 2, 3 \quad (3)$$

and the constraints implied by (2) are:

$$P_\tau^i x_{\sigma_2}^i = P_\tau^i x_{\sigma_1}^i = 0 \quad (4a)$$

$$P_\tau^2 = T^2 \{ x_{\sigma_1}^2 x_{\sigma_2}^2 - (x_{\sigma_1} \cdot x_{\sigma_2})^2 \} \quad (4b)$$

The co-ordinates x^i are periodic functions of the variables σ_1, σ_2 for a convenient compact range: $(\sigma_1, \sigma_2) \in D$. Here $\tau \in \mathcal{R} = (-\infty, +\infty)$.

The symmetry of the system is the full reparametrization group $\text{Diff}(D \times \mathcal{R})$ and the constraints (4a, 4b) are consequences of that. There is also an $O(D)$ rotational symmetry.

At the classical level some interesting solutions have been found and the quadratic fluctuations of the membrane around them have been studied. The conclusion is that the bosonic⁶⁾ and the supermembrane⁷⁾ do not provide massless particles in any integer dimension and this restricts the elementary particle interpretation of membranes. For recent work on supermembrane see Refs. 8) and 9).

To canonically quantize the theory one has to fix the gauge so that the independent degrees of freedom can be extracted from the constraints¹⁰⁾⁻¹³⁾ [for covariant quantization see Refs. 14) and 15)]. Of course the quantization procedure should be checked for consistency.

The Hamiltonian for these independent (transverse) degrees of freedom contains non-linear interactions that cannot be gauged away contrary to strings¹³⁾. Moreover, because there is no harmonic terms, these interactions are not perturbative, like non-Abelian systems¹⁶⁾.

Now we come to the discussion of the notion of topological charge. For any dimensionality $D = n(n-1)/2$, $n = 3, 4, 5, \dots$ denote by f_{ijk} the structure constants of the $O(n)$ group. Then for any configuration $x^i(\sigma_1, \sigma_2, \tau)$, $k, j, i = 1, \dots, n(n-1)/2$ we can construct the quantity

$$\Omega(x) = \frac{1}{3!} \varepsilon^{abc} f_{ijk} x_a^i x_b^j x_c^k \quad (5)$$

which is a total divergence of a current j^a

$$\Omega(x) = \partial_a j^a$$

where

$$j^a = \frac{1}{3!} \varepsilon^{abc} f_{ijk} x^i x_b^j x_c^k \quad (6)$$

Then the integral of the quantity $\Omega(x)$ over the embedded (in $\mathbb{R}^D$) three-volume swept out by the membrane does not depend on the configuration of the membrane in the between "times", say $\tau_1 = -\infty$, $\tau_2 = +\infty$, but it depends only on the boundary configurations

$$\varphi_1(\sigma_1, \sigma_2) = X(\sigma_1, \sigma_2, \tau_1), \quad \varphi_2(\sigma_1, \sigma_2) = X(\sigma_1, \sigma_2, \tau_2) \quad (7)$$

This is true because of the periodicity in σ_1, σ_2 :

$$\int d\sigma_1 d\sigma_2 d\tau \Omega(x) = Q(+\infty) - Q(-\infty) \quad (8a)$$

where

$$Q(\tau) = \int d\sigma_1 d\sigma_2 j^\tau(\sigma_1, \sigma_2, \tau) \quad (8b)$$

One can define a topological number for any configuration $x = x(\sigma_1, \sigma_2, \tau)$

$$N[x] = \frac{1}{V[x]} \int d\sigma_1 d\sigma_2 d\tau \Omega(x) \quad (9)$$

where $V[x]$ is the swept out volume:

$$V[x] = \int d\sigma_1 d\sigma_2 d\tau \{ \det X_a^i X_b^i \}^{1/2}$$

Let us define the vacuum configurations to be the zero action solutions of the equation of motion. It is easy to see that the time-independent embeddings of all closed surfaces of any topology and orientability belong to this class (there are also strings and points like vacuum configurations). Here we have an analogy with Yang-Mills theories where the vacuum configurations satisfy $\hat{F}_{\mu\nu} = 0$ and the gauge potentials $A_{\text{vac}} = g^{-1} \partial g$ can be classified according to their winding number. The membrane topological charge (9) classifies the membrane vacuum configurations and one is led to the question of the existence of instanton-like objects. These objects will play the role of connecting two vacuum configurations with different topology and we call them self-dual membranes. To study the self-dual membranes we shall introduce a notion of self-duality equation which looks like the self-duality equation of Yang-Mills theories.

For any completely antisymmetric constant tensor f_{ijk} we define self-dual membranes as the embeddings $x^i(\sigma_1, \sigma_2, \tau)$ which satisfy the property

$$P_i^a = \frac{1}{2} \epsilon^{abc} f_{ijk} x_b^j x_c^k \quad (10)$$

We see immediately that they satisfy the equation of motion (3) and the constraints (4a) but not the constraints (4b). For this it would be enough to have the relation:

$$f_{ijk} f_{ilm} \cong \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl} \quad (11)$$

which holds only for the $O(3)$ group.

