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STRING MODELS WITH $c < 1$ COMPONENTS

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ABSTRACT

We construct part of the current algebra of d dimensional strings from statistical mechanical models with $c < 1$. General properties of such models are discussed and a few examples are given. In the process, a fermionic field theoretic realization of the $c < 1$ modular invariant models emerges. Bosonization aspects of the systems are discussed.

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1. - INTRODUCTION

The mechanical picture of a string propagating in a space-time has led to the uncovering of the algebraic structure underlying the motion of extended objects, the Virasoro algebra. This algebraic structure to which some extra requirements are appended can be used to construct string ground states without relying on the specific realization of the Virasoro algebra¹⁾.

The abstraction is done by requiring that the conformal string theory describing the ground state of a string theory has the following two-dimensional properties.

- a) It should be a unitary realization of the algebra.
- b) To maintain Weyl and general co-ordinate invariance, the central charge associated with the realization of the left- or right-handed Virasoro algebras should be $c = 10$ or 26 .
- c) To maintain the world sheet structure, the system should be invariant under two-dimensional general co-ordinate transformations and thus have no global gravitational anomaly, that is, it should be modular invariant²⁾.
- d) The Hilbert space should allow factorization and target space unitarity (the only seemingly non two-dimensional requirement). In the presence of an explicit construction, factorization and all string multiloop modular invariance seem to suffice³⁾. Given an explicit construction, the closure of the operator product expansion (OPE) is required for consistency⁴⁾.
- e) If low energy string theory is related to the near future and present day phenomenology extra requirements may be imposed. These seem to relate to specific properties of some ground states but not a full string theory, in any case we do not discuss them here.

The general list of the two-dimensional field theories obeying the constraints is not known, it is also not known if all such ground states are indeed ground states of only one string field theory. Many results have been obtained by using the ten-dimensional heterotic models as starting points which are conformal string theories and modifying them without changing the constraints⁵⁾.

The compactification on various lattices or on torii, orbifolds and Calabi-Yau spaces are a physicist's way of generating interesting realizations of the Virasoro algebra.

In other cases strings were made to move on group manifolds⁶⁾. For the case when the central charge c is less than one⁷⁾ (or $3/2$ in the superconformal case), unitarity restricts the allowed values of c to a discrete sequence⁸⁾. For a given value of $c < 1$, there are only a finite number of allowed irreducible representations. That is, there is only a finite number of allowed values of $h(\bar{h})$, the eigenvalue of the Virasoro operator $L_0, (\bar{L}_0)$. All such cases have been related to statistical mechanical spin models and it was also shown that all allowed values of c and h can participate in a field theoretic realization on the two-dimensional plane.

In this paper, we study conformal string models which are constructed by performing tensor products of models in the discrete sequence⁹⁾. The total central charge of the tensor product should add up to an integer smaller or equal to the critical one, the missing integer (such as $c = 4$ for the conventional four-dimensional strings) is realized by the mechanical string. The tensor product is a realization in terms of spinlike systems. Our interest in these objects which are part string and part spin goes beyond the zoological classification aspect.

We have yet no clear physical picture of the sense in which some or all of the extra dimensions can be described as space. Even if they indeed have such a meaning, part or all of them may be so small that a space description is lacking. Spin systems may become more welcome at that scale. This class of models has an a priori starting point rather distinct from the usual description of space. We shall show that such models are finite in number and thus correspond to a finite set of points in the space of conformal field theories with a definite integer value of c . While in some cases spacelike properties clearly reappear upon tensoring, one wishes to test in general if the tensor product model is isolated, that if there exists no conformal model in their small neighbourhood.

In Section 2, we describe some general properties of the tensored models and illustrate them in the case where the basic building block is the $c = \frac{1}{2}$ (Ising) model⁹⁾. It has been an old practice^{10),11)} in string theories to study current algebra properties in terms of free fermions. In order to obtain a field theoretic and modular invariant realizations of the models with $c < 1$ starting from a free fermion model, we need to describe various previous constructions.

In Section 3, we review the Goddard, Kent and Olive (GKO)¹²⁾ construction of models with $c < 1$ on which we heavily rely. In Section 4, we review the GKO and Kent¹³⁾ construction in terms of free fermions emphasizing aspects which serve us to construct modular invariant models. In Section 5, we illustrate the increasing machinery to study yet again the $c = \frac{1}{2}$ case. In Section 6, we review and use the methods developed in string theory¹⁴⁾ to study modular invariant models constructed out of free fermions. The field theoretic fermionic description is very useful in obtaining $c < 1$ modular invariant solutions. In Section 7, we review the construction of modular invariant $c < 1$ representations obtained by Gepner¹⁵⁾ and Cappeli, Itzykson and Zuber¹⁶⁾ in which properties of finite groups and number theory were used. The problem was originally posed by Cardy¹⁷⁾. In Section 8, we demonstrate, relying on results of Refs. 15)-17), that the field theoretic construction starting from free fermions leads to the known modular invariant sequences and suggests new solutions with a field theoretic meaning extracted from theories with space-time fermions. In Section 9, we discuss bosonic aspects of the construction.

2. - THE TENSORING OF THE VIRASORO ALGEBRA GENERALITIES

The basic building block is a unitary conformal model belonging to a discrete sequence. For concreteness, we discuss those models whose central charge is smaller than one. These models do not contain a $(1,0)$ field and therefore possess no continuous Kac-Moody symmetry. A large number of properties of the model are already determined by its central charge.

a) The central charge

The allowed values of c in the unitary conformal sequence are⁸⁾

$$c(m) = 1 - \frac{6}{(m+1)(m+2)} \quad m = 2, 3, \dots \quad (2.1)$$

The allowed values of h corresponding to a given $c(m)$ are given by⁸⁾:

$$h(h, q) = \frac{[(m+2)h - (m+1)q]^2 - 1}{4(m+1)(m+2)} \quad 1 \leq h \leq m, \quad 1 \leq q \leq h \quad (2.2)$$

A general tensored model is characterized by a set of integers $(n_2, n_3, \dots, n_m)$, only a finite number of which is non-zero. n_m indicates the number of times, a

$c(m)$ representation appears in the tensor product. The total central charge associated with the set $\{n_m\}$ is the integer $c(\{n_m\})$ given by:

$$c(\{n_m\}) = \sum_{m=2}^{\infty} n_m c(m) \quad (2.3)$$

We are interested in all conformal models whose central charge is an integer equal or less than 26 (or superconformal ones with $c < 10$). The central charge may always be completed to its critical dimension by a mechanical string or by another tensored structure(s) with the appropriate value of the central charge. (In this paper we do not tensor the ghost sector.)

For a given integer value of c , there is only a finite number of models $\{n_m\}$ reproducing that value of c . We prove this in Appendix I, the result is applicable to tensoring any discrete rational sequence with an accumulation point ($c = 1$ in the conformal case). There are other known discrete sequences; one associated with the superconformal models with $c(m) = 3/2[1-8/m(m+2)]$ ⁸⁾, another with parafermions and self-dual z_n models with $c(m) = 2(m-1)/m+2$ ¹⁸⁾, yet another relates models with $N = 2$ superconformal symmetry in which case $c(m) = 3m/m+2$ ¹⁹⁾. In general, any level m representation of a compact Lie group forms a sequence with an accumulation point. Any combination of such models can be tensored and a finite number of them will lead to a finite number of tensored models. Note, however, that all but the $c < 1$ and $c < 3/2$ sequences are discrete sequences on a continuous background of allowed values of c . We have found experimentally that any integer $c \leq 26$ (or $c \leq 15$ in the superconformal case) is realized in a non-trivial way, that is it is obtained by a tensor product in which the value of c is non-integer at each intermediate step. We term such models as basic models. A $d = 4$ mechanical string theory would arise for example from the product $(14/15)^9(34/35)$ ¹⁴⁾.

Once c is chosen, the allowed values of $h(c)$ are determined as well. A choice of the participating values of h specifies the spectrum of the model. The question of whether given an allowed $h(c)$ there always exists a field theoretic Hamiltonian whose spectrum contains $h(c)$ was answered in the affirmative by GKO¹²⁾. We shall review their work in Section 3. The question of the $h(c)$ spectrum of known statistical mechanical models was turned by Cardy¹⁷⁾ into the question of constructing the one-loop modular invariant representations of the Virasoro algebra with $c < 1$. We impose this world sheet requirement on the tensored models.

b) Modular invariance

Modular invariance of the model may be achieved in many ways, starting for example by preparing each $c < 1$ building block to be modular invariant on its own and ending by correlating the mechanical string and its "spin part" in such a way that only the complete model manages to be modular invariant. It has been shown that under certain conditions there is only a finite known number of ways in which to construct modular invariant models with $c < 1$ ²⁰⁾. If that indeed is generally the case, there is most probably only a finite number of ways to have modular invariant tensored models for a given integer value of the central charge.

In Section 6, we shall review the construction of the $c < 1$ modular invariant models. These constructions are not directly derived from a field theory although they close under the operator product expansion and are therefore consistent theories. We find it useful to derive these modular forms using as a starting point the free quark model with $O(4n)_L \times O(4n)_R$ symmetry. This was first suggested by KGO¹²⁾ and further elaborated on by Kent¹³⁾. The techniques for studying modular invariant models built out of free fermions were improved in the study of string theory^{5),9),14)}. We show in Section 8 how to use them in order to obtain the field theoretic construction of the modular invariant $c < 1$ models.

c) Operator content and fixed lines

From the information accumulated on $c < 1$ models, one knows explicitly the spectrum of all tensored models built out of modular invariant $c < 1$ building blocks. We do not have yet the general classification of all the modular invariant tensored models. However we shall show how to construct many such models. Each $c < 1$ is primitive in the sense that the states have been striped of all their quantum numbers related to continuous Kac-Moody symmetries, the states are labelled only by their $L_0(\bar{L}_0)$ eigenvalues.

For a given value of integer c , many of the models contain $(1,0)$ and $(0,1)$ states and thus a Kac-Moody algebra²¹⁾. For the critical values of c , these turn into gauge symmetries. On the other hand, many models remain without these states and symmetries. The finite number of models formed by tensoring furnishes a set of points in the space of all string conformal field theories with a given value of c .

