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**MULTILOOP MODULAR INVARIANCE IN SPONTANEOUSLY
BROKEN SUPERSTRINGS^{*)}**

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ABSTRACT

The proof of modular invariance for a recently constructed class of four-dimensional superstrings exhibiting spontaneous breakdown of space-time supersymmetry is extended to any loop order.

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In a recent paper [1] we proposed a new class of solutions for superstrings models in which supersymmetry in the target space is spontaneously broken with arbitrary-order parameters (gravitino masses). The actual construction is the analogue, in string theory, of the Scherk-Schwarz mechanism [2] of extended supergravities. The same model can be reinterpreted [3] as arising from the coupling of a world-sheet non-propagating U(1) gauge field to (complex) world-sheet fermions; there, the order-parameters are the U(1) charges of the fermions.

In this present note we extend the above construction to Riemann surfaces of arbitrary genus, which proves the consistency of these models at any string loop.

Let us remember that the new models under consideration are obtained [1] from the five-dimensional superstrings in the fermionic construction by two modifications of the one-loop partition function. The first one is the compactification of the fifth bosonic coordinate; this amounts to replacing the partition function of an uncompactified boson:

$$Z_{Bos} = |\eta(\tau)|^{-2} \tau_I^{-1/2} \quad (1)$$

($\tau = \tau_R + i\tau_I$ is the modular parameter of the torus) with the function Z_{mn} , which is the contribution of the compactified boson with winding numbers n and m :

$$Z_{mn}(\tau) = Z_{Bos}(\tau) e^{-\frac{\pi R^2}{\tau_I} (m^2 + |\tau|^2 n^2 - 2\tau_R m n)}. \quad (2)$$

The second modification amounts to changing the fermion partition function, by introducing a dependence from the winding numbers m, n . Specifically we write:

$$Z^F \left(\begin{smallmatrix} a \\ b \end{smallmatrix} \right) (v, u, \tau) = \eta^{-N_L} \bar{\eta}^{-N_R} \prod_{p=1}^{N_L} e^{-i\pi u_L \theta \left[\begin{smallmatrix} a_p \\ b_p \end{smallmatrix} \right] (v_L^p, \tau)} \prod_{q=1}^{N_R} e^{i\pi u_R \bar{\theta} \left[\begin{smallmatrix} a_q \\ b_q \end{smallmatrix} \right] (v_R^q, \bar{\tau})}, \quad (3)$$

where N_L and N_R are the numbers of left-moving and right-moving complex fermions, respectively. The quantities v_L^p , v_R^q , u_L , and u_R are defined as follows:

$$\begin{aligned} v_L^p &= e_L^p (m - \tau n), \quad v_R^q = e_R^q (m - \bar{\tau} n), \\ u_L &= \sum_{p=1}^{N_L} e_L^p e_L^p n (m - \tau n), \quad u_R = \sum_{q=1}^{N_R} e_R^q e_R^q n (m - \bar{\tau} n), \end{aligned} \quad (4)$$

where e_L^p and e_R^q are arbitrary parameters.

With the definitions (2) and (3), the partition function of the model is

$$Z(e_L, e_R | \tau) = \sum_{m, n \in \mathbb{Z}} Z_{mn}(\tau) \sum_{\{a, b\}} C \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] Z^F \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (v, u | \tau) |\eta|^{-4} \tau_I^{-1}. \quad (5)$$

The sum over the spin structures $\{a, b\}$ from the various fermions of the model is chosen in order to have modular invariance for $e_L^p = e_R^q = 0 \forall p, q$ [4, 5].

It is easily verified that Eq. (5) is invariant under the modular transformation

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

As shown in ref. [1], the mass spectrum of the model specified by $Z(e_L, e_R|\tau)$ is that of a spontaneously broken supersymmetric string. In particular, the four-dimensional gravitino masses are proportional to the charges e_L^p or e_R^q , as are the masses of some of the gauge bosons. Once the one-loop partition function (5) is given, and is recognized to be modular-invariant, it is not difficult to find its N-loop generalization.