Below we shall restrict our consideration to the simplest dimensionality of the form $D = n(n-1)/2$ that is $D = n = 3$. In this case, because $f_{ijk} = \epsilon_{ijk}$ - the completely antisymmetric three-index tensor in three dimensions - we can see that the topological number $N[x]$ is the winding number of the mapping $x^i(\sigma_1, \sigma_2, \tau)$ $i = 1, 2, 3$ ¹⁷⁾.

In the case of the $O(3)$ group we have found two solutions which satisfy the self-duality equations (10) and hence the equation of motion as well as the constraints. In order to discuss these solutions we go to the gauge where the induced metric $g_{ab} = x_a^i x_b^i$ is diagonal.

In this gauge the Lagrangian is

$$L = T \{x_{\sigma_1}^2 x_{\sigma_2}^2 x_z^2\}^{1/2} \quad (12)$$

and the momenta P_i^a ,

$$P_i^a = L \cdot \frac{x_i^a}{x^{a2}} \quad (13)$$

Then the self-duality equation (10) becomes

$$\sqrt{x_{\sigma_1}^2 x_{\sigma_2}^2} x_z^i = \sqrt{x_z^2} \varepsilon^{ijk} x_{\sigma_1}^j x_{\sigma_2}^k \quad (14)$$

Let us define the unit orthogonal vectors

$$e_i^a = \frac{x_i^a}{\sqrt{x^{a2}}} \quad (15)$$

and transform (14) in a symmetric form

$$e_i^a = \frac{1}{2} \varepsilon^{abc} \varepsilon_{ijk} e_j^b e_k^c \quad (16)$$

To see the group theoretical structure of the self-duality equation (16) we note that if we define the matrices e^a ,

$$(e^a)_{jk} \equiv -e_i^a \varepsilon_{ijk} \quad (17)$$

equation (16) becomes

$$[e^a, e^b] = \varepsilon^{abc} e^c \quad (18)$$

so (18) or (16) define an embedding of $D \times \mathbb{R}$ into the group manifold of the $O(3)$ group that is the S^3 sphere.

As in the case of Yang-Mills theory, the self-duality equations (16) to (18) are first order differential equations instead of (3). The non-linear equations (14) to (18) give the possibility to construct non-perturbative solutions.

Coming back to our gauge choice we note that there remains invariance under reparametrization: $\tau \rightarrow \tilde{\tau}(\tau)$, $\sigma_1 \rightarrow \tilde{\sigma}_1(\sigma_1)$ and $\sigma_2 \rightarrow \tilde{\sigma}_2(\sigma_2)$. This can be seen from the invariance of the diagonal form of the metric g^{ab} . Indeed if $\xi^a \rightarrow \tilde{\xi}^a = \xi^a + \varepsilon^a(\xi)$, where $\varepsilon^a(\xi)$ are infinitesimal, then $\varepsilon^a(\xi)$ must satisfy

$$\partial_1 \varepsilon^2 g^{11} + \partial_2 \varepsilon^1 g^{22} = 0 \quad (19a)$$

$$\partial_1 \varepsilon^3 g^{11} + \partial_3 \varepsilon^1 g^{33} = 0 \quad (19b)$$

$$\partial_2 \varepsilon^3 g^{22} + \partial_3 \varepsilon^2 g^{33} = 0 \quad (19c)$$

The only g^{ab} -independent solution of (19) is:

$$\varepsilon^1 = \varepsilon^1(\xi^1), \quad \varepsilon^2 = \varepsilon^2(\xi^2), \quad \varepsilon^3 = \varepsilon^3(\xi^3) \quad (20)$$

Let us present now the solutions which we have found: a sphere and a torus evolving in time τ . First, for the sphere, we use the co-ordinate $-1 \leq \sigma_2 \leq 1$ that plays the role of the z-co-ordinate and similarly $0 \leq \sigma_1 \leq 2\pi$ for the ϕ -polar co-ordinate. We shall use a reparametrization of σ_1 inducing winding $\phi(\sigma_1) = n\sigma_1$ (n integer) and a reparametrization for σ_2 : $\phi: [-1,1] \rightarrow [-1,1]$. The solution of Eq. (16) is

$$X(\sigma_1, \sigma_2, \tau) = R(\tau) \left(\sqrt{1 - \phi'^2(\sigma_2)} \cos \phi(\sigma_1), \sqrt{1 - \phi'^2(\sigma_2)} \sin \phi(\sigma_1), \phi(\sigma_2) \right) \quad (21)$$

for arbitrary ϕ , R and ϕ . The action is

$$\begin{aligned} S &= \frac{1}{3} \cdot \int_{-\infty}^{+\infty} d\tau \frac{d}{d\tau} R^3(\tau) \int_0^{2\pi} d\sigma_1 \phi'(\sigma_1) \int_{-1}^1 d\sigma_2 \phi'(\sigma_2) \\ &= \frac{1}{4} \cdot (\text{swept out volume}) \times (\text{winding number of the sphere}). \end{aligned} \quad (22)$$