Are these points indeed isolated or do there exist other conformal field theories in their infinitesimal neighbourhood? In statistical mechanics language the question is: do there exist marginal $(1,1)$ operators at the isolated point and in its neighbourhood. Such operators may draw fixed lines or surfaces through the isolated point. In string language these operators correspond to the existence of dilatons related to flat directions in terms of the parameters describing internal dimensions.

The models we describe with $c = 1$ have indeed fixed lines and so will, of course, models containing $c = 1$ as a building block. We have checked many basic models and as yet have found a $(1,1)$ operator in all cases. In Appendix II we show as an example that all self-tensored models contain a $(1,1)$ operator. We are led to the conjecture that at least one $(1,1)$ operator will appear in any basic model. The existence of a $(1,1)$ operator in a model does not ensure that a $(1,1)$ operator exists also in the neighbourhood of a point, it is actually very unlikely²²⁾ for basic models with $c > 1$. For example, the $(1,1)$ operators discussed in the Appendix cannot serve to draw fixed lines from the conformal point. We are currently investigating this issue.

It would be very nice if all string conformal models at a definite value of c could be built by the tensored discrete series serving as nucleation seeds. Fixed surfaces passing through the non-isolated seeds completing the conformal space. These generalities will be illustrated in the following, $c = \frac{1}{2}$ case.

d) The $c = \frac{1}{2}$ case - a review

Before developing the general formalism, we review the construction for the $c = \frac{1}{2}$ case⁹⁾. The statistical model associated with $c = \frac{1}{2}$ is the Ising model. For $c = \frac{1}{2}$ the allowed values of h and $\bar{h}$ are $\bar{h}, h = 0, 1/16, \frac{1}{2}$. $h, \bar{h}$ correspond to ground state eigenvalues of $L_0, \bar{L}_0$.

There are two ways to tensor the Ising model and obtain a modular invariant model with central charge $c = 1$ [for a treatment of the case of larger integer values of c , see Ref. 9)]. The first way is to ensure that the Ising itself is modular invariant¹⁷⁾ and tensor it. The second is to construct a combination of the Ising model which is only modular invariant after tensoring. In both cases, there exists a field theoretical description of the model in terms of Majorana fermions. However, in this section we start by using the Virasoro representation language.

In the first case the spectrum is given by

$$\left((0,0) + \left(\frac{1}{16}, \frac{1}{16}\right) + \left(\frac{1}{2}, \frac{1}{2}\right) \right) \otimes \left((0,0) + \left(\frac{1}{16}, \frac{1}{16}\right) + \left(\frac{1}{2}, \frac{1}{2}\right) \right)$$

where $(h, \bar{h})$ denotes a representation of $(L_0, \bar{L}_0)$. The spectrum contains a $(1,1)$ state but not a $(1,0)$ or $(0,1)$ state. In the second case, the spectrum is

$$(0,0)(0,0) + \left(\frac{1}{2}, 0\right)\left(0, \frac{1}{2}\right) + \left(0, \frac{1}{2}\right)\left(\frac{1}{2}, 0\right) + \left(\frac{1}{2}, \frac{1}{2}\right) + \left(\frac{1}{16}, \frac{1}{16}\right)$$

Here the spectrum contains the states $(0,1)$, $(1,0)$ and $(1,1)$. In the space of $c = 1$ conformal string theories, there seem to be two disconnected points. However, it is known that the first model is equivalent to a string propagating on a z_2 orbifold with radius, $R_{\text{orbifold}} = 1$, while the second model is equivalent to a string moving on a circle (a Gaussian model) of radius $R = 1$. As each model of this kind belongs to a one parameter family of $c = 1$ conformal models, the first to the family of string moving on orbicircles of general radius R , the second to the family of string moving on circles with radius R . The $(1,1)$ state appearing in the spectra of both models was indeed a dilaton related to the two flat directions R_{circle} and R_{orbifold} . The marginal operator $\propto \partial_z \phi \bar{\partial}_{\bar{z}} \phi$ draws the fixed lines. In general, the existence of a $(1,1)$ state which will turn into a massless scalar in the critical dimension is a necessary condition that the models are not isolated in the conformal space. The $(1,0)$, $(0,1)$ states generate the $U(1) \times U(1)$ gauge symmetry in the circle case at the critical dimension. This symmetry is absent (spontaneously broken) when the orbicircle has $R = 1$. Nevertheless, it turns out that a theory of a string moving on a circle of radius $R_{\text{circle}} = \sqrt{2}$ is equivalent to that of a string moving on an orbicircle of radius $R_{\text{orbifold}} = 1/\sqrt{2}$. That is, one can continuously interpolate between the two models resulting from the modular invariant tensoring of the $c = \frac{1}{2}$ representations. In this case, we see clearly that the tensoring construction does not lead to any exceptional conformal theories, by tensoring the discrete nature of the degrees of freedom is smeared and the only trace of it is the orbifold, the models have an interpretation as strings moving on manifolds or orbifolds. In Fig. 1 we sketch the conformal space. The same results can be obtained by a much more involved construction, which is, however, useful for the general case. We thus return to the $c = \frac{1}{2}$ case in Section 5. We turn now to describe the ingredients of the general construction.

3. - CONSTRUCTION OF MODELS $C < 1$

The coset construction

GKO¹²⁾ have shown that all allowed values of $c < 1$ and $h(c)$ appear in field theoretic models, we review their construction. Given²³⁾ a Lie group g , one has an associated Kac-Moody algebra $\hat{g}$ depending on k . The generators T_n^a fulfil the algebra

$$[T_m^a, T_n^b] = if^{abc} T_{m+n}^c + km\delta^{ab}\delta_{m,-n} \quad (3.1)$$

$m, n \in \mathbb{Z}$; a, b, c run from over the adjoint representation of g , f^{abc} are the structure constants of g . To any such algebra, there is an associated Virasoro algebra given by the Sugawara construction. The Virasoro generators L_n^g are given by:

$$L_n^g = \frac{1}{2k+Q_g^g} \sum_{a,m} :T_{m+n}^a T_{-m}^a: \quad (3.2)$$

where the normal ordering is with respect to the m index of T_m^a and Q_g^g is the value of the quadratic Casimir in the adjoint representation of g . The operator L_n^g fulfils the algebra

$$[L_m^g, L_n^g] = (m-n)L_{m+n}^g + \frac{C_k^g}{12} m(m^2-1)\delta_{m,-n} \quad (3.3)$$

The central charge c_k^g is given by:

$$c_k^g = \frac{2k \dim g}{2k+Q_g^g} \quad (3.4)$$

If s is a subgroup of g one can construct on top of the algebras of L_n^g and L_n^s a third Virasoro algebra²⁴⁾ denoted $K_n^{g/s}$, where

$$K_n^{g/s} = L_n^g - L_n^s \quad (3.5)$$

The central charge associated with this algebra is given by (suppressing the level index)

$$c^{g/s} = c^g - c^s \quad (3.6)$$

Moreover $K_n^{g/s}$ commutes with s . For $c^{g/s} = 0$, it was shown that one can decompose any representation of g^α of g in terms of a finite number of (in principle reducible) representations s^β of s . That is

$$\hat{g}^\alpha = \sum_{\beta} n_{\beta}^{\alpha} \hat{s}^{\beta} \quad (3.7)$$

The sum runs on the finite number of representations of s and the integer coefficients n_{β}^{α} are finite.

For $0 < c < 1$ the multiplicities become infinite but factor out in terms of representations of the Virasoro algebra $K^{g/s}$ [there is only a finite number of $h(c)$ for $c < 1$]. Thus

$$\hat{g}^\alpha = \sum_{h, \beta} n_{h, \beta}^{\alpha} v_c^h \hat{s}^{\beta} \quad (3.8)$$

where v^h is the appropriate representation of $K^{g/s}$.

Examples:

KGO discuss two ways to build the discrete series.

1. Consider $g = SU(2) \times SU(2)$ and $s = SU(2)$ diagonal in g , one chooses the n level representation of one $SU(2)$ ($SU(2)_n$) and the level one representation of the other $SU(2)$ group. This choice induces a level $n+1$ representation of s . The

central charge $c^{g/s}$ sweeps the whole discrete sequence

$$c = 1 - \frac{6}{(n+2)(n+3)} \quad n = 1, 2, \dots \quad (3.9)$$

Moreover picking an isospin ℓ representation of $SU(2)_n$ (in general ℓ is bounded by $2\ell \leq n$) and a level ϵ ($\epsilon = 0, \frac{1}{2}$) representation of the second $SU(2)$, one obtained

$$\hat{g}^{(\ell, n) \otimes (\epsilon, 1)} = \sum_{\ell} v_{c(n)}^{h(\ell, n)} \otimes \hat{s}^{(\frac{1}{2}[\epsilon-1], n+1)} \quad (3.10)$$

where $p = 2l+1$ and the sum is over q such that $p-q$ is even ($\varepsilon = 0$) or odd ($\varepsilon = \frac{1}{2}$). In performing this decomposition GKO have done a major part of the calculation necessary for inferring the modular properties of the theory.

2. A choice of a level one representation of $g = sp(n)$ and $s = sp(n-1) \times sp(1)$ led to the same values of $c(n)$.

3. For the purpose of starting from a free quark model we found it useful to follow the suggestion of KGO and enlarge g to $SO(4n)$ and s to $sp(n-1) \times sp(1) \times SU(2)_n$. We now elaborate in detail on this construction.

