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OPERATOR APPROACH TO CONFORMAL INVARIANT QUANTUM FIELD THEORIES
AND RELATED PROBLEMS

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Conformal quantum field theory is analyzed from a global point of view. The use of the covering transformations leads to a global decomposition theory in which the basis building blocks are non-local quantum fields which live on each light cone separately. As an application of the new formalism the n-point function of the Ising field theory are explicitly calculated. The global viewpoint also sheds new light on the "unitary problem" of quantum field theory on Riemann surfaces.

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Introduction

In this paper I present an alternative viewpoint on conformal invariant two-dimensional quantum field theory and string theory which I call "global". I propose the thesis that the representation of the covering group of the circular diffeomorphisms: $\widetilde{Diff}(S^1) \times \widetilde{Diff}(S^1)$ is the important structure directly related to the Hilbert-space formulation of the theory.

This picture leads on the one hand to a more concrete realization of "light cone fields" i.e. objects which are (nonlocal) fields dependent on one light cone coordinate. These interpolate between the families of representations of the Virasoro algebra belonging to one value of the central extension constant c . On the basis of such fields the n -point function of the Ising model is calculated.

On the other hand the global point of view is very useful in order to study properties of field theories on Riemann surfaces. For free fields one clearly sees the existence of a distinguished one-dimensional subset resembling a Julia set on which positivity (i.e. "unitarity") holds.

The paper also contains a section on the behaviour away from conformal invariance (around critical points). We have included this mainly, because the formalism developed in the first section allows us to make some amusing observations. For example the set of all local composite fields is larger than in the conformal theory. In addition to those appearing already in conformal theories there are "shadow operators".

Part of the paper is a review of known results but in a more field theoretical language. In the first section we give a field theoretic (functional-integral) derivation and generalization of the Luther-Peschel¹ formula, which was originally obtained by comparing the (lattice) Ising model with the xyz model.

The second section is devoted to a review of the global Möbius-transformation properties of "quasi-primary" fields.² In the third section we finally come to the rôle of the universal covering group of the circular diffeomorphisms. Primary fields² are presented from this global point of view and the existence of "light cone fields" obeying an "exchange algebra" is discussed. As an application, the correlation functions of the Ising model are explicitly constructed. The fourth section contains a brief discussion on the relation of conformal invariant models with integrable quantum field theories.

Finally I come to the operator approach for conformal quantum field theories on Riemann surfaces. Certain submanifolds of an important subgroup of $\widetilde{Diff}(S^1)$ represent the Teichmüller deformations. The operator approach does not contain any reference to a metric, since it relates the flat space field theory and its Virasoro algebra directly to correlation functions on Riemann surfaces. It constitutes in some respect a generalization of the Hawking-Unruh³ construction of temperature correlation functions and horizons by (acceleration) transformations. The "quantization" is done for the flat space conformal theory, afterwards only transformations are carried out.

The Teichmüller deformation parameters correspond to a generalization of the Hawking-Unruh temperature. I have included a quick review of the better known Feynman path integral method because I believe that the most fruitful insight will be obtained by a confrontation of the two approaches.

I. The use of ordinary and chiral Aharonov-Bohm potentials in field theory.

Consider a euclidean functional-integral involving Majorana fermions. The selfadjoint Dirac operator has the form

$$D = \begin{pmatrix} o & -w^+ \\ -w & o \end{pmatrix} = \begin{pmatrix} o & \partial_z \\ -\partial_z & o \end{pmatrix} \quad (I.1)$$

where the Weyl-operator $W = \partial_z = \frac{1}{2}(\partial_x + i\partial_y)$ is the Cauchy Riemann operator acting on spinors (=half-forms). The euclidean action for the massive fermion has the form:

$$S = \int (\chi^+ D \underline{\psi} + iM \chi^+ \underline{\psi}) d^2x \quad (I.2)$$

where the ψ and χ are Grassmann valued. They will be decomposed with respect to a conveniently chosen orthogonal (continuous) basis-functions.

$$\begin{aligned} \underline{\psi}(x) &= \int_n a_n \psi_n(x) \\ \chi^+(x) &= \int_n b_n \psi_n(x) \end{aligned} \quad (I.3)$$

For the measure one takes the product of the Grassmann variables a_n and b_n . The euclidean Majorana condition is:

$$\underline{\psi}^c = C \underline{\psi}^* \quad , \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (I.4)$$

where L is a Lorentz-scalar operator bilinear in the creation and annihilation operators of the free massive Majorana field. For the calculation of the momentum space kernel of L it is sufficient to compare with the Minkowski-boundary value of $G_+(x, y)$:

$$L = \frac{-i}{4\pi} \int d\theta_p d\theta_q \left\{ 2 \operatorname{ctgh} \frac{\theta_p - \theta_q + i\varepsilon}{2} \cdot c_p^+ c_q e^{i(p-q)x} + \operatorname{tanh} \frac{\theta_p \theta_q}{2} [c_p^+ c_q^+ e^{i(p+q)x} + h.c.] \right\} \quad (I.13)$$

In deriving this result we have ignored a subtle and important point. The massive euclidean Dirac operator with boundary condition has one highly normalizable zero mode⁴.

$$\psi_o(x) = \begin{pmatrix} e^{\frac{i}{2}(\phi + \frac{\pi}{2})} & K_{\frac{1}{2}}(mr) \\ e^{-\frac{i}{2}(\phi + \frac{\pi}{2})} & K_{\frac{1}{2}}(mr) \end{pmatrix} \quad (I.14)$$

The K function (=Hankel function for imaginary argument) with index $\frac{1}{2}$ is still L^2 integrable at zero. This eigenfunction is unpaired i.e. it either projects on $\mathcal{H}_+$, the (+) selfconjugate wave functions or on $\mathcal{H}_-$ (for example if one multiplies (I.14) by i). This schizophrenic behaviour justifies to count it either as a C odd state, in which case we obtain the previous formula for μ , or as C even state. In the latter case the determinant of the massive Dirac operator vanishes and one obtains a nonvanishing 1-point function:

$$\int \underline{\psi}(x) [d\underline{\psi}] e^{S_-} = \psi_o(x) = \langle \sigma(o) \psi(x) \rangle \quad (I.15)$$

We called the new object the "order operator" σ . It has a very close relations to μ , because apart from the zero mode, the classical Green's function did not change. The operator-identification is done in a similar way as before with the result

$$\sigma(x) =: \frac{1}{\sqrt{4\pi}} \int (c_P e^{ipx + \frac{i\pi}{4}} + h.c.) d\theta_P \mu(x) : \quad (I.16)$$

The zero mode appearing in this problem should not be confused with a zero mode of the standard Atiyah-Singer type. Rather it is more like a global anomaly in the sense of Witten. It does not lead to " θ -vacua" but rather to a "transmutation of quantum numbers": the $(-1)^F$ multiplicative fermion number of a Majorana fields gets "hooked onto" a boson.

The functional formalism offers two ways to construct multi-point (dis)order correlation functions. Either via massive fermion determinants or through an operator identification with the help of classical Green function and subsequent Wick-contraction techniques^{4,5}. We have chosen the latter construction. Even though we ended up with a rather compact formula in a Majorana Fock space, it would be rather difficult to investigate their massless limit. Establishing the existence of $\lim_{M \rightarrow 0} \hat{\mu}$ and $\lim_{M \rightarrow 0} \hat{\sigma}$ with

$$\hat{\mu} = M^{\frac{1}{2}} \mu, \quad \hat{\sigma} = M^{\frac{1}{2}} \sigma \quad (I.17)$$

is not an easy matter for the following reason. The formula (I.12) and (I.16) become infrared-divergent (and meaningless) and the $\hat{\mu}$ and $\hat{\sigma}$ "leave the Fock-space" i.e.

$$\langle \hat{\mu}(o) \underline{\psi}(x) \underline{\psi}^+(y) \rangle \xrightarrow{M \rightarrow 0} 0 \quad (I.18)$$

A significant computational and conceptual simplification is obtained by "doubling" i.e. introducing a complex Dirac field. In that case one is allowed to work with a more general "quasiperiodic" circulation condition:

$$\hat{\psi}(r, \phi + 2\pi) = e^{i\alpha 2\pi} \hat{\psi}(r, \phi) \quad (I.19)$$

The "doubled Ising model" corresponds to $\alpha = \frac{1}{2}$. The complex spinor bundles with nontrivial winding around zero are topologically trivial: the trivialization is done with the help of Aharonov-Bohm potentials⁴:

$$A_\mu(x; o) = 2\pi \int_{path, o}^{\infty} \varepsilon_{\mu\nu} \delta(x - \xi) d\xi^\nu \quad (I.20)$$

$$D_\alpha = i\gamma^\mu (\partial_\mu + i\alpha A_\mu) \quad (I.21)$$

The corresponding operators μ_α are easily constructed from the classical Aharonov-Bohm Green-function. They are^{4,5}

$$\mu_\alpha =: e^{L_\alpha} : \quad (I.22)$$

$$\begin{aligned} L_\alpha = \frac{\sin\pi\alpha}{2\pi} \int d\theta_p d\theta_q & \left\{ \frac{e^{-\alpha(\theta_p - \theta_q)}}{\cosh\frac{1}{2}(\theta_p - \theta_q)} [e^{i(p+q)x} a_q b_p + h.c.] \right. \\ & - \frac{e^{-\alpha(\theta_p - \theta_q)}}{\sinh\frac{1}{2}(\theta_p - \theta_q + i\varepsilon)} e^{i(p-q)x - i\alpha\theta} a_p^+ a_q \\ & \left. + \frac{e^{-\alpha(\theta_p - \theta_q)}}{\sinh\frac{1}{2}(\theta_p - \theta_q - i\varepsilon)} e^{-i(p-q)x + i\alpha\pi} b_q^+ b_p \right\} \end{aligned}$$