We must choose for finite action

$$R(+\infty) - R(-\infty) < \infty.$$

We pass now to the torus solution:

$$X(\sigma_1, \sigma_2, \tau) = \left\{ (R + r(\tau)(1 - \phi^2(\sigma_2))^{1/2}) \cos \psi(\sigma_1), (R + r(\tau)(1 - \phi^2(\sigma_2))^{1/2}) \sin \psi(\sigma_1), r(\tau) \phi(\sigma_2) \right\} \quad (23)$$

This solution represents a torus lying on the x-y plane at a constant distance R from the origin and with an interior radius r(t) evolving in time. The functions $\phi(\sigma_2)$, $\psi(\sigma_1)$ are the same as in the previous case and r(t) is a function which must be chosen so that the action is finite.

$$S = T \left\{ \int_{-\pi}^{+\pi} d\tau \frac{R}{2} \frac{d}{d\tau} r^2 \int_{-1}^{+1} d\sigma_2 \frac{\phi'(\sigma_2)}{\sqrt{1 - \phi^2}} \int_0^{2\pi} d\sigma_1 \psi'(\sigma_1) + \int_{-\pi}^{+\pi} d\tau \frac{1}{3} \frac{d}{d\tau} r^3 \int_{-1}^{+1} d\sigma_2 \phi'(\sigma_2) \int_0^{2\pi} d\sigma_1 \psi'(\sigma_1) \right\} \quad (24)$$

A more general analysis of the self-dual equation is under investigation.

In conclusion we want to stress that the dynamics of Yang-Mills gauge fields and closed bosonic membranes seems to be very similar. The free-closed string does not cut itself unless we impose interactions. So it is similar to the electric and magnetic force lines of Faraday. On the other hand, the Yang-Mills theory in 1/N expansion looks like the closed membrane. Indeed closed membranes interact with quartic potentials, they do not have perturbative limits and their Hamiltonian is similar to the Yang-Mills one. The study of the topology and dynamics of the closed membranes may give new ideas for the solution of Yang-Mills gauge theories. The notion of self-dual membrane is perhaps helpful for the realization of this programme. The classification of all possible self-dual membranes will be the starting point in the understanding of the non-perturbative nature of membrane-gauge vacua.

ACKNOWLEDGEMENTS

The authors would like to thank L. Alvarez-Gaumé, I. Antoniadis, S. Coleman, M.J. Duff, C. Gomez, T. Inami, H. Leutwyler, S.G. Martinyan, P. Minkowski, H.B. Nielsen, H.P. Nilles, D. Olive, C.N. Pope, R. Stora, I. Volovich and B. Zumino for discussions. Two of the authors (F.G.F.) and (G.K.S.) thank the CERN Theoretical Division for its hospitality.

REFERENCES

- 1) A.M. Polyakov, "Directions in string theory", Lecture presented at the 1986 Mathematics Congress, Berkeley (1986).
- 2) J.H. Schwartz, "Superstrings, the first fifteen years of superstring theory", World Scientific 1985, Singapore.
- 3) E. Witten, Nucl. Phys. B268 (1986) 253, B276 (1986) 291;
M.J. Bowick and S.G. Rajeev, MIT preprint (1987);
H. Hata, K. Itoh, T. Kugo, H. Kunitomo and K. Ogawa, Phys. Lett. 175B (1986) 138.
- 4) Encyclopedic Dictionary of Mathematics, S. Iyanaga and Y. Kawada eds., MIT Press (1974) 1294;
See also F.A. Bais, C. Gomez and V. Rubakov, Nucl. Phys. B282 (1987) 531.
- 5) P.A.M. Dirac, Proc. Roy. Soc. 268A (1962) 57.
- 6) K. Kikkawa and M. Yamasaki, Progr. Theor. Phys. 76 (1986) 1379.
- 7) L. Mezincescu, R.I. Nepomechie and P. van Nieuwenhuizen, UMTG-139, ITP-SB-87-43 preprints (1987).
- 8) J. Hughes, J. Liu and J. Polchinski, Phys. Lett. 180B (1986) 370.
- 9) M.J. Duff et al., CERN preprint TH-4731/87 (1987).
- 10) P.A. Collins and R.W. Tucker, Nucl. Phys. B112 (1976) 150.
- 11) J. Hoppe, Aachen preprint PITHA 86/24 (1986).
- 12) M.E. Lasiev and G.K. Savvidy, preprint EFI-957 (7) (1987).
- 13) G.K. Savvidy, preprint EFI-972 (22) (1987) and CERN preprint TH.4779/87 (1987).
- 14) M. Hennaux, Phys. Lett. 120B (1983) 179.
- 15) K. Fujikawa and J. Kubo, Stony Brook preprint ITP-SB-87-36 (1987).
- 16) G.Z. Baseyan, S.G. Matinyan and G.K. Savvidy, Pis'ma v Zh. Eksp. Teor. Fiz. 29 (1979) 641;
G.K. Savvidy, Nucl. Phys. B246 (1984) 302.
- 17) B. Doubrovine, S. Novikov and A. Fomenko, "Géométrie Contemporaine" vol.I, II, ed. Moscow (1982).