4. - STARTING FROM FREE QUARKS

In this case $g = SO(4n)$ and $s = sp(n-1) \times sp(1) \times SU(2)_n$. A level one representation of $SO(4n)$ has a central charge

$$C_{SO(4n)} = 2n \quad (4.1)$$

The explicit construction is done in terms of $4n$ Majorana-Weyl fermions. We denote the fermions as $\psi^{i,\mu}$, $i = 1, n$, $\mu = 0, 1, 2, 3$. Each fermion can be expanded as

$$\psi^{i,\mu}(z) = \sum_{n=-\infty}^{\infty} z^{n+\varepsilon} \psi_n^{i,\mu} \quad ; \quad \{\psi_n^{i,\mu}, \psi_m^{i',\mu'}\} = \delta_{n,-m} \delta_{ii'} \delta_{\mu\mu'} \quad (4.2)$$

where ε may be either 0 (Ramond fermions) or $\frac{1}{2}$ (Neveu-Schwarz fermions). We shall return later to the distinction between these cases. The $O(4n)$ generators can be written as

$$T_n^a = \frac{1}{2} \sum_{i,j=1}^n \sum_{\mu,\nu=0}^3 \sum_k \psi_{n-k}^{i,\mu} M_{ij,\mu\nu}^a \psi_k^{j,\nu} \quad (4.3)$$

where the normalization conventions are:

$$[M^a, M^b] = i f^{abc} M^c \quad (4.4)$$

$$\sum_{ab} f^{abc} f^{abd} = Q_f \delta^{cd} \quad (4.5)$$

and $M_{i,j}^a$ is defined as

$$M_{ij}^{\alpha\beta} = \frac{i}{\sqrt{2}} [\delta_{i\alpha}\delta_{\beta j} - \delta_{i\beta}\delta_{\alpha j}] \quad \alpha > \beta \quad (4.6)$$

where i stands for (i, μ) and (α, β) describes a . In terms of $O(4n)$, the Kac-Moody algebra is

$$[T_n^a, T_m^b] = if^{abc} T_{n+m}^c + kn\delta_{n,-m}\delta_{ab} \quad (4.7)$$

with $k = \frac{1}{2}$ and $Q_\psi = 4n-2$. The corresponding Virasoro generators are

$$\begin{aligned} L_n^g &= \frac{1}{2k+Q_\psi} \sum_{a,m} :T_{m+n}^a T_{-m}^a: = \\ &= \frac{1}{4} \sum_{\substack{k \ell n \\ i r}} (k-1) \delta_{k+\ell, n} \psi_k^{i,r} \psi_\ell^{i,r} + \delta_{\varepsilon, 0} \delta_{n,0} \frac{1}{16} \end{aligned} \quad (4.8)$$

where the N.S. vacuum was shifted to zero (in the tensoring it may be reshifted). The Virasoro algebra

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12} n(n^2-1) \delta_{n,-m} \quad (4.9)$$

has indeed a central charge $c = 2n$. This is a case of the correspondence between the Sugawara construction and the free fermion construction. We next classify the fermions and construct the generators in terms of the subgroups $s = SU(2)_n \times sp(n-1) \times sp(1)$. The group of $4n$ fermions was thus organized in n groups of four fermions. The n groups are split into a group of $(n-1)$ members ($j = 1, \dots, n-1$) within which the $sp(n-1)$ group acts on the j index and on μ from the left and a group with one member on which $sp(1)$ acts on μ from the left. The $SU(2)_n$ group acts simultaneously on all n groups by acting on the index μ from the right. We now describe explicitly this induced representation. The induced $SU(2)_n$ representation is the following. Defining

$$\psi^i = \sum_r \psi^{i,r} G^r \quad ; \quad \bar{\psi}^i = \sum_r \psi^{i,r} \bar{G}^r \quad (4.10)$$

where

$$G^r = (1, i\vec{\sigma}) \quad ; \quad \bar{G}^r = (1, -i\vec{\sigma})$$

the induced $SU(2)_n$ Kac-Moody generators are

$$T_n^i = -\frac{1}{8} \sum_{k,j} \text{Tr} \{ \bar{\Psi}_{m-k}^j \Psi_k^j \sigma_i \} \quad i=1,2,3 \quad (4.11)$$

where the trace is on the two-dimensional space spanned by the σ^μ matrices. The operators indeed act within each j and in a j independent manner. The $SU(2)$ transformation, R , acts on ψ^i, r from the right, that is, $\psi^i \rightarrow \psi^i R$. Thus, we have two spin- $\frac{1}{2}$ combinations given by $(\psi^{i,0} + i\psi^{i,3}, i\psi^{i,1} + \psi^{i,2})$ and $(i\psi^{i,1} - \psi^{i,2}, \psi^{i,0} - i\psi^{i,3})$. The Kac-Moody algebra is given by:

$$[T_\ell^i, T_m^j] = i \varepsilon^{ijk} T_{\ell-m}^k + \frac{1}{2} n \ell \delta_{ij} \delta_{\ell-m} \quad ; \quad k = \frac{n}{2} \quad (4.12)$$

The central charge of the Virasoro algebra is:

$$C = \frac{3n}{2+n} \quad (4.13)$$

and

$$L_r = \frac{1}{\beta} \sum_m : T_{m+r}^i T_{-m}^i : \quad i=1,2,3 \quad \beta=2k+2=n+2 \quad (4.14)$$

The induced $sp(n-1)$ level one representation is given by the generators

$$T_m^{ab,r} = \frac{1}{4} \sum_k \sum_{i,j=1}^{n-1} \text{Tr} \{ \bar{\Psi}_{m-k}^i M_{ij}^{ab,r} \Psi_k^j \} \quad (4.15)$$

where

$$M_{ij}^{ab,r} = \frac{1}{\sqrt{2}} \{ \delta_{ai} \delta_{bj} \sigma_r - \delta_{aj} \delta_{bi} \bar{\sigma}_r \} \quad a \neq b$$

$$M_{ij}^{aa,r} = \frac{1}{2} \delta_i^a \delta_j^a (\sigma_r - \bar{\sigma}_r) \quad (4.16)$$

The number of generators is $2\ell^2 + \ell$; $k = \frac{1}{2}$; $Q_\psi = \ell+1$ for $sp(\ell)$. Thus the central charge of the operators

$$L_n = \frac{1}{4(\ell+2)} \sum_a : T_{n-m}^a T_{-m}^a : \quad (4.17)$$

is

$$C = \frac{\ell(2\ell+1)}{\ell+2}$$

For $\lambda = n-1$

$$C = \frac{(n-1)(n-1)}{n+1} ; L_0 = \frac{1}{n+1} \sum_{a,n} : T_n^a T_{-n}^a : \quad (4.18)$$

For $\lambda = 1$

$$C=1 ; L_0 = \frac{1}{3} \sum_{a,n} : T_n^a T_{-n}^a : \quad (4.19)$$

consistent with the fact that $sp(1) \approx SU(2)$.

Collecting all this together, one obtains that the central charge of the coset indeed runs over the discrete sequence. The next stage is to study some features of the decomposition of the Ramond and N.S. sectors in terms of the $SU(2)_n$ representations. In general, level one $SO(2n)$ has four irreducible representations; they are denoted o, v, s, t and correspond respectively to (NS, even), (NS, odd) (R, even), (R, odd) fermionic Fock spaces, where even or odd relates to the even or odd fermion number projection operator (the GSO projector) $(1 \pm (-1)^F)$.

First consider the N.S. sector. The N.S. fermion operator $\psi_n^{i,\mu}$ is a spin $\frac{1}{2}$ representation of $SU(2)_n$. The index μ transforms as a sum of two spin- $\frac{1}{2}$ representations. Thus the sector (0) contains only integer angular momentum states while the sector (v) contains only half-integer angular momentum states.

The Ramond sector is complicated by its zero modes. The projection operator as expressed in terms of the oscillators is appended by the projection $(1 \pm \gamma_I)$ on the zero modes. The operator γ_I is given by $\gamma_I = \prod_i \gamma_5(i)$, $i = 1, \dots, n$. Since the $SU(2)_n$ generators in the R sector are given by:

$$T = \sum_i \frac{1 \pm \gamma_5(i)}{2} G(i) \quad (4.20)$$

The fermions $(1 \pm \gamma_5^{(i)}) U_i$ carrying spin 0 and $\frac{1}{2}$ are respectively. U_i denotes the zero modes. We choose the projection $(1 \pm \gamma_I)$ such that (s) contains only an integer angular momentum and (t) only half integer angular momentum in correspondence to (0) and (v)*). The Ramond zero mode sector contains all possible angular momentum states from 0 to $n/2$.

*) In the case of $O(4n)$ this will require that s is obtained by the projection $1 + (-1)^n \gamma_I$.

KGO¹²⁾ and Kent¹³⁾ have suggested and developed another aspect of the coset construction. The identification of representations implicit in a g/s structure can be re-expressed as a gauge projection. Given a group g and a subgroup s , one gauges s , in particular for $g = O(4n)$ one gauges the subgroup $s = sp(n-1) \times sp(1) \times SU(2)_n$ by defining appropriate covariant derivatives for the fermions. In the absence of kinetic energy terms for the gauge bosons, conformal invariance is maintained. Also in the absence of such a term, the gauge potential serves as a Lagrange multiplier causing the fermionic currents to annihilate the vacuum.

It has been shown²⁵⁾ that a consistent quantization of gauged fermions (by using Dirac brackets where necessary) would lead back to the same central charge as obtained in a Sugawara construction. In Ref. 25), it was actually shown that fermions with an $O(N) \times O(k)$ symmetry where $O(N)$ is a global flavour symmetry and $O(k)$ is a gauged symmetry, have the same central charge as a string moving on the $O(N)$ group manifold in the presence of a Wess-Zumino term with an integer coefficient k . This was achieved by using the conformal embedding²⁶⁾ of $O(N)_k + O(k)_N$ in $O(kN)$ to calculate the central charge of the gauged group. An operator R_n was defined²⁵⁾ by:

$$R_n = L_n^{O(N)_k} + L_n^{O(k)_N} \quad (4.21)$$

Its central charge is kN and its free field commutators are all identical to the free field commutators of the energy momentum tensor. Recalling that by the equations of motion for A_μ^I ($I = 1, \dots, \dim O(k)$), $L_n^{O(k)_N}$ annihilates the vacuum, the central charge of the gauged system was found to be that of $O(N)_k$. For the coset case, the conformal embedding to be used is that of $s + v^{g/s}$ [where $s = sp(n-1) \times sp(1) \times SU(2)_n$ are gauged] into $g = O(4n)$. This leads to an identical result, the central charge of the gauged system is identical to the one obtained by the coset construction.

In general, calculating and identifying the value of the central charge alone in two different models does not prove their quantum equivalence²³⁾, but for the unitary $c < 1$ case at hand, there is only a finite number of allowed values of h . Moreover, once modular invariance is imposed, there are only two or three²⁰⁾ allowed representations and the quantum equivalence is easy to establish given that both theories are indeed conformal invariant. In Section 9, we use a functional approach to show that the gauged system is conformal invariant and has the coset value of the central charge. We do believe that the results hold also

for $c > 1$. Some part of our analysis, such as the construction of modular invariant $c < 1$ models relies mainly on the coset structure, nevertheless we glean some intuition from the gauged version that is also necessary for a Lagrangian interpretation and useful for bosonization.