There is a more suggestive way of writing this result:

$$\begin{aligned} \mu_\alpha &= :e^{i\alpha\phi}:, \bar{\psi}\gamma^\mu\psi = \frac{1}{\sqrt{\pi}} \varepsilon_{\mu\nu} \partial^\nu \phi \\ \phi &= \frac{i}{4\sqrt{\pi}} \int d\theta_p d\theta \left\{ \frac{-1}{\sinh\frac{1}{2}(\theta_p - \theta_q - i\varepsilon)} e^{i(p-q)x} [a_p^+ a_q - b_p^+ b_q] \right. \\ & \left. + \frac{1}{\cosh\frac{1}{2}(\theta_p - \theta_q)} [e^{i(p+q)x} a_p^+ b_q^+ + h.c.] \right\} \end{aligned} \quad (I.23)$$

In this way the connection with Mandelstam's⁶ and Coleman's bosonization⁷ becomes close. However the normal products of Coleman cannot be used because it turns singular as a result of "cumulative mass effects"⁸ for $\beta \geq 4\pi$, using the β -parametrization of Coleman. The case of $\alpha = \frac{1}{2}$, i.e. the "doubled Ising" model has very special features. In that case the circulation condition is C-invariant and there exist in addition to the Bohm-Aharonov Green's function

$$G_{\pm\frac{1}{2}}(x, y) = \lim_{\alpha \rightarrow \pm\frac{1}{2}} G_{\alpha}(x, y) \quad (1.24)$$

two other Green functions which are selfconjugate are antiselfconjugate under C. From the eigenfunction representation one concludes

$$G_{\pm} = \begin{cases} \frac{1}{2}(G_{\frac{1}{2}} & +G_{\frac{1}{2}}) \\ \frac{1}{2i}(G_{\frac{1}{2}} & -G_{\frac{1}{2}}) \end{cases} \quad (1.25)$$

For the corresponding operation this yields in the representation (I.23)

$$\mu_D = \frac{1}{2}(:e^{\frac{1}{2}\phi}: + :e^{-\frac{1}{2}\phi}:)$$

$$\sigma_D = \frac{i}{2}(:e^{\frac{1}{2}\phi}: - :e^{-\frac{1}{2}\phi}:) \quad (1.26)$$

Triple dots denote the removal of all vacuum expectation values⁸, not just that of the two point function as in Coleman's work. The first formula describes the doubled disorder operator of the Ising model:

$$\mu_D = \mu^{(1)} \cdot \mu^{(2)} \cdot \text{mixing term} \quad (1.27)$$

Here $\mu^{(i)} = 1, 2$ are the disorder operators of two independent Ising models. The mixing terms do not contain fluctuation terms and turn out to be irrelevant for the calculation of the vacuum-expectation values.^{9,5}

This results from the observation that the C-selfconjugate Green's function of the Dirac problem is twice the Ising Green function or in operator terms the L_D in the operator representation is a sum of two Ising L 's. For the σ_D the argument is more involved. In one of our papers we have verified the quadratic relationship between (I. 27) and (I.26) without the use of functional integration directly via operator identities.⁹ The σ_D as the σ itself carries a even-odd selection rule. Now we are prepared to take the zero mass limits. According to our Bohm-Aharonov formulation we just have to evaluate the Schwinger determinant:

$$\text{Det } D(A_\mu) = \exp - \frac{1}{8\pi} \int \int F(x) G(x-y) F(y) d^2 x d^2 y \quad (\text{I.28})$$

with

$$F = \epsilon^{\mu\nu} F_{\mu\nu} \text{ and } \Delta G(x) = \delta^{(2)}(x)$$

on Bohm-Aharonov configuration of total flux zero.

$$A_\mu^{\text{tot}}(x) = \alpha \sum_{i=1}^n A_\mu(x; x_i) - \alpha \sum_{i=1}^n A_\mu(x; y_i) \quad (\text{I.29})$$

with $A_\mu(x, x_i)$ given by (I.29). For nonvanishing total flux the zero mode of the Atiyah Singer index theorem forces the determinants to vanish. A trivial renormalization obtained by omitting all selfinteraction of the Bohm-Aharonov fluxes yields the result

$$\langle \mu_\alpha(x_1) \dots \mu_\alpha^+(y_1) \dots \rangle = \left(\frac{\prod_{i < k} |x_i - x_k|^2 |y_i - y_k|^2}{\prod_{i,j} |x_i - y_j|^2} \right)^{\alpha^2} \quad (\text{I.30})$$

Combining (I.30) with (I.29) and using

$$\mu_{Ising} = \sqrt{\mu_D}, \sigma_{Ising} = \sqrt{\sigma_D} \quad (I.31)$$

$$i.e. \langle \mu_I \dots \sigma_I \dots \rangle = \langle \mu_D \dots \sigma_D \dots \rangle^{\frac{1}{2}}$$

we have succeed to rederive and generalize a result which many years ago Luther and Peschel¹ obtained on the basis of the relation between the Ising model and the so called xyz model of statistical mechanics. Since this result is of importance for the later sections in this paper, I tried to give a completely field theoretic derivation based on the well known technique of euclidean functional integration. One more comment is in order. I could have completely bypassed the massive model and directly represented the Ising model $\mu\sigma$ - mixed correlation function in terms of square roots of linear combination of Schwinger determinants. In that case we would have to convince ourselves that the $\mu - \sigma$ expectation values of the Ising model and its doubled version are also determinants. This can indeed be shown. The Dirac operator is the free one acting on section with (-) boundary conditions around certain points and with the additional restriction that the wave functions appearing in the Grassmann decomposition of ψ and χ^+ have certain (anti)selfconjugate properties under C near each circulation point.

It is remarkable that these determinants (Pfaffians) are the square roots of a linear combination of Schwinger determinants. The explicit drawing of the square roots leads to Ising correlation functions as sums of products of "light cone correlation functions". The calculation requires a good understanding of conformal invariance and will be done in a later section. There are no Pfaffians for $\alpha \neq \frac{1}{2}$.

There are some more interesting lessons to be learned from the Bohm-Aharonov construction of the μ_α in the massive theory. Whereas the original euclidean functional integral is completely periodic in α with period 1, the resulting operator (I.22) just naively continued from small values to $\alpha > 1$ is only periodic "modulo local terms":

$$\begin{aligned}\mu_\alpha &=: \mu_{\alpha_r} \text{ polyn. } (\psi, \bar{\psi}) : \\ \alpha_r &= \alpha \text{ reduced to the intervall } (0, 1)\end{aligned}\tag{I.32}$$

These "shadow operators" with their "would be" dimension $d = \alpha^2 + \text{integer}$ disappear for $mass \rightarrow 0$ and hence do not appear in the "conformal block" of Belavin, Polyakov and Zamolodchikov². Their existence makes the construction of massive theories from conformal invariant ones (see section IV.) extraordinarily subtle.

In order to obtain a still deeper insight into the power of the Bohm-Aharonov method in the case of massless fermions, let us briefly look at what turns out to be an alternative method: a bosonic (gaussian) model with (imaginary) electric, and magnetic (vortex) charges:*

$$\begin{aligned}\langle \tilde{\mu}_{a_1}(x_1) \dots \tilde{\sigma}_{b_1}(y_1) \dots \rangle &= \int [d\phi] \exp - \int (\partial_\mu \phi \partial^\mu \phi - \phi \cdot q) d^2 x' \\ q(x'; x_1 \dots y_1 \dots) &= i \sum a_i \delta(x' - x_i) + \sum_i b_i \int_{\nu_i}^\infty \varepsilon_{\mu\nu} \partial^\nu \delta(x' - \xi) d\xi^\mu\end{aligned}\tag{I.33}$$

The solution of the classical equation

$$\Delta \phi(x') = q(x'; x_1 \dots, y_1 \dots)\tag{I.34}$$

* This formalism has been (apparently independent) worked out in references (10) and (11). The field theoretic notation used here is that of (11)

"saturates" the functional integral (the remaining fluctuations just integrate to an x_i, y_i' - independent constant). In more careful words: one first solves the problem in a finite volume with (Dirichlet) boundary conditions and performs a trivial renormalization accounting for the infinite selfinteractions of the sources. The thermodynamical limit $V \rightarrow \infty$ only exists in the case of charge neutrality

$$\sum_i a_i = 0 \quad \sum_i b_i = 0 \quad (1.35)$$

There is a subtle connection between the existence of the thermodynamical limit and the positivity of the Wightman-functions. It was known before in the early work of Klaiber¹² that these correlation functions go with a positive definite Hilbert-space; he simply found a representation in which a suitable noncovariant infrared cutoff in the Fock-space operators is cancelled in the final formulae.