On an arbitrary Riemann surface, we can define an homology basis given in terms of the cycles a_i and b_i , where $i = 1, \dots, g$ (g = genus). As usual, the one-forms dual to a_i and b_i will be denoted by α_i and β_i , which verify the normalization conditions:

$$\int_{a_j} \alpha_i = \delta_{ij}, \quad \int_{b_j} \alpha_i = 0, \quad \int_{a_j} \beta_i = 0, \quad \int_{b_j} \beta_i = \delta_{ij}.$$

A global diffeomorphism acts on the homology basis $\begin{pmatrix} b \\ a \end{pmatrix}$ in the following way:

$$\begin{pmatrix} b \\ a \end{pmatrix} \rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_g \end{pmatrix}, \quad \underline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_g \end{pmatrix}. \quad (6)$$

The matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathcal{G}$ is an element of $Sp(2g, \mathbb{Z})$. The dual one-forms are transformed according to:

$$(\beta, \alpha) \rightarrow (\beta, \alpha) \mathcal{G}^{-1} = (\beta, \alpha) \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}. \quad (7)$$

A compactified boson gives rise to many winding sectors, now labelled by a set of integers $\underline{m} \in \mathbb{Z}^g$ and $\underline{n} \in \mathbb{Z}^g$, the classical solution in the sectors $(\underline{m}, \underline{n})$ is:

$$dX_{\underline{m}, \underline{n}} = \sum_{i=1}^g m_i \beta_i + n_i \alpha_i = \underline{m}^t \beta + \underline{n}^t \alpha.$$

The contribution of this sector to the N-loop partition function is:

$$Z_{\underline{m}, \underline{n}} = \left(\frac{\det' - \nabla^2}{\int \sqrt{g}} \right)^{-1/2} e^{-S_{\underline{m}, \underline{n}}}, \quad (8)$$

where $\det' - \nabla^2$ is the partition function of the uncompactified boson, g denotes the two-dimensional metric, and

$$S_{\underline{m}, \underline{n}} = 2\pi \int dX_{\underline{m}, \underline{n}} \wedge^* dX_{\underline{m}, \underline{n}}$$

is the classical action for the soliton $X_{\underline{m}, \underline{n}}$ [6]. The modular parameter τ , has a multiloop generalization also. Its equivalent is a symmetric $g \times g$ complex matrix Ω , transforming under the modular group $Sp(2g, \mathbb{Z})$ as:

$$\Omega \rightarrow (A\Omega + B)(C\Omega + D)^{-1}. \quad (9)$$

On a genus g Riemann surface we have to specify the boundary conditions (periodic or antiperiodic) for any fermion, along all the homology cycles a_i and b_i . We denote this choice with the matrices $\begin{pmatrix} a_L \\ b_L \end{pmatrix}$ and $\begin{pmatrix} a_R \\ b_R \end{pmatrix}$, with N_L and N_R columns respectively, and $2g$ rows, whose entries have value 0 (anti-periodic b.c.) or $1/2$ (periodic b.c.). It is known that the partition function of a complex Weyl fermion on a genus g Riemann surface is, for boundary conditions $\begin{pmatrix} a \\ b \end{pmatrix}$, equal to

$$\int [d\psi]_{\begin{pmatrix} a \\ b \end{pmatrix}} \exp - \int d^2x \bar{\psi} \not{\partial} \psi = c \vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (0 | \Omega), \quad (10)$$

where c is independent of the spin structure and

$$|c|^2 = \left(\frac{\det' - \nabla^2}{\int \sqrt{g} \det \operatorname{Im} \Omega} \right)^{-1/2}.$$

Here $\vartheta \left(\begin{smallmatrix} a \\ b \end{smallmatrix} \right) (\underline{v}, \Omega)$ is the multiargument ϑ -functions defined as in Refs. [6] to [8],

$$\vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (\underline{v}, \Omega) = \sum_{n \in \mathbb{Z}^g} e^{\{ i\pi(\underline{n} + \underline{a})\Omega(\underline{n} + \underline{a}) + 2\pi i(\underline{n} + \underline{a})^t(\underline{v} + \underline{b}) \}}.$$

The last quantities to be defined are the $\underline{v}_{L,R}$ and $\underline{u}_{L,R}$. They are

$$\underline{v}_L^p = e_L^p (\underline{m} - \Omega \underline{n}), \quad u_L = \sum_{p=1}^{N_L} e_L^p e_L^p (\underline{v}_L^p)^t \underline{n}, \quad (11a)$$

$$\underline{v}_R^q = e_R^q (\underline{m} - \bar{\Omega} \underline{n}), \quad u_R = \sum_{q=1}^{N_R} e_R^q e_R^q (\underline{v}_R^q)^t \underline{n}. \quad (11b)$$