The first intuition one gains is that the addition of the gauge fields causes many representations to be projected out of the Hilbert space. By confining representations, the gauge fields reduce the effective number of degrees of freedom and with them the value of the central charge. The confinement is so strong in the $c < 1$ case that the surviving particles have lost all their $O(4n)$ continuous quantum numbers and their identities are maintained only through the value of L_0 , the energy momentum tensor. In the Lagrangian formulation the appropriate reduction of the central charge is obtained through the negative contributions of the ghosts associated to the gauge fixing, in Section 9, we give a general derivation of this result. A derivation of the value of c in the presence of a factor $U(1)$ field was given in Ref. 27). The second intuition gained concerns modular invariance, using vector (non-chiral) free fermions as a starting point, one can easily construct $O(4n)_L \times O(4n)_R$ modular invariant models. Gauging the non-anomalous diagonal groups mentioned before, no gravitational or gauge anomalies are introduced, the resulting model is thus also modular invariant with the correct value of c . Modular invariant representations of the Virasoro algebra with $c < 1$ are produced from modular invariant representations of $O(4n)_L \times O(4n)_R$. We use this extensively.

For concreteness and for future use, we discuss here in some more detail the consequences of gauging the diagonal subgroup. The gauging is done by adding terms of the form $A_\mu j_\mu$ to the free Lagrangian. The currents j_μ were defined in Eqs. (4.11)-(4.15), the gauge fields A_μ correspond to $s = sp(1) \times sp(n-1) \times SU(2)$. Varying with respect to A_μ , one obtains $j_\mu = 0$. In an operator formulation, this condition is weakened to read $j_\mu^{(+)} |0\rangle = 0$, where $(+)$ are the positive frequency modes. In more detail, one can decompose

$$A_\mu(\sigma, \tau) = A_\mu^{(0)} + A_\mu^r(\sigma + \tau) + A_\mu^l(\sigma - \tau) \quad (4.22)$$

where $A^{(0)}$ is the zero mode. Varying with respect to A^r and A^l results in:

$$j_\mu^{(r)(\theta)} |0\rangle = 0 \quad ; \quad j_\mu^{(l)(\theta)} |0\rangle = 0 \quad (4.23)$$

Varying with respect to the zero mode leads to

$$(j_r^{(0)r} + j_r^{(0)l}) |0\rangle = 0 \quad (4.24)$$

This correlates in particular the left- and right-handed $SU(2)$ angular momentum k carried by the various ground states of the Kac-Moody algebras to be projected out. One solution of (4.24) is that left- and right-handed Kac-Moody representations will carry the same angular momentum. That is the gauging and integration over the vector A_μ is equivalent to projecting out left and right Kac-Moody representations with the same value of k . The states are $\sum_k |kv\rangle_r |k'v\rangle_l$. k and k' are the projected out Kac-Moody representations and v the surviving Virasoro quantum numbers. k and k' should add up to zero subgroup quantum numbers, this is the way to tie the left-right knot.

At this stage we have gathered enough formalism so that it may be worthwhile to work out an example, the $c = \frac{1}{2}$ case.

5. - THE $c = \frac{1}{2}$ CASE, ONCE MORE

The $c = \frac{1}{2}$ case appears in this construction by considering the coset $SO(8)/sp(1) \times sp(1) \times SU(2)_2$. The values of h in the Virasoro algebra will be obtained by

$$h = L_0(SO(8)) - \frac{1}{4} j(j+1) - \frac{1}{3} l_1(l_1+1) - \frac{1}{3} l_2(l_2+1) \quad (5.1)$$

where (j, l_1, l_2) are the angular momentum under $(SU(2)_2, sp(1), sp(1))$ [recall $sp(1) \approx SU(2)$]. Given a representation (i) of $SO(8)$, it is easy to see that in its decomposition

$$\hat{SO}(8)^{(i)} = \sum_{j, l_1, l_2, h} n_{j, l_1, l_2, h}^i \quad SU(2)_2^j \otimes sp(1)^{l_1} \otimes sp(1)^{l_2} \otimes \sigma_{\frac{1}{2}}^h \quad (5.2)$$

all the multiplicities n_α^i are finite. The reasoning is the following: all the groups have a maximally allowed value for their ground state vacuum (in this case $j \leq 1$, $l_1, l_2 \leq \frac{1}{2}$), h has only three allowed values $(0, 1/16, \frac{1}{2})$, thus the maximal value of $L_0(SO(8))$ which could give rise to a new ground state of the subgroup of a new value of h is

$$L_0(SO(8)) \leq \begin{pmatrix} 0 \\ \frac{1}{16} \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (5.3)$$

That is, one need not go to check an $SO(8)$ excitation in the representation (i) with $L_0(SO(8)) > 3/2$, this leads to finite values of n_α^i . All states with $L_0(SO(8)) > 3/2$ contribute only to excitations above the vacuum of either $s^{(j)}$ or v_c^h or both. In a Table we collect the decomposition of levels.

Note that o and s contain indeed only integer values of j and thus through the GKO decomposition [Eq. (3.10)] contribute only to odd values of q , while v and t contain only half integer values of j and thus contribute to even values of q . All allowed values of h appear in the Table and summing over all representations of $SO(8)$, one obtains all the values of h with an equal multiplicity four. Summing over the NS or Ramond sector would also give all values of h . In Appendix C, we collect some of the additional ingredients which went in constructing the Table. We next discuss how to construct modular invariant building blocks.

6. - MODULAR INVARIANT MODELS WITH $c < 1$

In this section we describe how to obtain the modular invariant representations starting from the free fermion model. The problem of constructing modular invariant models based on free fermions was studied in detail in the context of $d = 4$ string theory model building⁵⁾. In most of our discussion we address only vector models, the formalism developed and reviewed in the former sections used only a chiral notation, such models are in general not modular invariant, however, there are less restrictions on vector models. We thus study $O(4n)_L \times O(4n)_R$ free fermion theories once these models are constructed to be modular invariant, the coset construction respects the modular invariance and therefore the resulting Virasoro representation is modular invariant. This was argued using the gauged version of the coset construction. Moreover, if the $O(4n)_L \times O(4n)_R$ system is constructed to obey the factorization and all loop modular invariances, so should the Virasoro algebra.

Modular invariant models of free fermions

There are a few versions of composing modular invariant representations of $(N)_L \times (\bar{N})_R$ fermions. We follow here the construction of Schellekens and Warner¹⁴⁾. On a torus a fermion may be characterized by its periodicity properties in the two available directions that determine its spin structure. There are thus $4^{N+\bar{N}}$ possible partitions of the spin structures. The $4^{N+\bar{N}}$ space is reducible under modular transformations. Each irreducible representation is characterized by choosing some partition of the $N+\bar{N}$ fermions into at most four groups, this forms at most a combination of six terms. Their coefficients are determined by starting with one partition and forming its orbit by performing three times the sequence of the two transformations $\tau \rightarrow \tau+1$ and $\tau \rightarrow -1/\tau$, where τ is the complex parameter which specifies the world sheet metric on the torus. Taking into account the consistency requirement that the combined six transformations reproduce the identity operator Schellekens and Warner obtain conditions on $m_i = \lambda_i - \bar{\lambda}_i$ where $\lambda_i (\bar{\lambda}_i)$ is the number of left (right) handed particles with spin structure i ($i = 1, 2, 3, 4$).

$$\begin{aligned} m_i &= 0 \pmod{4} \\ 2m_1 + 2m_2 - m_3 - m_4 &= 0 \pmod{24} \end{aligned} \tag{6.1}$$

where 1 is the $(++)$ spin structure. In the case that two or more of the m_i are zero then

$$m_i = 0 \pmod{8} \tag{6.2}$$

These are the cases discussed in this work. Details on how the various modular transformations act on the Kac-Moody and Virasoro characters appear in Refs. 15)-17). The solution of the requirement of factorization and modular invariance to all loops appear in Antoniadis, Bachas and Kounnas work⁵⁾. As the constructions described in this paper solve these constraints trivially we do not review them here.

$c < 1$ - The generic solutions

We will show that one obtains all generic modular invariant representations by choosing the following representations of $SO(4n)_L \times SO(4n)_R$

1. For any $n \geq 2$

$$(0,0) + (v,v) + (s,s) + (t,t) \quad (6.3)$$

2. In addition for any $n \geq 4$, there exists another modular invariant representation

a) $n = 4k$

$$Z_{so(4k) \times so(4k)} = (0,0) + (s,s) + (0,s) + (s,0) = (0+s, 0+s) \quad (6.4)$$

b) $n = 4k+1$

One treats separately the $16k$ fermion and the four fermions to obtain a representation of $O(4n)_L \times O(4n)_R$ in which the symmetry is spontaneously broken down to $O(4n-4)_L \times O(4n-4)_R$ ⁹⁾. The first bracket describes $4n-4$ fermions while the second one describes 4 fermions

$$Z_{so(4k+1) \times so(4k+1)} = [(0+s, 0+s)] [(0,0) + (v,v) + (s,s) + (t,t)] \quad (6.5)$$

c) $n = 4k+2$

$$Z_{so(4k+2) \times so(4k+2)} = (0,0) + (s,s) + (v,t) + (t,v) \quad (6.6)$$

d) $n = 4k+3$

The first bracket describes $4n-4$ fermions while the second one describes 4 fermions.

$$Z_{so(4k+3) \times so(4k+3)} = [(0,0) + (s,s) + (v,t) + (t,v)] [(0,0) + (v,v) + (s,s) + (t,t)] \quad (6.7)$$

analogous to the $n = 4k+1$ case.

All models are modular invariant to all orders and obey target space factorization. This completes the list of known generic modular invariant solutions with only positive contributions to the partition function. In order to prove the claim, we shall need more machinery.