The local operators $\tilde{\sigma}$ and $\tilde{\mu}$, which have multivalued mixed euclidean correlation functions, simply fulfil relative space-like duality commutation relations with respect to each other and the dimension of these bosonic fields turns out to be:

$$\dim \tilde{\mu} = \frac{a^2}{4\pi} \quad , \quad \dim \tilde{\sigma} = \frac{b^2}{4\pi} \quad (1.36)$$

The pure $\tilde{\mu}$ or $\tilde{\sigma}$ correlation functions are identical to the previous fermionic ones with

$$\alpha^2 = \frac{a^2}{4\pi} \quad , \quad \beta^2 = \frac{b^2}{4\pi} \quad (1.37)$$

$$\tilde{\mu}_a = \mu_a \quad , \quad \tilde{\sigma}_b = \sigma_b$$

The advantage of this bosonic (gaussian) description is that there are no problems in the construction of the mixed correlation functions with their ramification points in the euclidean region. In

fact, if one defines (leading terms in short-distance expansions):¹¹

$$\lim \mu\sigma(x) = \psi^{a,b}(x) \quad (I.38)$$

one obtains:

$$\dim\psi = \frac{a^2 + b^2}{4n} \quad , \quad \text{spin}\psi = \frac{ab}{2\pi} \quad (I.39)$$

In order to interpret these correlation functions as the "complete bosonization" of a fermionic functional integral, we have to extend our "Bohm-Aharonov-Schwinger" formalism of fermion determinate to " γ_5 -Bohm-Aharonov potentials". The pure γ_5 situation does not give any problem. However if we look at a situation in which simultaneously ordinary and γ_5 potentials are present we run into a subtle difficulty. In order to understand this in a slightly more general context consider the QCD_2 determinant in a form in which it was calculated prior¹³ to Polyakov and Wiegman.¹⁴

$$\begin{aligned} \ln \text{Det} D &= - \int_0^g dg' \int (tr A_+(x) j_-(x, g') + h.c) d^2x \\ A_{\pm} &= A_1 \pm i A_2 \\ J_{\pm}(x, g') &= - \frac{g'}{4\pi} A_{\pm}(x) + \text{non-local fct. } F(A_{\pm}, g') \end{aligned} \quad (I.40)$$

The non-local function F is explicitly representable in terms of a classical object: the Green function of the Dirac operator in the adjoint representation¹³. Looking at (40) one realizes that apart from a local "obstruction" term $A_+ A_-$, the determinant has a holomorphic factorization into A_+ and A_- . For the more general situation of chiral unsymmetry, the coupling constants for A_+ and A_- will be different. This "obstruction" i.e. clash between holomorphy and gauge invariance has been analysed in a very general mathematical setting by Quillen¹⁵. He found a mathematical framework in

which holomorphic Weyl (chiral) determinants can be defined as natural mathematical objects. The issue of "complete bosonization" has nothing to do with gauge invariance. In order to achieve it one has to take Quillen's definition of Weyl determinants. In the older literature it was erroneously claimed that the functional derivation of Witten's (nonabelian) bosonization is obstructed by local counterterms.

We cannot resist to point out one more amusing application of our formalism. Consider the normalizable state (pure imaginary time)

$$| \mu \rangle = \mu(t = i, o) | o \rangle \quad (1.41)$$

where μ is the previously constructed disorder operator in the massless Ising model. The spinor two-point function in this state is:

$$\langle \mu | \psi(u) \psi^+(u') | \mu \rangle = \frac{1}{2\pi i} \frac{u + u'}{u - u' - i\epsilon} \frac{1}{\sqrt{uu'}} \quad (1.42)$$

as compared to the standard vacuum expectation value

$$\langle o | \psi(u) \psi^+(u') | o \rangle = \frac{1}{2\pi i} \frac{1}{u - u' - i\epsilon} \quad (1.43)$$

(42) is called in the string literature the Ramond two point function and the usual free field spinor function (43) carries the name Neveu-Schwarz. In our picture of trying to understand everything from flat space field theory *, there exists only the free spinor field of the textbooks, namely the one which comes from Wigner's analysis of representations of the Poincaré group. An extension of this point of view along the lines indicated in section 4 leads to the "general spin

* This point of view was taken in [16]

structures" on Riemann surfaces. For $g > 0$ one has to consider in addition to (41) the covering aspect of the transformation property of a flat spinor under Fuchsian transformation (special subgroup of Möbius transformations) in order to account for all spin structures as just being different expectation values of the flat "Wigner spinor".

II The Global Aspects of Möbius-Invariance

It is well known that every quantum field theory possesses analytic properties following from spectrum condition and the analytic extension via complex Lorentz-transformation. This natural and schlicht analytic domain which bears the name of Bargman Hall and Wightman¹⁷ may be easily visualized in two dimensions because it can be projected onto the light cone variables $u = t + x$ and $v = t - x$. The natural domain of analyticity of an $n + 1$ -point function:

$$\langle A(x_1) \dots A(x_{n+1}) \rangle \quad (\text{II.1})$$

consists of n points

$$\begin{aligned} u_i &= \xi_1^0 + \xi_i^1 & \xi_i &= x_i - x_{i+1} \\ v_i &= \xi_i^0 - \xi_o^1 \end{aligned} \quad (\text{II.2})$$

which lie in halfplanes. The u -halfplane is rotated by the (complex Lorentz) angle α as compared to the upper halfplane whereas in the case of the v -plane the rotation is opposite.

Locality enlarges this domain in a rather complicated way¹⁷ and the resulting region is only expressible in terms of conditions which couple the u 's with the v 's. In all these domains the expectation values are univalued. Allowing also "dual" fields in two dimensions (as i.e. the (dis)order fields of the Ising model) one finds branch lines or more appropriately "branch hypersurfaces" (since each point consists of two complex variables namely u and v).

Something dramatic happens if we pass to conformal quantum field theories. Before looking at the analyticity, let us first consider the algebraic aspects of Möbius-invariance. Two dimensional Möbius-transformations are generated by supplementing the Poincaré transformations with

dilations and special conformal transformations:

$$x \rightarrow \frac{x - bx^2}{\sigma(b, x)} \quad , \quad \sigma(b, x) = 1 - 2bx + b^2x^2 \quad (\text{II.3})$$

The full Möbius group factors in lightcone coordinates:

$$u \rightarrow \frac{\alpha u + \beta}{\gamma u + \delta} \quad , \quad v \rightarrow \frac{\tilde{\alpha} v + \tilde{\beta}}{\tilde{\gamma} v + \tilde{\delta}} \quad (\text{II.4})$$

$$\text{i.e. } SL(2, \mathcal{R})_L \times SL(2, \mathcal{R})_R$$

referring to left and right movers.

In quantum physics symmetries become unitary operators and the least one has to do, in order to get representation theory into the picture, is to introduce the universal covering of the symmetry group

$$\widetilde{SL(2, \mathcal{R})} \rightarrow SL(2, \mathcal{R}), \quad \pi_1(SL(2, \mathcal{R})) = \mathbb{Z} \quad (\text{II.5})$$

The reason for this structure can be seen most suitably by passing to the compact description:

$$u \rightarrow z = \frac{u - i}{u + i} = e^{i\varphi} \quad (\text{II.6})$$

This complex conformal transformation maps the upper half-plane into the interior of the unit circle in such a way that the real axis is transformed into the circumference with the normalization:

$$(\infty, -1, 0, 1, \infty) \rightarrow (1 + i\varepsilon, i, -1, -i, 1 - i\varepsilon) \quad (\text{II.7})$$

Instead of $SL(2, R)$ one obtains the isomorphic group $SU(1, -1)$. This noncompact group clearly contains a (maximal) abelian subgroup:

$$\hat{K} = \left\{ \begin{pmatrix} e^{\frac{i\alpha}{2}} & 0 \\ 0 & e^{-\frac{i\alpha}{2}} \end{pmatrix}, \alpha \in R \right\} \quad (II.8)$$

which in original language corresponds to

$$K = \left\{ \begin{pmatrix} \cos \frac{\alpha}{2} & \sin \frac{\alpha}{2} \\ -\sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{pmatrix}, \alpha \in R \right\}$$

A transformation with $\alpha = 2\pi$ is clearly the identity as a Möbius transformation (II.4). By "rolling" the angle one obtains an explicit description of $SL(\widetilde{2}, R)$ which is not a matrix group.

There is a very simple physical argument why local quantum field theory is in need of $SL(\widetilde{2}, R)$. This is the so called Einstein-causality paradox¹⁸ of conformal quantum field theory.

Theorem : In any local field theory with "reverberation", i.e. nonvanishing commutators for time-like separations, a transformation law:

$$U(b)A(x)U^+(b) = \left\{ \frac{1}{\sigma(b, x)} \right\}^{d_A} A\left(\frac{x - bx^2}{\sigma(b, x)} \right) \quad (II.10)$$

is inconsistent with causality.

This transformation law is of course the one which one would had naively written down on the basis of the infinitesimal conformal generator commutation relation. To prove the theorem, consider any continuous path that passes once through a zero of $\sigma(b, x)$ respectively through a zero of the denominator in (II.4). Such a transformation links a point which is space-like separated from the origin i.e. $uv < 0$ to a point with a time-like separation $uv > 0$. With the commutator

vanishing for space-like separation one obtains by using (II.8) an outright contradiction with the non-vanishing (reverberation) time-like commutator.

To make a long story short , the correct transformation law can be described in terms of fields $A_n(x)$.^{19,20,21}

$$A_n(x) = Z^n A(x) Z^{n+} \quad (\text{II.11})$$

Z is a nontrivial unitary operator which represents the $\alpha = 2\pi$ transformation . It relates the original field $A(x) \equiv A_o(x)$ to fields in "heaven" ($n > o$) and "hell". The minimal central element Z lies on a one parametric subgroup A whose generator is called the conformal hamiltonian

$$L_o = \frac{1}{2}(P + K) \quad , \quad L_o \geq 0 \quad (\text{II.12})$$

Here P is the light-cone hamiltonian $P = P_o + P_1$, and $K = K_o + K_1$ is the generator of the proper conformal transformations (I.3). This subgroup is similar to angular momentum in that it is compact. To be more precise, it is the covering of the circular group. The situation is like angular momentum but without the univalence requirement of wave function in quantum mechanics. The dramatic aspect is that we have found a hermitean operator with discrete spectrum. The spectrum of this "conformal hamiltonian " which is bounded from below by zero may however be quite irregular unlike that of a bona fide angular momentum operator. In all (up to now) physically relevant cases the mod 1 spectrum is given in terms of a finite set of rational numbers. This means that the "conformal clock" , which measures conformal time, returns to its original position after a time lapse which is given by the smallest rational divisor of the reciprocal spectrum. The first models, which had been investigated in the early seventies from this global viewpoint, also admitted nonrational numbers.¹⁹ They are in modern parlance described by vertex operators. As

a general model-independent statement one obtained in those days the following formula for the spectrum.¹⁹

$$\begin{aligned} \text{spec } Z &= \{e^{-i2\pi\theta}\} \\ \theta &= \frac{1}{2}(d_A + d_C - d_B) \pmod{1} \end{aligned} \tag{II.13}$$

where the d's range over the dimensional spectrum of all operators which communicate through nonvanishing 3-point functions:

$$\langle 0 | C^{d_C} A^{d_A} B^{d_B} | 0 \rangle \neq 0$$

The tacit hypothesis was that these values, which certainly must appear in the Z-spectrum, also exhaust the spectrum. This minimality hypothesis had been checked in models.¹⁹

Here, and in the following, we just looked at the u-part of the theory. The total center contains that of both chiral worlds. The reader can easily write down the combined relations. Note however that a conformal transformation needs not be chirally symmetric. As will be clear later, it can for example change a disorder-field of the Ising model into an order field.