Under $\operatorname{Sp}(2g, \mathbb{Z})$ the above quantities transform as

$$\underline{v}_L^p \rightarrow (\Omega C^t + D^t) \underline{v}_L^p, \quad u_L^p \rightarrow \underline{v}_L^{p^t} (C\Omega + D)^{-1} (C \underline{v}_L^p + u_L^p), \quad (12a)$$

$$\underline{v}_R^q \rightarrow (\bar{\Omega} C^t + D^t) \underline{v}_R^q, \quad u_R^q \rightarrow \underline{v}_R^{q^t} (C\bar{\Omega} + D)^{-1} (C \underline{v}_R^q + u_R^q), \quad (12b)$$

when the $\underline{m}$ and $\underline{n}$ are transformed according to

$$\begin{pmatrix} \underline{m} \\ \underline{n} \end{pmatrix} \rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \underline{m} \\ \underline{n} \end{pmatrix}.$$

We are now ready to extend the one-loop partition function given in Eq. (5) to any loop. The partition function is given by

$$Z_g(e_L, e_R | \Omega) = \sum_{\underline{m}, \underline{n} \in \mathbb{Z}^g} K \sum_{\underline{m}, \underline{n}} \sum_{\{a, b\}} \zeta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] Z_g^F \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (\underline{v}, \underline{u} | \Omega). \quad (13)$$

With K we denote all the contributions to the partition function which are independent of m, n and of the charges e_L, e_R . This includes, in particular, the uncompactified bosonic coordinates and the longitudinal component of X^μ and Ψ^μ , as well as the two-dimensional gravitino determinant. The $Z_g^F \left[\begin{smallmatrix} \underline{a} \\ \underline{b} \end{smallmatrix} \right] (\underline{v}, u | \Omega)$ is

$$Z_g^F \left[\begin{smallmatrix} \underline{a} \\ \underline{b} \end{smallmatrix} \right] (\underline{v}, u | \Omega) = \frac{N_L - N_R}{C} e^{-i\pi u_L} i\pi u_R \prod_{p=1}^{N_L} \sigma \left[\begin{smallmatrix} \underline{a}^p \\ \underline{b}^p \end{smallmatrix} \right] (\underline{v}_L^p | \Omega) \prod_{q=1}^{N_R} \bar{\sigma} \left[\begin{smallmatrix} \underline{a}^q \\ \underline{b}^q \end{smallmatrix} \right] (\underline{v}_R^q | \Omega). \quad (14)$$

The multiargument ϑ -functions transform in the following way [6, 8]:

$$\begin{aligned} \vartheta \left[\begin{smallmatrix} \tilde{\underline{a}} \\ \tilde{\underline{b}} \end{smallmatrix} \right] ((\Omega C^t + D^t)^{-1} \underline{v} | (A\Omega + B)(C\Omega + D)^{-1}) &= \\ = u \exp \left\{ \pi i \underline{v}^t (C\Omega + D)^{-1} C \underline{v} \right\} \det(C\Omega + D)^{1/2} \vartheta \left[\begin{smallmatrix} \underline{a} \\ \underline{b} \end{smallmatrix} \right] (\underline{v} | \Omega) & \quad (15) \\ \begin{pmatrix} \tilde{\underline{a}} \\ \tilde{\underline{b}} \end{pmatrix} &= \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} \underline{a} \\ \underline{b} \end{pmatrix} + \frac{1}{2} \begin{bmatrix} (C D^t)_d \\ (A B^t)_d \end{bmatrix}. \end{aligned}$$

For every matrix M , M_d is the column vector obtained by taking the diagonal elements of the matrix M .

Under the modular transformation

$$\Omega \rightarrow (A\Omega + B)(C\Omega + D)^{-1}, \quad \begin{pmatrix} m \\ n \end{pmatrix} \rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}$$

the compactified boson contribution $Z_{m,n}$ is left invariant, $Z_{m,n} \rightarrow Z_{m,n}$, and, by using Eqs. (15) and (12a,b), one verifies that the $Z_g^F(\underline{v}, u | \Omega)$ given in Eq. (14) transforms in the same way for any value of e_L^p, e_R^q . Therefore Eq. (13) is modular-invariant for any value of e_L^p, e_R^q whenever it is invariant in the supersymmetric limit ($e_L^p = e_R^q = 0 \forall p, q$).

Notice that all the properties of the N -loop partition functions are valid for every e_L^p, e_R^q : Factorization, for example, holds automatically since it relies only on the property

$$\Omega \rightarrow \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix}, \quad \underline{v} \rightarrow \begin{pmatrix} \underline{v}_1 \\ \underline{v}_2 \end{pmatrix}$$

on a non-separating knot of the Riemann surface [6-8].

In this letter we have shown how to construct string models depending on several free parameters which are consistent at every string loop.

Our technique allows us to obtain a class of spontaneously broken models in four-dimensions for every supersymmetric five-dimensional string constructed in the fermionic formulation.

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