7. - MODULAR INVARIANT VIRASORO REPRESENTATIONS

Cardy¹⁷⁾ first posed the problem of constructing a modular invariant set characterized by a matrix of integers $N_{h\bar{h}}$

$$Z = \sum_{h,\bar{h}} N_{h\bar{h}} \chi_h \chi_{\bar{h}}^* \quad (7.1)$$

where χ_h ($\chi_{\bar{h}}$) are the partition function of an $h(\bar{h})$ representation of $L_0(\bar{L}_0)$. Cardy¹⁷⁾ found the diagonal solution

$$N_{h,\bar{h}} = \delta_{h,\bar{h}} \quad (7.2)$$

which is valid for any n . Cardy¹⁷⁾ also showed that $h-\bar{h}$ must be an integer and found some non-diagonal solutions. Gepner¹⁵⁾ showed that by constructing modular invariant $SU(2)_n$ or $SO(3)_{2k}$ models, one induces a construction of modular invariant Virasoro representations. Considering a modular invariant $g_L \times g_R$ model

$$Z_{g_L \times g_R} = \sum_{\lambda, \bar{\lambda}} a_{\lambda \bar{\lambda}} \chi_{\lambda} \chi_{\bar{\lambda}} \quad (7.3)$$

where g is $SU(2)_n$ or $SO(3)_n$ and χ_{λ} ($\chi_{\bar{\lambda}}$) are the characters of $g_L(g_R)$ in the representation of $g_L(g_R)$ based on a vacuum with spin $\lambda(\bar{\lambda})$. Then, a model invariant Virasoro representation will result from the choice:

$$N_{h,\bar{h}} = N_{h,1;\bar{h},1} = \frac{1}{2} a_{h,\bar{h}} b_{h,\bar{h}} \quad (7.4)$$

where a - relates through Eq. (7.3) to a level $k = m-2$ representation of $g_L \times g_R$ and b - to a level $k = m-1$ representation. The central charge of the Virasoro model is $c = 1 - 6/m(m+1)$ ($m = 3, 4, \dots$). Cappelli, Itzykson and Zuber¹⁶⁾ have found the general construction of g invariants and pointed out a correspondence between the solutions and simply laced Lie algebras. The formula in terms of the modular invariant Virasoro representations (in addition to the diagonal solution) are:

$$n = 4k \quad (n = m-1)$$

$$Z_c^v = \sum_{j=1}^n \sum_{\substack{q=1 \\ \text{odd}}}^k (j,1; j,1) + \frac{1}{2} \sum_{j=1}^n \sum_{\substack{q=1 \\ \text{odd}}}^{n+1} (j,1; j, m+1-q) \quad (7.5)$$

$$n = 4k+1$$

$$z_c^v = \sum_{\substack{j=1 \\ j \text{ odd}}}^n \sum_{q=1}^j (j, q; j, q) + \frac{1}{2} \sum_{j=1}^n \sum_{q=1}^{n+1} (j, q; m-j, q) \quad (7.6)$$

$$n = 4k+2$$

$$z_c^v = \sum_{j=1}^n \sum_{\substack{q=1 \\ q \text{ odd}}}^j (j, q; j, q) + \frac{1}{2} \sum_{j=1}^n \sum_{\substack{q=2 \\ q \text{ even}}}^{n+1} (j, q; j, m+1-q) \quad (7.7)$$

$$n = 4k+3$$

$$z_c^v = \sum_{\substack{j=1 \\ j \text{ odd}}}^n \sum_{q=1}^j (j, q; j, q) + \frac{1}{2} \sum_{j=1}^n \sum_{\substack{q=2 \\ q \text{ even}}}^{n+1} (j, q; m-j, q) \quad (7.8)$$

For $m(n) = 11(10), 12(11), 17(16), 18(17), 29(28), 30(29)$ an additional solution exists. The great advantage of this approach is that the representation content is explicit. Its drawback from our point of view is the lack of a field theoretic realization. For the purpose of obtaining the partition function of the tensored models, Eqs. (7.4)-(7.8) are all that is needed. We wish to prove that the modular invariant $O(4n)_L \times O(4n)_R$ forms listed in Eqs. (6.3)-(6.7) lead after using the coset construction to the Virasoro representations listed in Eqs. (7.2) and (7.5)-(7.8). To that end let us first be more general.

8. - FROM $SO(4n)$ TO THE VIRASORO REPRESENTATION - GENERAL CONSTRUCTION

To simplify notation we start by decomposing a chiral case in several stages.

a) Consider $SO(4n)/SO(4n-4) \times O(4)$

The resulting central charge is zero, the partition functions are related by

$$z_{\hat{SO}(4n), r} = \sum_{k, k'} z_{\hat{SO}(4n-4), k} z_{\hat{SO}(4), k'} n_{kk'}^r \quad (8.1)$$

(r, k, k') are representations of $(SO(4n), SO(4n-4), SO(4))$. $n_{kk'}^r$ is given by:

$$\begin{aligned} 0 &= (0,0) + (v,v) & ; & & v &= (0,v) + (v,0) \\ s &= (s,s) + (t,t) & ; & & t &= (s,t) + (t,s) \end{aligned} \quad (8.2)$$

b) Consider $SO(4)/SU(2) \times sp(1)$

The partition function relation of this $c = 0$ case is

$$z^{\hat{s}^0(n), k'} = \sum_{l, l'} z^{\hat{s}^0(n), l} z^{\hat{s}^1(n), l'} \eta_{ll'}^{k'} \quad (8.3)$$

$n_{pp}^{k'}$ is given by: $(0) = (0,0)$; $(v) = (\frac{1}{2}, \frac{1}{2})$; $(s) = (0, \frac{1}{2})$; $(t) = (\frac{1}{2}, 0)$

c) Consider $SO(4n-4)/sp(n-1) \times SU(2)_{n-1}$

Once again $c = 0$ and

$$z^{\hat{s}^0(4n-4), k} = \sum_{u, u'} z^{\hat{s}^1(n), u} z^{\hat{s}^0(2n), u'} \eta_{uu'}^k \quad (8.4)$$

The determination of n_{uu}^k is a question of calculating Kac-Moody coefficients, we have neither succeeded in obtaining it nor in finding it in the literature^{*}). Its knowledge is the only obstacle for finding directly the Virasoro content and calculating n_c , the number of copies of the representations of the Virasoro algebra obtained in the next stage.

d) Consider $SU_{n-1}(2) \times SU(2)/SU(2)_n$

This leads to all $c < 1$ values

$$z^{\hat{s}^0(2n), u'} : z^{\hat{s}^1(n), l} = \sum_{l, h} z^{\hat{s}^0(n), l} \chi^h \eta_{lh}^{u'} \quad (8.5)$$

The rather complicated matrix $n_{q,h}^{u',p}$ was obtained by GKO. We make extensive use of their results. The final form of the decomposition is

^{*}) After the paper was completed we have found the matrix n_{uu}^k , it will be reported in D. Altschuler, K. Bardacki and E. Rabinovici, to appear.

$$Z^{\hat{S}o(h),r} = \sum_{h,k,l,l',u,u',q,l} n_{hk}^r n_{ll'}^{k'} n_{uu'}^k n_{ql}^{u'} Z^{\hat{S}o(h),l} Z^{\hat{S}o(h-1),u} Z^{\hat{S}o(l),l'} \chi_{(8.6)}^L$$

We will deal with each of the generic cases

The diagonal sequence

We now show that all allowed values of h appear in the decomposition of $(0+0)+(v,v)+(s,s)+(t,t)$. We first concentrate on one chirality: the Ramond states (s) and (t) contain together, as we have seen from the point of view of $SU(2)_n$, all allowed spins (integers and half-integers) that is by the GKO decomposition all values of the allowed $h(c)$ will indeed appear^{*)}. We expect them all to appear with an equal multiplicity once (0) and (v) are accounted for as is the case $n = 2$ studied previously. The analysis above shows the separate left and right representation content, the correlation between them is enforced by the fact that total (left-right) angular momentum should be zero. For the (R,R) , (NS,NS) cases, this leads to the diagonal representations. In the gauged version of the construction, this results from the fact that the gauge field zero mode couples to both left and right current zero modes coupling their total angular momentum to zero. Indeed it has been proven²⁰⁾ that the only modular invariant partition function containing all the values of h is the diagonal one, thus completing the correspondence between the diagonal $O(4n)_R \times O(4n)_L$ modular invariant representation and that of the diagonal Virasoro representation. The knowledge of the matrix $n_{qh}^{u'}$ would lead directly to the modular form.

We have also checked using the GKO decomposition that summing over all allowed representations of the chiral $SU(2)_{n-1} \times SU(2)$, produces all allowed values of h with equal multiplicity. An independent argument that summing over all diagonal representations of a $g_L \times g_R$ will lead after the coset construction to a modular invariant Virasoro representation is contained in a proof by Goddard and Bowcock²⁶⁾.

^{*)}This can be verified also by examining the $SU(2)_{n-1} \times SU(2)$ content.

The complementary sequence

$$n = 4k$$

The modular invariant solution chosen for $SO(16k) \times SO(16k)$ is $(o-s)(o+s)$. This leads for example to a $E_8 \times E_8$ lattice for $k = 1$. From the $SU(2)_n$ point of view, (o) and (s) contain only integer angular momentum states, that is when fed down to $n_{qh}^{u,p}$ [Eq. (8.5)], all half integer angular momenta are all absent even in the diagonal terms. The final result is modular invariant, with positive coefficients relying again on the uniqueness of the solution it should coincide with the results known for $n = 4k$. As the angular momentum is even, only $h(p,q)$ and $\bar{h}(p,q)$ corresponding to odd values of q should appear as is indeed the case [Eq. (7.5)]. Once again we cannot verify the multiplicities directly. From the point of view of Gepner, we have a fermionic realization of $SO(3)_{2k} \times SU(2)_{2k+1}$.

$$n = 4k+2$$

The representation chosen for $O(4n) \times O(4n)$ is $(o,o) + (s,s) + (v,t) + (t,v)$. The diagonal terms carry integer $SU(2)_n$ spin, that is led to odd values of q , while the off diagonal terms contain only half-integer $SU(2)_n$ spin leading to even values of q . This again corresponds to the known result [Eq. (7.7)].

$$n = 4k+1, 4k+3$$

The case $n = 4k+1$ is studied in a similar fashion from the point of view of the $SU(2)_{n-1}$ group, which will restrict p to obtain odd values in agreement with the known results, analogous results hold for $n = 4k+3$. The coset methods allows a construction of many (and we believe at least all generic cases) modular invariant tensored models which do not consist of modular invariant building blocks. This is done by starting from the modular invariant $O(4K)_L \times O(4K)_R$ model where $K = \sum_{i=2}^{\infty} n_i$ (n_i were defined in Section 2) and considering in addition all its allowed spontaneously broken forms⁹⁾.