It is now simple to change the description and trade the "heaven" and "hell" fields A_n which are relatively local in each "sheet" in favour of nonlocal but conformally simple (irreducible) fields:

$$\begin{aligned} A_\xi^d(x) &= \sum_n Z^n A^d(x) Z^{+n} e^{i\pi n(d-2\xi)} \\ A^d(x) &= \sum_\xi A_\xi^d(x) \end{aligned} \tag{II.14}$$

In order to get agreement with the notation of the older literature¹⁹, we have shifted the θ by d . The ξ spectrum is to be counted modulo 1. By construction the fields A_ξ^d are irreducible and

transform globally almost as (II.8)

$$U(b)A_{\xi}^d(x)U^+(b) = \left(\frac{1}{\sigma_+(b,x)}\right)^{d-\xi} \left(\frac{1}{\sigma_-(b,x)}\right)^{\xi} A_{\xi}^d\left(\frac{x-bx^2}{\sigma(b,x)}\right) \quad (\text{II.15})$$

with the always understood u-v split version of this formula : $d \rightarrow (d, \tilde{d}), \xi \rightarrow (\xi, \tilde{\xi})$. The global fields are in mathematical language operator valued sections of fiberbundle over S^1 .

Now we return to the expectation values. For the two- and three-point function the contributing ξ values are completely fixed:¹⁹

$$\langle A A \rangle = \langle A_{\xi=d_A} A_{\xi=o} \rangle \quad (\text{II.16})$$

$$\langle C A B \rangle = \langle C_{\xi=d_C} A_{\xi_A} B_{\xi=o} \rangle \quad (\text{II.17})$$

with ξ_A given by :

$$\xi_A = \frac{1}{2}(d_A + d_B - d_C) \mod 1 \quad (\text{II.18})$$

However the 4-point function and all higher functions are reducible under global conformal transformations. The irreducible components are given by the following decomposition:

$$\langle ABCD \rangle = \sum_{\xi} \langle A_{d_A} B_{\hat{\xi}(\xi)} C_{\xi} D_o \rangle \quad (\text{II.19})$$

The sum extends over the ξ -phase spectrum of C whereas the $\hat{\xi}$ values of B are determined by a sum rule following from the Z-invariance of the vacuum. According to our previous arguments these irreducible pieces are certainly analytic in (rotated) u-v halfplanes. In order to obtain their

conformal analytic extension we define a holomorphic semi-group extension of

$$SL(\widetilde{2}, R) = \tilde{K}AN \quad (II.20)$$

In other words one starts from the Iwasawa decomposition with

$$\begin{aligned} \tilde{K} &= \{\alpha, \alpha \in R\} \\ A &= \text{dilation or L - transformation} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \in R - \{0\} \\ N &= \text{light cone translation} = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}, w \in R \end{aligned} \quad (II.21)$$

The complexification is done by allowing λ to be any complex number different from zero and choosing α and w in the upper halfplane. This generates a "holomorphic semigroup" $SL(\widetilde{2}, R)_C$. It projects onto a subset of $SL(2, C)$ but remains essentially different from $SL(2, C)$ itself. The group $SL(2, C)$ turns out to be the invariance group of the correlation functions restricted to euclidean points i.e. $v^* = u$. It is not related to any operator aspect of quantum field theory, whereas the $SL(\widetilde{2}, R)_C$ transformations are represented by (non-unitary) operators. In order to understand their action one must also introduce the universal covering of a pointed plane. Start from an analyticity point of an n -point function $w(u_1 \dots u_n)$ with $Im u_1 > Im u_2 > \dots$. Removing the points of coincidence, one obtains a manifold on which $SL(\widetilde{2}, R)_C$ acts in a natural way²¹. This semigroup can transform one point around another into a different sheet. In other words the heaven and hell picture is maintained in the analytic continuation. The Bargman-Hall-Wightman domain which contains the euclidean points $u = z, v = z^*$ is smaller than that obtained by conformal extension.

It would have been very disappointing if the reduction theory (II.19) only serves to resolve the causality paradox and plays no other useful role. However the existence of "global operator

expansions" shows that this is not so. Their validity was first established in solvable models.²²

Applied to the vacuum state they have the following form:

$$A_{\xi_A}^{d_A}(x) B_O^{d_B}(y) | 0 \rangle = \sum_{C^{d_C}} \int K_{\xi_A}^{d_A d_B d_C}(x - x', y - x') C_O^{d_C}(x') d^2 x | 0 \rangle \quad (\text{II.22})$$

Here the K's are similar to 3-point functions but with more complicated phases:

$$K_{\xi_A}^{d_A d_B d_C}(x_1, x_2, x_3) = C_{ABC} \frac{1}{(-x_{12}^2)^{(\lambda_1, \xi_1)}} \frac{1}{(-x_{13}^2)^{(\lambda_2, \xi_2)}} \frac{1}{(-x_{23}^2)^{(\lambda_3, \xi_3)}} \quad (\text{II.23})$$

with

$$\frac{1}{(-x^2)^{(\lambda, \xi)}} = \frac{1}{(-x_+^2)^{\lambda - \xi}} \frac{1}{(-x_-^2)^{\xi}}$$

when the $\pm$ boundary values are obtained by taking positive or negative imaginary parts in the time-components. The values of the λ 's and ξ 's are:

$$\lambda_1 = \frac{1}{2}(d_A + d_B + d_C) - 1, \quad \lambda_2 = \frac{1}{2}(d_A - d_B - d_C) + 1 \quad (\text{II.24})$$

$$\lambda_3 = \frac{1}{2}(d_B - d_A - d_C) + 1$$

$$\xi_3 = 0 \bmod 1, \quad \xi_1 = \lambda_1 \bmod 1, \quad \xi_2 = -d_C \bmod 1$$

The sum in (II.18) extends over all "quasi-primary" * fields C i.e. fields which are covariant and transform homogenously under Möbius-transformations (II.14). There are always (even in free field theories) infinitely many terms but in contradistinction to Wilson expansions the expansion is convergent rather than asymptotically convergent. Sometimes e.g. if one of the operators is a canonical current or the energy momentum tensors this global (integration over all of Minkowski-space) expansion can also be written in a local form. The global vacuum expansion formula was

* The reason for this terminology becomes clear in the next section.

subsequently proven from field theoretical principles and conformal invariance by Lüscher and Mack²¹. In order to expand a multipoint state, one may apply this formula several times. Away from the vacuum state global expansions cannot have this simple form. Its nontrivial modification is related to the structure of the exchange algebra of the next section.

The aforementioned authors also realized that the expansion (IL22) provides a very powerful tool for the old conformal bootstrap program of Migdal and Polyakov. In an unpublished paper Mack and Lüscher (1978)* obtained some of the restrictions on the strength of the energy momentum tensor (inequalities for the C-values of the Virasoro algebra) which are special cases of the Friedan-Qiu-Shenker results based on Feigin-Fuchs representation theory. Rehren and I recently succeeded** to obtain the results of the Feigin-Fuchs approach in the framework of global expansions for the set of quasiprimary (i.e. Möbius-covariant) fields. The advantage is that unitarity enters very naturally: it is just the reality of 3 point normalization constants. There is no reference to a Kac determinant. The formalism is entirely field theoretic and in spirit very similar to the original Migdal-Polyakov conformal bootstrap.

* This paper is referred to in reference (23)

** Part of this material is contained in K.H. Rehren and B. Schroer "Quasiprimary Fields: An Approach to Positivity of 2D Conformal Quantum Field Theory" FUB preprint (1987).