Other solutions

We have not yet isolated the special properties of the $O(4n)$ groups related to the six exceptional solutions. We have also not yet understood how it comes about that all the many possible twistings of $O(4n)_L \times O(4n)_R$ lead to the same Virasoro representation as the untwisted cases. On the other hand, for $n = 4k+2$, we obtain another modular invariant representation involving a coset construction in a theory with space-time fermions. This model may not be very useful for internal dimensions, but it allows the classification

of modular invariant solutions with some negative coefficients, starting from a $(v-t, v-t)$ $O(4n)_L \times O(4n)_R$ theory one is led to the form

$$Z = \sum_{j=1}^n \sum_{\substack{q=2 \\ \text{even}}}^j (j, q; j, q) - \frac{1}{2} \sum_{j=1}^n \sum_{\substack{q=2 \\ \text{even}}}^j (j, q; j, n+1-q) \quad (8.7)$$

For $n = 4k$ the $O(4n)_L \times O(4n)_R$ partition function is no longer semi-positive definite. It is obtained from $(v+t, v+t) + 2(t, o) + 2(o, t) - 2(s, o+v) - 2(o+v, s)$ and contains the same diagonal terms as the $n = 4k+2$ case. This form is not obviously positive definite. The existence of such solutions is hinted at in Refs. 15) and 16).

For $n = 2$ the partition function is zero, its chiral parts are related to the superstring sector in the heterotic string construction

$$Z = \frac{(2\pi)^{1/3}}{(\theta_1')^{1/3}} \left(\theta_2^4 + \theta_4^4 - \theta_3^4 \right) = 0$$

This series contains only odd angular momentum, that is only even values of q . One step further would be to study purely chiral models, the case $n = 2$ is subtle because the partition function is not really modular invariant although it is zero. Under a modular transformation, it acquires the phase $\exp(i\pi/12)n$ and its vertex operators probably fail to give modular invariant results. In the heterotic string case, this phase is cancelled by the opposite chirality oscillators. In any case, for $n = 2$ we obtain after the coset construction

$$Z = 2 \left(\chi_{\frac{1}{16}} - \chi_{\frac{1}{16}} \right)$$

However, for $n = 12k$, there are truly chiral invariant $O(4n)$ models the coset construction may lead to an interesting result. (Our suspicion is phrased in the gauge field language, that as the gauge fields are anomalous, the theory may either not have a well-defined value of c or not be modular invariant.)

9. - BOSONIZATION

We now show that the gauged $O(4n)_L \times O(4n)_R$ fermionic system is conformal

invariant and carries a central charge equal to that obtained by the coset construction.

The starting point is the gauged $O(4n)_L \times O(4n)_R$ fermionic Lagrangian

$$\mathcal{L} = \sum_{j=1} i \bar{\psi}_j \not{\partial} \psi_j + \sum_k j_\mu^k A_\mu^k \quad (9.1)$$

where the sum over k represents a sum over the adjoint representations of the diagonal (vector) subgroup s of $O(4n)_L \times O(4n)_R$ that is gauged. It is anomaly free. j_μ^k are the respective fermionic currents coupled to the gauge fields. No kinetic energy term for the gauge field is added.

To be concrete we choose s to be the diagonal $SU(2)_n$ in $O(4n)_L \times O(4n)_R$. In this case the coset construction would yield the $sp(n)_1$ central charge:

$$C_{\frac{su(n)}{u(1)}} = C^{O(4n)} - C^{su(n)} = 2n - \frac{3n}{n+2} = \frac{2n^2+n}{2+n} \quad (9.2)$$

We wish to show that the gauged Lagrangian carries the same central charge. The generalization to an arbitrary s will be immediate. In our calculation we follow the spirit of Ref. 28). The first step is to fix the gauge to be $A_- = 0^{*)}$ in the model discussed, one has simultaneously $A_+ = ig^{-1}(\partial+g)$. g is a (3×3) matrix in the adjoint representation of $SU(2)$. We normalize the generators such that

$$(\lambda^a)^{ij} = -\frac{i}{\sqrt{2}} \varepsilon^{aj} \quad ; \quad \text{Tr}(\lambda^a \lambda^b) = \delta_{ab} \quad ; \quad [\lambda^a, \lambda^b] = if^{abc} \lambda^c \quad (9.3)$$

$$f^{abc} = \frac{1}{n} \varepsilon^{abc}$$

In this normalization the Casimir, $C_\psi(SU(2))$, for $SU(2)$ has a value 1 in the adjoint representation

$$A_\pm = \frac{1}{\sqrt{2}} (A_0 \mp A_1) \quad \partial_\pm = \frac{1}{\sqrt{2}} (\partial_0 \mp \partial_1) \quad (9.4)$$

The integration over the fermionic degrees of freedom, that is the evaluation of $\det(\partial+A)$, is given by $\det(\partial)\det(WZW)$ where $\det(WZW)$ refers to the Wess-Zumino-Witten²⁹⁾ system. The effective action is $I_{FF} + I_{WZW}$. The free fermion term (which can be bosonized) I_{FF} is still

$$I_{FF} = \int d^2x \sum_j \bar{\psi}_j i \not{\partial} \psi_j \quad (r^0 = r_1; r^1 = i r_2; r^2 = r_3) \quad (9.5)$$

*) We do not consider on the torus non-trivial boundary conditions on A_μ .

The value of the central charge, C_{FF} , of the Virasoro algebra associated with it is $C_{FF} = 2n$. The covariant WZW action is:

$$I_{WZW} = \frac{n_1}{4\pi} \int d^2x \text{Tr} (g^{-1} \partial_\mu g g^{-1} \partial_\mu g) - \frac{n_1}{24\pi} \int d^3x \epsilon^{\mu\nu\lambda} \text{Tr} [\partial_\mu g \partial_\nu g^{-1} \partial_\lambda g] \quad (9.6)$$

In the light-cone gauge the WZW Lagrangian has the following right-moving $SU(2)$ currents

$$j_+^a(x_+) = -\frac{in_1}{4\pi} \text{Tr} (\lambda^a (\partial_+ g) g^{-1}) \quad (9.7)$$

The right-moving currents satisfy a Kac-Moody algebra²⁹⁾

$$[j^a(x_+), j^b(x'_+)] = if^{abc} j^c(x_+) \delta(x_+ - x'_+) + i\delta^{ab} \frac{n_1}{4\pi} \delta'(x_+ - x'_+) \quad (9.8)$$

The Kac-Moody central charge is given by

$$k = -|n_1| = -\frac{n}{4} \quad (9.9)$$

The negative integer results from the kinetic energy term having a negative sign. The magnitude of the central charge is determined to be $1/4$ for each of the N Dirac fermions in the fundamental representation of $SU(2)$.

I_{FF} and I_{WZW} do not constitute however the full Lagrangian, first one has gauge fixed to $A_- = 0$ and second, as A_+ will be integrated over, there is a Jacobian related to the change of integration variable to the group element g . This Jacobian in turn will also (as $A_- = 0$) be represented by a ghost system.

The complete ghost system I_{FP} in the light-cone gauge is thus:

$$I_{FP} = \int d^2x \ 2 \left\{ \bar{c}^+ (i\partial_- c_+) + \bar{c}^- (i\partial_+ + A_+) c_- \right\} \quad (9.10)$$

(The factor of two is just for convenience.) The notation implies a sum over the index a , the dimension of the group. For example:

$$\begin{aligned} \bar{c}^+ i\partial_- c_+ &= \sum_{a=1}^3 \bar{c}_a^+ i\partial_- c_+^a \quad ; \quad (A_+ c_-)^a = if^{abc} A_+^b c_-^c \\ \sum_{a=1}^3 A_+^a \lambda^a &= ig^{-1} (\partial_+ g) \end{aligned} \quad (9.11)$$

$\bar{c}^a$ in a covariant form is a world-sheet vector (and thus has conformal weight $\lambda = 1$) replacing the gauge field A_μ^a . It is in the adjoint representation of $SU(2)$. The ghosts c_1^a, c_2^a represent the gauge parameters associated with the form $A_- = 0$ and $A_+ = ig^{-1}\partial_+g$ and as such are world-sheet scalars (with conformal weight $\lambda = 0$) and in the adjoint representation of $SU(2)_n$. In a covariant formulation I_{FP} is given by:

$$I_{FP} = \int d^2x \left\{ \bar{c}^\mu i (\partial_\mu + \xi_\mu^\nu \partial_\nu + A_\mu) c_1 + \bar{c}^\mu (i \partial_\mu - i \xi_\mu^\nu \partial_\nu + A_\mu) c_2 \right\} \quad (9.12)$$

where

$$\bar{c}^\pm = \frac{1}{\sqrt{2}} (\bar{c}^0 \pm \bar{c}^1)$$

Now the integral over the ghost system can be performed in analogy to the calculation of the fermionic determinant, the systems $\bar{c}^\mu$ and c_i are independent and are in the adjoint representation of $SU(2)$.