III Diffeomorphisms of the Circle and Primary Fields

A significant step towards the explicit construction of nontrivial conformal field theories was made at the beginning of the 80's by Belavin, Polyakov and Zamolodchikov. ² Part of this progress consisted in realizing that there exist very fundamental fields, called primary by the authors, which possess a much higher symmetry group than the Möbius-group. Consider a conformally invariant action i.e.

$$S = \frac{1}{2} \int \partial_\mu \varphi \partial^\mu \varphi = \int \partial_u \varphi \partial_v \varphi du dv \quad (\text{III.1})$$

It is evidently invariant under

$$u \rightarrow f(u), \quad v \rightarrow g(v)$$

i.e. under all diffeomorphisms:

$$\text{Diff}(R) \simeq \text{Diff}(S^1) \quad (\text{III.2})$$

where the equivalence is given in terms of the transformation (II.6). A general field theoretical argument reveals that it is not possible to have an infinite dimensional continuous group which leaves the vacuum invariant. So barring a Nambu-Goldstone symmetry breaking, which certainly does not happen in the above model, the only remaining possibility is that the diffeomorphisms change the vacuum. The standard situation to which we will adhere in the following is the one in which the Möbius-subgroup is the maximal invariance group of the vacuum. According to arguments of the previous section, it is always the universal covering group $\widetilde{\text{Diff}}(R)$ which will be unitarily represented in quantum field theory, however special fields will often only use a finite

covering. Fortunately we do not need any new construction: all the winding in $\widetilde{Diff}(R')$ can be pushed into the Möbius-subgroup or more precisely into the maximal compact subgroup K . Diffeomorphism groups for a physicist are difficult to handle since almost everything he learned about the relation to the Lie-algebras via exponential maps becomes invalid. The only thing which remains is the possibility of approximating every element by an infinite product of one-dimensional subgroups. These subgroups are conveniently taken as:

$$G_n = h_n^{-1} SL(\widetilde{2}, R) h_n \quad (\text{III.3})$$

where h_n is the "standard" n -fold covering transformation

$$z \rightarrow z^n \quad (\text{III.4})$$

which for convenience we wrote in the circular coordinates rather than the light-cone u . The covering transformation (4) is an outer automorphism of $SL(\widetilde{2}, R)$. The unitary representatives of such outer automorphism generate the central extension $\widetilde{Diff}_{cent}$ as we will see below. We introduce the abbreviations:

$$L_o = \frac{1}{2}(P + K), \quad L_{\pm} = \frac{1}{2}(K - P) \pm iD \quad (\text{III.5})$$

where the P and D were already used in the Iwasawa decomposition of the previous section.

Define

$$\hat{L}_n = U^+(h_n) L_1 U(h_n) \quad (\text{III.6})$$

and

$$L_o^{(n)} = U^+(h_n)L_oU(h_n)$$

The $L_o^{(n)}$ all commute with $L_o = L_o^{(1)}$. The irreducibility postulate yields

$$L_o^{(n)} = \frac{1}{n}L_o + c_n 1. \quad (\text{III.7})$$

where the factor $\frac{1}{n}$ is a result of the n^{th} root operation h_n^{-1} in (III.3). The central extension aspect is solely reflected in c_n . With

$$\hat{L}_n = \frac{1}{n}L_n \quad (\text{III.8})$$

and the use of Jacobi-identities one finally obtains the Virasoro algebra.

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0} \quad (\text{III.9})$$

The last term results from the c_n and its n -dependence may be determined by a standard cocycle argument. The subgroup (III.3) generates the extremely important subgroup $\tilde{Q}(S^1)_{\text{cent}}$ of the central extensions of quasisymmetric $\tilde{Q}(S^1)_{\text{cent}}$ diffeomorphism. As expected on the basis of field theoretical arguments the "generator field"

$$T(u) = \frac{1}{8\pi} \left(\frac{1}{u+i} \right)^4 \sum_n L_{-n} \left(\frac{u-i}{u+i} \right)^{n+2} \quad (\text{III.10})$$

has the following commutator:

$$-i[T(u), T(u')] = -\delta'(u - u')(T(u) + T(u')) + \frac{c}{24\pi}\delta'''(u - u') \quad (\text{III.11})$$

Positive definite conformal quantum field theory never represents $Diff(R)$ but always $\widetilde{Diff}(R)_{cent}$. So therefore $T(u)$ cannot be an operator valued quadratic differential form (i.e. a (2,0) tensor) of $Diff(R)$ but rather of $\widetilde{Diff}(R)_{cent}$. This is the same as saying that $c > 0$. One may also call T a Schwarzian tensor because applications of $\widetilde{Diff}(s)$ add a Schwarzian derivative to the standard tensor transformation law.

The most important contribution of Belavin, Polyakov and Zamolodchikov is the recognition of the existence of "primary" fields which do transform as true tensors. In our global setting we must use the irreducible components

$$U(g_n)A_\xi(u, v)U^+(g_n) = |g'_n|^d e^{i\pi(d-2\xi)\varphi} A_\xi(u_{g_n}, v) \quad (III.12)$$

where φ is a step function which depends on u and the n^{th} level quasisymmetric transformation g_n .

The correct phase in the pre-factor for a general element of $\widetilde{Diff}(R)$ can be worked out by approximating that element as an infinite product

$$g = \lim_{N \rightarrow \infty} \prod_1^N f_n \quad (III.13)$$

The field $J(u) = \partial\varphi$ of the model (III.1) would be an example. The vertex operators which I discussed in the previous section also turn out to be primary fields. They are more interesting than $J(u)$ because they are reducible under global conformal transformations. The quantum mechanics of the zero mode determine the charge decomposition

$$V =: e^{i\alpha\varphi} := \sum_n e^{i\alpha\varphi^{(+)}} e^{i\alpha\varphi^{(-)}} e^{n\ell n u} \left(e^{i\alpha q} \right)_{n+1, n} \quad (III.14)$$

The phase factor in the global transformation results from the quantum mechanical (zero mode) contribution in

$$L_0 = p^2 + \text{oscillator contribution} \quad (\text{III.15})$$

The charge decomposition can be easily rewritten as a ξ decomposition^{19,20}. Unfortunately vertex models are still trivial regarding the structure of their correlation function. The resulting ξ -sum (I.14) in such models just degenerates to one term. Later we will construct the fields of a nontrivial model.

Most readers are familiar with the development of conformal quantum field theories since the work of BPZ. A systematic investigation based on the representation theory of the Virasoro-algebra as developed by Feigin and Fuchs revealed the existence of special families of "sectors" of representations for $c < 1$ with a finite rational spectrum of $L_0 \bmod 1$ given by the Kac formula. Among those representation families there exists a subset with c -values given by

$$c = 1 - \frac{6}{(n+1)(n+2)} \quad n = 1, 2, \dots \quad (\text{III.16})$$

which are the only ones qualifying as unitary, a fact pointed out first by Friedan, Qiu and Shenker²⁴. The unitarity issue for $c < 1$ was to some extent settled by Goddard and Olive²⁵ via the use of current-algebra techniques. The characters of these representation were calculated by Rocha-Caridi²⁶. Cardy and later Capella, Itzykson and Zuber found relations between the characters of different representation sectors belonging to one c -value ("modular relations"). Since formally partition functions defined by euclidean Feynman integrals on tori are independent of the lattice construction of these tori, they were motivated to construct modular invariant models of partition functions as bilinears in the Rocha-Caridi characters of the u and v representations.

By now there exists a (almost) complete classification of modular invariant partition functions with interesting interrelation with the representation theory of the $SU(2)$ loop algebra and finite dimensional Lie groups.²⁸ For many "would be" partition functions in the classification table, candidates for models at critical points in statistical mechanics have been found. Christe and Flume²⁹ showed that a similar analysis can be performed for the 4-point-functions of local quantum fields. Their approach is based on differential equations² resulting from the Kac-Moody and Feigin-Fuchs representation theory as well as on the Coulomb gas picture of Fateev and Dotsenko^{30,31}. It would be very desirable to show a more direct relation between these two methods.

We would like to add here a still different approach which we call the "representation theory of exchange algebras" and which fits in nicely with our global point of view. To motivate this approach, let us return to the vertex operators (III.13). They obviously fulfil commutation relations:

$$\begin{aligned} : e^{i\alpha\varphi(u_1)} :: e^{-i\alpha\varphi(u_2)} &= e^{-i\pi\alpha^2} : e^{-i\alpha\varphi(u_2)} :: e^{i\alpha\varphi(u_1)} : \\ : e^{i\alpha\varphi(u_1)} :: e^{i\alpha\varphi(u_2)} &= e^{i\pi\alpha^2} : e^{i\alpha\varphi(u_2)} :: e^{i\alpha\varphi(u_1)} : \end{aligned} \quad (\text{III.17})$$

and the complex conjugate relations. They are primary fields and relate, by iterative application, the vacuum sector of the $c = 1$ Virasoro-algebra to infinitely many other representation sectors. Note that the phase appearing in the above algebra is half the monodromy phase which appears in the vertex correlation function if one point circulates around by 2π . Fairly straightforward arguments * show that there is only one representation of the vertex algebra by operators in (positive) Hilbertspace if one requires the existence of vacuum states. This is of course the one in Fock-space in terms of oscillators and quantum mechanics. Therefore the "interpolating fields"

* This was explained to me by D. Buchholz.

between the different representation sectors of the Virasoro algebra for a given value of c may have simpler, more universal properties than the energy momentum tensor; their algebra may, as in the vertex case, have a unique irreducible positive definite vacuum representation. Assume for the moment that the Ising field theory could be built up from primary light cone fields $a(u)$ of dimension $\frac{1}{16}$. Our global analysis (I.14) would tell us for the disorder operator:

$$\mu(u, v) = a(u)a(v) + h.c. + \dots \quad (\text{III.18})$$

where $a(u)$ is the only operator which applied on the ket-vacuum gives a nonvanishing vector. The same is true of $a^+(u)$ with respect to the bra-vacuum. Therefore the 4 point-function ($u_1 > u_2 > u_3 > u_4$).

$$\langle 0 | a^+(u_1)a(u_2)a^+(u_3)a(u_4) | 0 \rangle \neq 0 \quad (\text{III.19})$$

But this cannot be all the story. In addition to (now indicating global phases):

$$a = a_{\xi=0}^{d=\frac{1}{16}} \quad (\text{III.20})$$

and

$$a^+ = a_{\frac{1}{16}}^{d=\frac{d}{16}}$$

there must exist an operator b with dimension $\frac{1}{16}$ and a phase $\xi = -3/8 \text{ mod } 1$. This is a consequence of

$$\langle 0 | m(x_1)\mu(x_2)\mu(x_3) | 0 \rangle \neq 0 \quad (\text{III.21})$$

where $m(x) = \bar{\psi}(x)\psi(x)$ is the (Majorana) mass operator with $d = (\frac{1}{2}, \frac{1}{2})$. The b and its hermitean adjoint annihilate the vacuum and hence

$$\langle o | a^+(u_1)b^+(u_2)b(u_3)a(u_4) | o \rangle \neq 0 \quad (\text{III.22})$$

is the only non vanishing 4-point function involving b 's. Since there is no more sector to be reached by two μ 's, there exists no reason for introducing further light-cone operators. To simplify the discussion of the relation of (III.18) and (III.21) one conveniently splits off a "vertex-factor" V which accounts for the dimension of the 4 point function:

$$\langle o | a^+(u_1)a(u_2)a^+(u_3)a(u_4) | o \rangle = V(u_1 \dots u_4)f(x) \quad (\text{III.23})$$

here

$$V(u_1 \dots u_4) = \langle o | e^{-i\varphi(u_1)} e^{i\alpha\varphi(u_2)} e^{-i\alpha\varphi(u_3)} e^{i\alpha\varphi(u_4)} | o \rangle$$

where

$$\alpha = \frac{\sqrt{2}}{4}, \quad \langle \varphi(u)\varphi(u') \rangle = -\ln(u - u' - i\varepsilon)$$

$$\text{i.e.} \quad \dim e^{i\alpha\varphi} = \frac{1}{16}$$

and

$$x_{1234} = \frac{(u_1 - u_2)(u_3 - u_4)}{(u_1 - u_3)(u_2 - u_4)} \quad (\text{III.24})$$

is the conformal invariant crossed ratio of four points. We call f the reduced 4-point function . The same vertex factor is extrated from (III.21) and the reduced function is called $g(x)$. Both functions have the large ramified analyticity domain of conformal field theory and we assume that for the chosen ordering of points we are outside the cuts. These functions must be related since otherwise the introduction of b would have served no purpose and the only possible relation is through analytic continuation yielding a monodromy matrix. We take the simplest possibility:

$$(M_{32}f)(x) = g(x) , \quad (M_{32}g)(x) = f(x) \quad (\text{III.25})$$

M_{32} denotes the monodromy operation which circulates u_3 clockwise once around u_2 . Such a circulation on the vertex-factor just produces a phase and without loss of generality we just look at its action on the reduced functions. The monodromy of u_3 around u_4 must be diagonal and since in the Ising model the square of all monodromies should be one, we write

$$M_{32} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$M_{34} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\text{III.26})$$

Choosing the relative (-) sign in M_{34} in this way, we obtain agreement with the monodromy relation of BPZ ² . The exchange of two adjacent coordinates is equivalent to a π circulation. For this operation we take the square root of the matrices. Hence the desired algebra is³² with

$$\sqrt{M_{32}} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-\frac{i\pi}{4}} & e^{\frac{i\pi}{4}} \\ e^{\frac{i\pi}{4}} & e^{-\frac{i\pi}{4}} \end{pmatrix} , \quad \sqrt{M_{34}} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \quad (\text{III.27a})$$

$$\begin{aligned}
\begin{pmatrix} a(u')a^+(u) \\ b^+(u')b(u) \end{pmatrix} &= \text{vertex phase} \sqrt{M_{32}} \begin{pmatrix} a(u)a^+(u') \\ b^+(u)b(u') \end{pmatrix} \\
\begin{pmatrix} a^+(u')a(u) \\ b(u')a(u) \end{pmatrix} &= \text{vertex phase} \sqrt{M_{34}} \begin{pmatrix} a^+(u)a(u') \\ b(u)a(u') \end{pmatrix} \\
b(u')b^+(u) &= \text{vertex phase} b(u)b^+(u') \quad , u > u'
\end{aligned} \tag{III.27b}$$

Here the first factor is the phase coming from the vertex function. This algebra looks quite strange as compared to algebras in local quantum field theory. It is not an algebra in which two operators are commuted but rather involves an exchange of coordinates only. For this reason the name "exchange algebra" is perhaps most appropriate. In terms of the reduced 4 point functions, the exchange algebra reads:

$$x^{1/4} \begin{pmatrix} f_+(\frac{1}{2}) \\ g_+(\frac{1}{x}) \end{pmatrix} = \sqrt{M_{32}} \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} \tag{III.28a}$$

$$(1-x)^{1/4} \begin{pmatrix} f_-(\frac{-x}{1-x}) \\ g_-(\frac{-x}{1-x}) \end{pmatrix} = \sqrt{M_{34}} \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} \tag{III.28b}$$

$$0 < x < 1$$

where $f_{\pm}, g_{\pm}$ denotes the value above (below) the respective cut. We now write ($A = a + a^+, B = b + b^+$).

$$\mu = A(u)A(v) + B(u)B(v) \tag{III.29}$$

The present operator formulation is more economical than the one of the doubled model. There is no separate σ -operator. Rather the difference between μ and σ is not an absolute one, but

only relative. If a pair of μ 's is suitably "conformally deformed" with the points at the end of the deformation falling on top of the original points the pair of μ 's inside a correlation function is transmuted into a pair of σ 's³². From the monodromy properties of the Ising correlation function we obtain the following representation:

$$\begin{aligned}
\langle \mu\mu\mu\mu \rangle &= \langle \sigma\sigma\sigma\sigma \rangle = V \cdot \bar{V} (f(x)f(\bar{x}) + g(x)g(\bar{x})) \\
\langle \mu\sigma\sigma\mu \rangle &= \langle \sigma\mu\mu\sigma \rangle = V \cdot \bar{V} (f\bar{g} + g\bar{f}) \\
\langle \mu\mu\sigma\sigma \rangle &= \langle \sigma\sigma\mu\mu \rangle = V\bar{V} (f\bar{f} - g\bar{g}) \\
\langle \mu\sigma\mu\sigma \rangle &= \langle \sigma\mu\sigma\mu \rangle = V\bar{V} i(f\bar{g} - g\bar{f})
\end{aligned} \tag{III.30}$$

$V = \text{vertex factor}$

For the doubled Ising correlation function we invoke our generalization of the Luther-Peschel formula derived in the first section which in the zero mass limit are described in terms of the bosonic Wick-ordering:

$$\mu = \frac{1}{2} (: e^{\frac{i}{2}\varphi} : + : e^{-\frac{i}{2}\varphi} :), \quad \sigma = \frac{1}{2i} (: e^{\frac{i}{2}\varphi} : - : e^{-\frac{i}{2}\varphi} :)$$

$$\langle \varphi(x)\varphi(x') \rangle = -\ln(x - x' - i\varepsilon)^2 \tag{III.31}$$

The equality of the squared Ising-correlation with the doubled model yields for the reduced part:

$$(f\bar{f} \pm g\bar{g})^2 = 1 \pm \sqrt{x}\sqrt{\bar{x}} + \sqrt{1-x}\sqrt{1-\bar{x}} \tag{III.32a}$$

$$(f\bar{g} \pm g\bar{f})^2 = 1 \pm \sqrt{x}\sqrt{\bar{x}} - \sqrt{1-x}\sqrt{1-\bar{x}} \tag{III.32b}$$

By solving with respect to f we obtain the formula of BPZ:²

$$f(x) = \sqrt{2}\sqrt{1 + \sqrt{1-x}} \quad , \quad g(x) = \sqrt{2}\sqrt{1 - \sqrt{1-x}} \quad (\text{III.33})$$

The exchange algebra (27) permits to extend this machinery to the n -point function i.e.

$$\langle a^+(u_1)a(u_2)a^+(u_3)a(u_4)a^+(u_5)a^+(u_6) \rangle = V f_1(x, y, z) \quad (\text{III.34})$$

$$x = u_{1234}, y = u_{3456}, z = u_{1256}$$

similarly:

$$\langle a^+ a a^+ b^+ b a \rangle = V f_2(x, y, z)$$

$$\langle a^+ b^+ b a a^+ a \rangle = V f_3(x, y, z)$$

$$\langle a^+ b^+ b b^+ b a \rangle = V f_4(x, y, z) \quad (\text{III.35})$$

There are 14 different mixed correlation functions i.e.

$$\langle \mu \mu \mu \mu \mu \mu \rangle = V(f_1 \bar{f}_1 + f_2 \bar{f}_2 + f_3 \bar{f}_3 + f_4 \bar{f}_4) \quad (\text{III.36})$$

$$\langle \mu \mu \mu \sigma \mu \sigma \rangle = V(f_1 \bar{f}_2 + f_2 \bar{f}_1 + f_3 \bar{f}_4 + f_4 \bar{f}_3)$$

etc.

Entirely analogous manipulations using the square of these functions and comparing them with

the extended Luther-Peschel correlation functions resulting from (III.32) give

$$f_1 = \sqrt{1 + \sqrt{1-x}\sqrt{1-y} + \sqrt{1-x}\sqrt{1-z} + \sqrt{1-y}\sqrt{1-z}} \quad (\text{III.37})$$

The extension of this analysis to the n point function is mainly a matter of clever notation. For the result, which is rather elegant, we refer to reference [32].

The only remaining issue is the question of (Wightman-)positivity. Unitary conformal transformations never transform an a into a b and therefore one may ask whether the a field by itself really belongs to a Hilbertspace. A global conformal transformation from the center changes the relative phase between a^+ and b . Hence there exist linear combination of μ and $Z\mu Z^+$ in which the b contributions cancel. Therefore operators as $a(u) a(v)$ can be readily extracted in the positive definite Ising model. Since u and v can be varied completely independently, the pure a -expectation values must obey positivity. I believe that the positive representation of the exchange algebra provided by the Ising model is the only positive vacuum representation of eqn. (III.27).