The resulting action contains again two parts, I_{FFP} ; the action of the free Faddeev-Popov ghosts and I_{WZWFP} , a WZW model formed by the interacting ghosts

$$I_{WZWFP} = \frac{n_2}{16\pi} \int d^2x \text{tr} (g^{-1} \partial_\mu g g^{-1} \partial_\mu g) - \frac{n_2}{24\pi} \int d^2x \epsilon^{\mu\nu} \text{tr} [g^{-1} \partial_\mu g g^{-1} \partial_\nu g g^{-1} \partial_\lambda g] \quad (9.13)$$

This term adds to the I_{WZW} term, $|n_2| = 1$ as the contribution of the ghost system is like that of one Dirac particle in the adjoint representation of $SU(2)$. As one is again considering a negative kinetic energy Lagrangian, the central charge in the Kac-Moody algebra contributed by I_{WZWFP} is (-1) . All in all, the WZW parts contribute a Kac-Moody central charge $-(n/4+1)$. The resulting central charge C_{WZW} of the Virasoro algebra induced by $I_{WZW} + I_{WZWFP}$ is given by

$$C_{WZW} = \frac{2k \dim}{2k + Q_\psi^{su(2)}} = \frac{-2(\frac{n}{4}+1)3}{-2(\frac{n}{4}+1)+1} = \frac{3(n+4)}{n+2} \quad (9.14)$$

where $Q_\psi^{su(2)}$ is the value of the Casimir in the adjoint representation of $SU(2)$. For a general group s the result would be

$$C_{WZW} = \frac{2(k+Q_4^2) \dim s}{2k - Q_4^2} \quad (9.15)$$

k is the Kac-Moody central charge and c_ψ the value of the Casimir in the adjoint representation. The contribution of I_{FFP} , c^{ghosts} to the Virasoro central charge can be calculated explicitly, however, as $\bar{c}^\mu$ has conformal weight $\lambda=1$ (and c_i has $\lambda=0$) and as they are anticommuting variables ($\epsilon=1$), their contribution to the Virasoro central charge can be calculated using the formula $C^{\text{ghost}} = -\epsilon(12\lambda^2 - 12\lambda + 2)$. This leads to $c = -2$ for each group index for $SU(2)$ the result is

$$C^{\text{ghost}} = -6 \quad (9.16)$$

and, in general,

$$C^{\text{ghost}} = -2 \dim s \quad (9.17)$$

The total central charge, c , is given by

$$C = C_{FF} + C_{WZW} + C^{\text{ghost}} = 2n - \frac{3n}{n+2} = \frac{2n^2 + n}{2+n} \quad (9.18)$$

for $SU(2)_n$ and, in general, by

$$C = 2n - \frac{2k \dim s}{2k + Q_4^2} \quad (9.19)$$

This is indeed equal to the central charge obtained by the coset construction. We have also verified this result in the large n limit where g can be replaced by ϕ^a defined by $g = \exp(i\lambda^a \phi^a)$. Note that the versions of the theories from which respective contributions to the central charge are obtained, are all explicitly conformal and decoupled. $O(4n)$ can be replaced by the appropriate WZW action for other groups g , the central charge associated with the coset construction resulting. The above discussion amounts (after bosonizing the free fermions) to a bosonization of the system. There are several other ways to discuss bosonization of the system.

At first, the free fermion model along with its $O(4n)_R \times O(4n)_L$ symmetry is considered. Such a model is bosonized by describing the motion of a string on a

parallelized $O(4n)_R \times O(4n)_R$ group manifold, that is the bosonic theory is an $O(4n)_R \times O(4n)_L$ non-linear sigma model in the presence of a Wess-Zumino term (with $k = 1$). The partial gauging of the fermionic model by the appropriate subgroup has its counterpart in the corresponding partial gauging of the Wess-Zumino model. As discussed by Witten³⁰⁾, this is possible as long as the gauged subgroup is anomaly free. In all of the vector examples, we discuss H which is anomaly free.

In general, in order to prove the equivalence of two conformal theories, it is not enough to show that their central charges are identical. For $c < 1$, however, due to the finite allowed values of $h(c)$, it is sufficient to show that the values of c are the same and rely on modular invariance to identify the bosonic and fermionic model. Guadagnini, Martellini and Mintchev³¹⁾ consider such conformal invariant theories of the form G/H with H gauged. (They actually add another WZW term instead of using the left-right structure, they prove conformal invariance for $G \times G'/H$ structures where either G/H or G'/H is a symmetric space). They show that up to two loops, the central charge of the gauged WZW model is $c^{G/H}$ obtained by the coset construction [Eqs. (3)-(6)]. As for $c < 1$ the allowed values of c are discrete, the two loop answer has to coincide with the exact answer. We also believe that the result should hold for all values of $c^{G/H}$ obtained. For the explicit form of the gauged WZW model we refer the reader to Refs. 30) and 31). From the bosonic point of view, this is the $o(4n): k = 1$ bosonized description of the $c < 1$ system, a string moving on $(SU(2)_n \times SU(2))_L \times (SU(2)_n \times SU(2))_R$ with a gauged $(SU(2)_{n+1})$ vector or a string moving on any of the above structures multiplied by a G'/H' with $c^{G'/H'} = 0$ ^{32), 26)} are all equivalent.

The tensoring $(n_2, \dots, n_m)$ is thus a theory of at least the global symmetry

$$G(2) \times Z(n_2) \times G(3) \times Z(n_3) \dots \times G(m) \times Z(n_m)$$

and a gauged symmetry $H(2) \times \dots \times H(m)$. This is a bosonization of all $c < 1$ models in particular the $c = \frac{1}{2}$. Indeed, the reader may have some difficulties recognizing the single Majorana fermion of the Ising model. This leads us to the second level of bosonization.

Consider as an example the Schwinger model, it consists of massless fermions coupled by an Abelian gauge field possessing an electric field term. This theory confines charge in two dimensions, the confinement is so successful that only a free massive boson emerges. This is the bosonization of the Schwinger model.

One could rather call it "bound stating" of the theory as it contains explicitly its infra-red degrees of freedom. In the extreme case, they are non-interacting, for the massive fermion case, the bound states include also soliton, antisoliton pairs which have complicated interactions.

The case here are somewhat intermediate, the confinement is obtained by projection of representations [which indeed leads to a Hamiltonian containing four fermi couplings²³⁾]. The infra-red degrees of freedom lose their Kac-Moody quantum numbers, with only their Virasoro structure serving to label them. These, in general, are interacting degrees of freedom. In that respect, the Ising model is rather unique, as the confined states are free massless canonical fermions. As we have discussed in some cases upon tensoring a space picture is resurrected.

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APPENDIX I

Finiteness of the number of different tensored models contributing an integral central charge

Prop.: Let $(c_m)_{m \in \mathbb{N}}$ be a monotonously growing sequence of positive rational numbers with limit c . Then, for any given $D \in \mathbb{N}$, there are finitely many different sequences $(N_m)_{m \in \mathbb{N}}$ of non-negative integers, such that

$$\sum_{m=1}^{\infty} N_m c_m = D \quad (I.1)$$

Proof:

Since (c_m) is monotonous, we have: $\forall m \in \mathbb{N}: c_1 \leq c_m < c$, and thus: $D/c < \sum_{m=1}^{\infty} N_m \leq D/c_1$ for any sequence (N_m) obeying (1). The non-zero members of (N_m) must, therefore, form a partition of one of the integers in the interval $[D/c, D/c_1]$. There is only a finite number of such partitions, and every admissible sequence (N_m) has only a finite number P of non-zero members. The set of these numbers P is bounded from above by $[D/c_1] + 1$. Let P_0 denote their maximum, and let Ω be the set of all sequences (N_m) obeying (1).

Assumption: $\forall M \in \mathbb{N} \exists m \in \mathbb{N} \exists (N_k) \in \Omega : (m > M) \wedge (N_m \neq 0)$

If this assumption is false, then:

$$\exists M \in \mathbb{N} \forall m \in \mathbb{N} \forall (N_k) \in \Omega : (m > M) \Rightarrow (N_m = 0)$$

There is only a finite number of ways to combine maximally $P_0 < M$ non-zero numbers $N_1, \dots, N_M$ with the first M members of (c_m) in an expression of the type $D = \sum_{i=1}^M N_i c_i$. This is evident from the fact that the integer $D \cdot p$, where p is the smallest common denominator of $c_1, \dots, c_M$ [here we use rationality of the sequence (c_m)], has only a finite number of integer partitions. Thus, if the assumption is false, we have proved our proposition. In the following, we consider the case where the assumption is true. Let $\varepsilon > 0$ be fixed. Since $(c_m) \rightarrow c$, we have:

$$\exists M_\varepsilon \in \mathbb{N} \forall m \in \mathbb{N} : m > M_\varepsilon \Rightarrow |c_m - c| < \varepsilon \quad (I.2)$$

For this particular M_ε , we have with our assumption:

$$\exists n, \varepsilon/N \exists (N_k) \in \Omega : (n_1 > H_\varepsilon) \wedge (N_{n_1} \neq 0) \quad (I.3)$$

Now,

$$D = \sum_{m=1}^{\infty} N_m c_m = N_{n_1} c_1 + \sum_{\substack{m=1 \\ m \neq n_1}}^{\infty} N_m c_m \quad (I.4)$$

Use (I-2) with $\varepsilon c_1/2D$ instead of ε , together with (I.4):

$$|D - N_{n_1} c - \sum_{\substack{m=1 \\ m \neq n_1}}^{\infty} N_m c_m| = |N_{n_1} c_{n_1} - N_{n_1} c| = N_{n_1} |c_{n_1} - c| < N_{n_1} \frac{\varepsilon c_1}{2D} < \varepsilon$$

Thus, for any $\varepsilon > 0$, we can choose $(N_k) \in \Omega$ such as to obtain a ε -approximation of D by the term $N_{n_1} c + \sum_{\substack{m=1 \\ m \neq n_1}}^{\infty} N_m c_m$. The same line of reasoning can be applied to the sequences selected in (3), and over again, at most P_0 times, to yield:

$$\forall \varepsilon > 0 \exists (N_k) \in \Omega \exists n_1, \dots, n_{P_0} \in \mathbb{N} : |D/c - \sum_{i=1}^{P_0} N_{n_i}| < \varepsilon \quad (I.5)$$

But: $\forall (N_k) \in \Omega : D/c < \sum_{m=1}^{\infty} N_m$

For the Virasoro algebra, $c = 1$ and $D+1 < \sum_{m=1}^{\infty} N_m$ which contradicts Eq. (I.5). For the super Virasoro algebra, $c = 3/2$ and $2D/3 < \sum_{m=1}^{\infty} N_m$ which contradicts Eq. (I.5), since all the N_m are integers. Therefore, our assumption must be false and the proof is complete.