In this treatment of $\widetilde{Diff}$ in quantum field theory we have not used the fact that the quantum mechanical diffeomorphism group automatically appears as part of the automorphism group of loop algebras. This structure $*$ which is responsible for the simplification of the partition function and 4-point functions²⁸ clearly should be added into any systematic program of the construction of exchange algebras and their representation theory. It is instructive to compare the global method with the statistical mechanics method of euclidean field theory. The euclidean subset $u = z, v = z^*$ is a two dimensional subset of the analytic domain. The symmetry group of the euclidean functions $SL(2, C)$ is not related to symmetry operators in the Hilbertspace. It does not reveal the sheet

* Since the automorphism group of the loop algebra of $SU(2)$ already contains $Diff(S^1)$, it is sufficient to look at this special situation [33]

structure of right- and left moving "hells and heavens". The most convincing argument is the possibility of conformal deformation of order into disorder operators, which the euclidean approach does not reveal.

IV. Behaviour "near" Conformally Invariant Field Theories.

Experimentally (in solid state physics) and also on purely theoretical grounds one is interested in models near scale invariant fixed points. The best-studied model in this respect has been the massive Thirring model and its Sine-Gordon reformulation. It had been known for some time that at certain values of the coupling or of the β -parameter in Coleman's⁷ language:

$$\beta^2 = 4\pi \quad \text{more generally}^* \quad \beta^2 = \frac{8\pi}{1 + \frac{1}{\nu}}, \quad \nu = 1, 2, \dots \quad (\text{IV.1})$$

special mass singularities appear. These β values coincide with the c-quantization in (III.16). These "cumulative" mass effects represent some sort of phase transition and already the lowest value leads to a breakdown of the bosonic Wick ordering (used by Coleman). They are also responsible for logarithmic modifications of the short distance algebra in the first nonleading term e.g. in the Ising model

$$\mu(x)\mu(y) \simeq \langle \mu(x)\mu(y) \rangle \{ 1 + F(x-y) : \bar{\psi}\gamma_5\psi : + \text{higher terms} \} \quad (\text{IV.2})$$

Here F can be computed from formula (IV.1). The method is based on Wick-contraction and is straightforward but tedious. As a general rule, the following picture emerges: only the leading term, which in this particular example is proportional to the unit operator is free of logarithmic correction, already the next-to-leading Wilson coefficient contain logarithms.

A more dramatic effect is the appearance of "shadow" primaries which have no place in the conformal invariant limit. For example for the product μ with ψ we find:

$$\mu(x)\psi_1(y) \simeq \frac{1}{(u_x - u_y - i\varepsilon)^{1/2}} \sigma(x) + : \mu(x)\psi_1(x) : \quad (\text{IV.3})$$

* These values also appeared in the S-matrix approach of Bogoliubov, Korepin and Izergin [34].

The “would be dimension” of the second $s = \frac{1}{2}$ composite field is $\frac{1}{8} + \frac{1}{2}$, a number which does not appear in the table of critical indices for $c = \frac{1}{2}$. Indeed:

$$\begin{aligned}
& \langle : \mu(x) \psi_1(x) : : \mu(y) \psi_1(y) : \rangle = \langle : \underbrace{\mu \psi_1} : : \mu \psi_1 : \rangle \quad (\text{IV.4}) \\
& + \langle : \underbrace{\mu \psi_1} : : \mu \psi_1 : \rangle \\
& = \langle \mu \mu \rangle \frac{i}{u - u' - i\varepsilon} - \left(\frac{e^{-i\pi/4}}{(u - u' - i\varepsilon)^{1/2}} \right)^2 \langle \mu \mu \rangle + \ell n m(u - u') \langle \mu \mu \rangle
\end{aligned}$$

The leading power behaviour completely cancels and the operator starts with nonleading behaviour containing logarithms. This is the reason why this operator can never appear in a scale invariant theory. It has to be constructed as a composite after the mass perturbation. A deeper model independent understanding of such objects would be highly desirable.

Many integrable systems with critical points have been realized recently. Apart from the Ising model, correlation functions have not been calculated. Fortunately there are simpler quantities than correlation functions, namely partition functions on a strip or on a torus. According to Blöte, Cardy and Nightingale³⁵ and Affleck³⁶ the value of the central charge as well as the dimensional spectrum of the field content can be seen from the asymptotic behaviour near T_c . The behaviour is continuous, but is expected to be of the form $(T - T_c)^2 \ell n^b(T - T_c)^2$ similar to the mass correction of nonleading term in the field theory.

In this connection a recent publication of de Vega and Karowski³⁷ is very relevant. They were able to determine all the quantized c -values from certain integrable six-vertex models modified by a “seam” (in order to account for the family of Potts- and SOS-models within the language of the

6-vertex model). More recently Karowski³⁸ obtained a dimensional spectrum which contains the Kac values for those central charges. The possible influence of shadow operators on such scaling limits is at the moment unclear.

Another interesting problem is the question whether global operator expansion with modified kernels survives in the neighbourhood of a scale invariant fixed point. Massive free fields do have global expansions and by using the techniques based on fermionic Wick-ordering used in references,⁵ one can show that this is also the case for the Ising model and the Federbusch model.³⁹ In more general complicated models as e.i. the massive Thirring- or Sine Gordon-model one seems to be able to control the removal of a certain natural ultraviolet and infrared cut off in the mass perturbation theory.⁴⁰ With such techniques one may hope to describe the mass perturbation and investigate the validity of global operator expansion. In the case of the Ising model it is easy to see that the mass term destroys the continuous diffeomorphism including the Möbius subgroup. However the diagonal subgroup of the center of the covering group

$$Z_{diag.} = \{a \times a^{-1}, a \in Z_u, a^{-1} \in Z_v\} \quad (IV.5)$$

remains unbroken. This is consistent with $\langle \mu \rangle \neq 0, \langle \sigma \rangle = 0$, and the observed modular invariance. This survival of a discrete subgroup of $\widetilde{Diff}(S^1)_{cent} \times \widetilde{Diff}_{cent}(S^1)$ is, on a more profound level, the explanation why even though $Diff(S^1)$ is broken, modular invariance remains intact. In the Ising model the preservation of modular invariance has been noted recently by Itzykson and Saleur⁴¹. The main reason for including this section into this (perhaps already overloaded paper) is to point out that any attempt to obtain massive integrable field theories from conformal invariant ones has to face the challenge of explaining the changes in the "algebra of

fields" (the appearance of shadows). Introducing masses is not just "dressing" conformal covariant fields.

V Quantum Field Theory on Riemann Surfaces and the Unitarity Problem

The modern covariant formulation of relativistic strings is based on perturbative Feynman integrals. The g -loop order requires the study of special two-dimensional conformal quantum field theories on Riemann surfaces of genus g and their dependence on Teichmüller parameters. Using the method of Jacobi, physicists have learned⁴³ how to calculate free field genus g correlation functions of arbitrary spin in terms of Jacobi Theta-functions. The price to be paid for this formal elegance was very high (up to now). There is no simple natural relation of this approach to the flat space conformal invariant field theory with its circular diffeomorphism group and the Virasoro algebra. From experience with the $g=1$ case and its temperature formalism, one knows that there exists a 1-dimensional "real" submanifold on which the free field correlation functions for spin $s \neq 0$ fields are positive definite in the sense of Wightman i.e. have positive definite Källen-Lehmann spectral functions. Using the θ -function representation it becomes however quite cumbersome to derive this unitarity statement even in the case $g=1$. In the $g > 1$ case I have not been able to see a generalization of this unitarity structure by just studying the explicit θ -function representation*. There exists however an alternative method which, although not leading (up to now) to such elegant compact formulas, nevertheless has the elegance of a field theoretic method. Let us first revisit the $g=1$ case. Consider first the special case of a $s=1$ field (a current) on a rectangular torus. By averaging over the lattice points (n,m) , I define a new field on the u -light cone:

$$J^\beta(u) = \lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \sum_{n,m}^N J_{(n,m)\beta}(u) \quad (V.1)$$

* I am indebted to R. Schrader for informing me that D. Friedan is working on this problem.

For positive m this is the action of the semigroup of the light cone hamiltonian on a free field. The sum (V.1) ranges over N lattice points with $N \rightarrow \infty$. The two point function

$$\langle J(u)J(u') \rangle = \frac{1}{2\pi i} \frac{1}{(u - u' - i\varepsilon)^2} \quad (V.2)$$

implies:

$$\begin{aligned} \langle J^\beta(u)J^\beta(u') \rangle &= \sum_{n,m} \langle J(u)J_{(n,m\beta)}(u') \rangle = \frac{1}{2\pi i} \sum_{n,m} \frac{1}{(u - u' + n + i\beta m)^2} \\ &= \frac{1}{2\pi i} \frac{1}{(u - u' - i\varepsilon)} + \frac{1}{2\pi i} \sum_{(n,m) \neq (0,0)} \left[\frac{1}{(u - u' + n + i\beta m)^2} - \frac{1}{(n + i\beta m)^2} \right] \end{aligned} \quad (V.3)$$

The last expression has been obtained by symmetric summation in m and represents the Weierstraß p -function restricted to the real light cone variable $z=u$.

The positivity follows either directly from (V.1) by smearing with (anti) analytic test functions or from the form of the Källen-Lehmann spectral function (the Fourier transform of (V.3)):

$$\rho^\beta(p) = \theta(p) p^{\frac{3}{2}} \delta_{per.}(p) \sum_{n \geq 0} (e^{\beta n p} + e^{-\beta n p}) \quad (V.4)$$

Three comments are in order:

(1) Spinors have a nontrivial transformation law under global conformal transformations: they live on a two fold covering of Minkowski-space. Appropriate signs in the sum (V.1) as well as the use of the disorder operator μ as explained at the end of the first section allow to reconstruct all spin structure on the torus. Again this multitude of spin structure should not be blamed on the existence of various spinors in the Hilbert-space. Rather there exists only one operator ψ but it admits several expectation values.