APPENDIX II

We show as an example that in all self-tensored models a (1,1) operator appears. For the series $c < 1$ the central charge is given by:

$$C(m) = 1 - \frac{6}{m(m+1)} \quad (II.1)$$

It actually can be written as

$$C(m) = \frac{p-1}{p} \quad (II.2)$$

for $m = 0, 2 \bmod 3$ where $p = m(m+1)/6$, or as

$$C(m) = \frac{q-3}{q} \quad (II.3)$$

for $m = 1 \bmod 3$ where $q = m(m+1)/2$. The integer values of c arising by self-tensoring are $c_p = p-1$ and $c_q = q-3$, the self-tensoring contains a product of p and q models respectively. The (0,0) block does not contain a (1,1) state, other non-primary fields have $h > 1$ so one need consider only the primary fields. They are given by

$$h(h,1) = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}$$

Thus for generic m , candidates for (1,1) operators should have $p-q \approx 1$. For the generic self-tensored series (1,1) operators with $p = q$ can be found. For $m = 0, 2 \bmod 3$

$$m=6 \quad p \cdot h(5,5;5,5) = (1,1)$$

$$m=3 \quad p \cdot h(2,1;2,1) = (1,1)$$

$$m=5 \quad (2h(2,1;2,1) + 3h(3,3;3,3)) = (1,1)$$

For $m = 1 \bmod 3$

$$q \cdot (h(3,3;3,3)) = (1,1)$$

All twisted modular versions retain these operators. These operators, however, generically do contain themselves in their self-operator product expansion. This yields a divergence in the two-point function²²⁾. These operators will thus not draw marginal directions. The twisted model $m = 7$ containing only even values of q nevertheless has a (1,1) operator. It is:

$$4h(6,6;6,6) + h(4,4;4,4) + 23h(2,2;2,2)$$

APPENDIX III

We collect here some of the ingredients which led to the decomposition of $SO(8)$.

1. The (0) sector

The ground state in the (0) sector is obviously decomposed in terms of only singlets of the gauged subgroup.

The first excited state also contributed to the ground state of a representation of the subgroup. The N.S. operators $\psi^{i,\alpha\beta} = (\psi^{i,\mu_\sigma})^{\alpha\beta}$ ($i = 1, 2$) have $(\frac{1}{2}, \frac{1}{2}, 0)$ and $(\frac{1}{2}, 0, \frac{1}{2})$ quantum numbers under $(SU(2)_2, SU(2)^a, SU(2)^b)$. The reason is that β transforms under $SU(2)_2$ for both $i = 1$ and 2 . The index α is acted upon by $SU(2)^a$ for $i = 1$ and $SU(2)^b$ for $i = 2$. Their product $\psi_{-\frac{1}{2}}^{i,\alpha\beta} \psi_{-\frac{1}{2}}^{i',\alpha'\beta'}$ creates the excited $L_0 = 1$ state.

From the 32 states $(\frac{1}{2}, \frac{1}{2}, 0) \times (\frac{1}{2}, 0, \frac{1}{2})$ and $(\frac{1}{2}, 0, \frac{1}{2}) \times (\frac{1}{2}, \frac{1}{2}, 0)$ only 16 remain because of the antisymmetrization they are $(1, \frac{1}{2}, \frac{1}{2})$ and $(0, \frac{1}{2}, \frac{1}{2})$. They are given by $\psi_{-\frac{1}{2}}^{1,\alpha\beta} \psi_{-\frac{1}{2}}^{2,\alpha'\beta'} |0\rangle$ with β and β' symmetrized in the first case and antisymmetrized in the second. The states $(\frac{1}{2}, \frac{1}{2}, 0) \times (\frac{1}{2}, \frac{1}{2}, 0)$ formed by $\epsilon_{\beta\beta'} \psi_{-\frac{1}{2}}^{1,\alpha\beta} \psi_{-\frac{1}{2}}^{1,\alpha'\beta'}$ and $\epsilon_{\beta\beta'} \psi_{-\frac{1}{2}}^{2,\alpha\beta} \psi_{-\frac{1}{2}}^{2,\alpha'\beta'}$ are $(0, 1, 0)$ and $(0, 0, 1)$. These are not allowed ground states for $SU(2)$ level 1 representations and are indeed as seen by the explicit construction $SU(2)^a$ and $SU(2)^b$ excitations T_{-1}^i and as such join the tower built upon the $\ell = 0$ state in $(0, 0, 0)$. It should not be counted as a new source of a Virasoro representation. The $(1, 0, 0)$ states are formed with $\epsilon_{\alpha\alpha'} (\psi_{-\frac{1}{2}}^{1,\alpha\beta} \psi_{-\frac{1}{2}}^{1,\alpha'\beta'} \pm \psi_{-\frac{1}{2}}^{2,\alpha\beta} \psi_{-\frac{1}{2}}^{2,\alpha'\beta'}) |0\rangle$. The combination with a plus sign is an $SU(2)_2$ excitation and should not be counted, the combination with a minus sign has an allowed spin one ground state of $SU(2)_2$. This is a new ground state upon which a new tower of the Kac-Moody algebra will be constructed. It gives rise to a new Virasoro representation. In the 0 sector the next value of L_0 is two while $L_0^{\max} \leq 3/2$ so finally $0 = (0, 0, 0) \times (0) + (1, \frac{1}{2}, \frac{1}{2}) \times (0) + (0, \frac{1}{2}, \frac{1}{2}) \times (\frac{1}{2}) + (1, 0, 0) \times (\frac{1}{2})$.

2. The (v) sector

The $L_0 = \frac{1}{2}$ vacuum contains by construction the states with the quantum numbers of $\phi^{1\mu}$ and $\phi^{2\mu}$. As the angular momentum is half-integer, q can only obtain the value 2 and h the value 1/16, therefore $L_0^{\max}$ in the v sector is 1/16+7/8, the next level in v has $L_0 = 3/2$ and thus need not be considered.

3. The Ramond sector

In the Ramond sector the generators of $(SU(2)_2, Sp(1), Sp(1))$ are respectively

$$\frac{1-\gamma_a}{2} G^a, \frac{1-\gamma_b}{2} G^b, \frac{1+\gamma_a}{2} G^a, \frac{1+\gamma_b}{2} G^b$$

where $\gamma_a = \gamma_1\gamma_2\gamma_3\gamma_4$; $\gamma_b = \gamma_5\gamma_6\gamma_7\gamma_8$. The Ramond vacuum contains the states

$$(1 \pm \gamma_a) U_a \otimes (1 \pm \gamma_b) U_b$$

The total projection is relative to $1 \pm \gamma_9$; $\gamma_9 = \gamma_a \gamma_b$, the allowed states need to have $\gamma_9 = -1$ that is $\gamma_a = -\gamma_b = \pm 1$.

Thus the states $(1-\gamma_a)U_a \times (1+\gamma_b)U_b$ are $(\frac{1}{2}, 0, \frac{1}{2})$ and the states $(1+\gamma_a)U_a \times (1-\gamma_b)U_b$ are $(\frac{1}{2}, \frac{1}{2}, 0)$. Its $SU(2)_2$ quantum number comes adding the angular momentum. In the sector s, $\gamma_9 = 1$ and from the states $(1+\gamma_a)U_a \times (1+\gamma_b)U_b$, one obtains $(0, \frac{1}{2}, \frac{1}{2})$ and from the states $(1+\gamma_a)U_a \times (1-\gamma_b)U_b$ one obtains $(0, \frac{1}{2}, \frac{1}{2})$ and from states $(1-\gamma_a)U_a \times (1-\gamma_b)U_b$ one obtains the states $(0, 0, 0)$ and $(1, 0, 0)$.

We still need to search the $L_0 = 3/2$ Ramond sector. In that sector, by Eq. (5.1) only the state $(1, \frac{1}{2}, \frac{1}{2}) \times (\frac{1}{2})$ may appear in the decomposition. Such a state may only arise by the application of ϕ_{-1} on the (t) sector as the resulting angular momentum must be an integer. At first it looks like there are only two states: $\phi_{-1}^{1, \alpha\beta} |0\rangle \times (1+\gamma_5)U_1 \times (1-\gamma_5)U_2$ and $\phi_{-1}^{2, \alpha\beta} |0\rangle \times (1-\gamma_5)U_1 \times (1+\gamma_5)U_2$. Defining as before $\phi_{-1}^i = \phi_{-1}^{1, \mu} \sigma_\mu$, $\gamma_{(i)} \equiv \gamma_{(i)}^\mu \sigma_\mu$, $i = 1, 2$, these states can be rewritten as $\text{Tr}(\bar{\phi}_{-1}^1 \gamma_{(1)} \sigma_\alpha) (1-\gamma_5)U_1 \times (1-\gamma_5)U_2 |0\rangle$ and $\text{Tr}(\bar{\phi}_{-1}^2 \gamma_{(2)} \sigma_\alpha) (1-\gamma_5)U_1 \times (1-\gamma_5)U_2 |0\rangle$, where the trace is taken in the σ space. Their sum is the (-1) moment of the $SU(2)_2$ generator given by $\sum_i \sum_m \text{Tr}(\bar{\phi}_{n-m}^i \phi_m^i \sigma_\alpha)$ acting on the ground state (recall $\phi_0^{i, \mu} \equiv \gamma_{(i)}^\mu$). This is just an excited state. The difference of the states leads to the state $(1, \frac{1}{2}, \frac{1}{2})$ and to $h = \frac{1}{2}$.

Table

The decomposition of o, v, s, t of $SO(8)$ in terms of $sp(1) \times sp(1) \times SU(2)_2$

$g^{(i)}$	$L_0(SO(8))$	$(j, \lambda_1, \lambda_2)$	h	p	q
o	0	$(0, 0, 0)$	0	1	1
o	1	$(1, \frac{1}{2}, \frac{1}{2})$	0	1	1
		$(0, \frac{1}{2}, \frac{1}{2})$	$\frac{1}{2}$	2	1
		$(1, 0, 0)$	$\frac{1}{2}$	2	1
v	$\frac{1}{2}$	$(\frac{1}{2}, 0, \frac{1}{2})$	1/16	2	2
		$(\frac{1}{2}, \frac{1}{2}, 0)$	1/16	2	2
s	$\frac{1}{2}$	$(0, 0, 0)$	$\frac{1}{2}$	2	1
		$(1, 0, 0)$	0	1	1
		$(0, \frac{1}{2}, \frac{1}{2})$	0	1	1
s	3/2	$(1, \frac{1}{2}, \frac{1}{2})$	$\frac{1}{2}$	2	1
t	$\frac{1}{2}$	$(\frac{1}{2}, \frac{1}{2}, 0)$	1/16	2	2
		$(\frac{1}{2}, 0, \frac{1}{2})$	1/16	2	2

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FIGURE CAPTION

$c = 1$ conformal models