(2) The correlation functions of J^τ for τ complex is not positive definite. However if one adds the conjugate situations $-\tau^*$, positivity is restored. For string theoretic purpose this is sufficient for getting positivity after integration over the Teichmüller parameter τ .

(3) The resulting torus correlation functions have the remaining L_0 -symmetry. The existence of an abelian symmetry implies the positivity on each real line parallel to the light cone line. This means that the (thermal) reference state, which is obtained via the Gelfand-Segal reconstruction of the Hilbert space from the set of positive expectation values, is not a lowest weight state of the symmetry generator.⁴⁴ To be more precise, there are two operators L_0 . One is obtained by integrating the hamiltonian density. This operator is bounded below, but the thermal reference state is not an eigenstate. The symmetry generator on the other hand cannot be obtained in the field algebra. Its construction requires the additional use of the nontrivial commutant of the field algebra, which turns out to be (anti)isomorphic to the field algebra⁴⁴. These two copies exist also for the total Virasoro algebra.

The generalization of this construction to $g > 1$ is conceptually simple but computationally involved. Take a Fuchsian group G , i.e. a discrete subgroup of $SL(2, \mathbb{R})$ generating the particular compact Riemann surface.⁴⁵ Such a group is generated by a finite number of elements. These elements correspond to the fundamental domain which has a hyperbolic polygon as its boundary. Such a polygon can be chosen to touch the light cone. A polygon side which lies on the light cone (i.e. the real axis) is called a free side and the others are inner sides. Inner sides are pairwise equivalent by the generators of the Fuchsian group but free sides remain distinguished. Let us again consider a $s=1$ free field $J(u)$ which has a trivial transformation law under the covering

$SL(\widetilde{2}, R)$. In analogy to the previous case we define

$$J^G(u) = \lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \sum_{g_i \in G}^N U(g_i) J(u) U^+(g_i) \quad (V.5)$$

Now the U 's are true unitary operators in the Fock-space. The N^{th} approximant clearly leads to a positive definite two point function and this state of affairs cannot change in the limit $N \rightarrow \infty$.

The resulting two point-function *

$$\langle J^G(u) J^G(u') \rangle = \sum_{g_i \in G} \langle J(u) U(g_i) J(u') \rangle > 0 \quad (V.6)$$

depends on the particular Fuchsian group. The free side of the polygon lying on original light cone is the real subset on which one has Wightman positivity. It is distinguished and unlike the torus case there exists no hamiltonian which shifts it to other places. The free line represents points which in the hyperbolic metric are at infinity i.e. such a situation describes a noncompact surface. For a compact Riemann surface without boundary, a point on the light cone line can be represented by a sequence of points on the surface, i.e. the distinguished set for $g > 1$ is not a smooth one dimensional submanifold, but rather a quite complicated topological limiting subset with possible Hausdorff dimension larger than one. The field theoretic generation of the Teichmüller parameters proceeds by using a construction similar to section III. Consider special non-Möbius diffeomorphisms of $\text{Diff}(R) \simeq \text{Diff}(S^1)$ generated by the subgroups belonging to the infinitesimal generators L_0, L_n, L_{-n} . These transformations have the form

$$u \rightarrow f_n(u) = (h_n^{-1} \circ A \circ h_n)(u) \quad (V.7)$$

* The sums are only conditionally convergent if the spin is < 2 .

h is a covering transformation which in the compact (S^1) language corresponds to:

$$h : z \rightarrow z^n$$

whereas

$$A = \frac{au + b}{cu + d} \quad (\text{V.8})$$

The result of all three transformations is a bona fide (non-covering) transformation from $\text{Diff}(\mathbb{R})$. In addition to demanding $f(+\infty) = +\infty$ we normalize the f 's at two finite points say $f(0)=0$ and $f(1)=1$. For odd n the only normalized maps of the form (7) are the identity. For $n=4m+2$, the Möbius part consists of just translation, whereas for $n=4m$ one obtains all affine Möbius-transformations. A straightforward calculation reveals that for $u_2 > 0$

$$\frac{1}{c_n} < \frac{f(u_1 + u_2) - f(u_1)}{f(u_1) - f(u_1 - u_2)} < c_n \quad (\text{V.9})$$

where c_n depends on n but not on u_i . Diffeomorphisms with this property are called (normalized) "quasisymmetric" transformation. They form a subgroup of $\text{Diff}(\mathbb{R})$ and possess a deep relation to Beltrami-form and "quasiconformal" transformations⁴⁶. As a result of their normalization they constitute a contractable subset and form the points of the universal infinite dimensional Teichmüller space used by Bers⁴⁶. The smallest c_n is a measure from the distance to a Möbius transformation; $c_n = 1$ is only obtained for a Möbius transformation. The finite dimensional contractible Teichmüller spaces of compact Riemann surfaces are subspaces of normalized quasisymmetric diffeomorphisms fulfilling

$$(fGf^{-1})(u) = \text{Möbius - transformation} \quad (\text{V.10})$$

where G represents the previously introduced Fuchs group. If the set of Möbius-transformations on the right hand side are again elements of the group G , i.e. for automorphisms f of G one obtains the so called modular transformations. Dividing out this discrete group of diffeomorphisms one descends from the Teichmüller space of all "G-covariant" diffeomorphism (V.10) to the Riemann moduli space. It is fairly easy to see that the level n quasimetric transformations of the form (V.7) can never have property (V.10); one needs infinite products

$$f(u) = \prod_n f_n(u) \quad (V.11)$$

The Ansatz (V.11) may be converted into a Virasoro algebra relation by using the fact that the f_n lie on one parametric subgroups. The only finite transformation is such a relation in the Virasoro algebra are the unitary representors of the Fuchsian group G . We hope to return to some explicit algebraic calculation identifying elements of the Virasoro algebra (involving infinite high level elements) with the tangent vectors to the origin of Teichmüller space. It is interesting to note that in pure mathematics there seems to be no simple model of $\widetilde{Diff}_{cent}$ besides the physicist's model of the integrated Virasoro algebra.

Our main purpose in this note was to point out that there exists a thoroughly quantum field theoretical approach to Riemann surfaces and Teichmüller-spaces in which the link with the flat space conformal field theory with its Virasoro generators is essentially used.

In some sense our approach resembles the Hawking-Unruh³ construction of generating new correlation functions with horizons and a Hawking temperature ($\rightarrow$ Teichmüller - parameters?) by doing just (acceleration) transformations on flat space correlation functions. Related mathematical topics are the determination of the commutant algebra for the J^G and the possible connection with so called "modular properties" of the local algebras of quantum field theory⁴⁷.

It is instructive to compare this operator approach with the treatment in the literature based on Feynman integrals. The general, say, right moving spin fields are described by a first order bilinear b-c Lagrangian which contains the Cauchy Riemann operator acting on tensor (spinors) of type λ . Their flat space two point function is a for the case say of spin $\frac{1}{2}$

$$\langle \psi(u)\psi^+(u') \rangle = \frac{1}{2\pi i} \frac{1}{u - u' - i\epsilon} \quad (\text{V.12a})$$

The Feynman-integral on a Riemann surface leads to a two point function with two properties.

(1) It has a singularity at coinciding point which exactly agrees with (6).

(2) It satisfies the free field equation which in holomorphic coordinates reads:

$$\begin{aligned} \partial_{\bar{z}} G(z, w) &= \delta(z - w) + \text{possible zero modes} \\ \partial_w G(z, w) &= -\delta(z - w) + \text{possible zero modes} \end{aligned} \quad (\text{V.12b})$$

An object with these two properties which has no additional poles is uniquely determined and may be written in terms of θ -functions and the "prime differential" $E(z, z')$ ⁴³. The formulas are different for different spin structures. For the spin structure corresponding to $(+, +)$ boundary conditions on the torus it reads:

$$G(z, w) = \frac{1}{E(z, w)} \frac{\theta[\alpha](z - w)}{\theta[\alpha](o)} \quad \alpha = \left(\frac{1}{2}, \frac{1}{2} \right) \quad (\text{V.13})$$

Since the object obtained on the basis of formula (V.6) fulfils the same properties (V.1) and (V.2), the two point functions (V.13) and the one computed à la (V.6) must be identical up to a normalization constant. The energy-momentum tensor T is a composite field and its expectation values may be calculated by a limiting procedure. As for the induced current in the Schwinger model there are two possible definitions:

The analogy with the better known situation of two different induced currents in the Schwinger model facilitates the discussion

(a) A diffeomorphism-covariant tensor which in the analogy to the Schwinger model corresponds to the gauge invariant current of Schwinger obtained by attaching a line integral (parallel-transport) before the limit

(b) a "holomorphic" tensor corresponding to the subtraction of the singularity for coinciding points. In the Schwinger model case this would be the "holomorphic" current which depends only on A_z .

The partition function of chiral spinors on a Riemann surface corresponds to an integration of a variational equation in which the "holomorphic" energy momentum tensor is used. For the details we refer the reader to the literature. We would use the same steps if we were to calculate partition functions from our representation of correlation functions. Of course we expect to obtain the same partition function just in another representation. An educated guess leads one to expect a representation of the Selberg type. In the simplest case of $g = 1$ these different representations for partition functions are known to be related by Poisson resummation. The existence of two different methods, the operator method and the one based on Feynman integrals is expected to be a source of very rich mathematical identities especially if the model is a nontrivial one.

VI. Concluding Remarks and Outlook

In this paper I presented various aspects of conformal field theory from a unified point of view. The "global operator method" is not entirely new. The beginnings of a global decomposition theory for $SL(\widetilde{2}, R)$ date back to the 70's. But only after the pioneering work of Belavin, Polyakov and Zamolodchikov, in which the notion of primary fields and the possibility of having models with a finite set of fields was clearly recognized, it became feasible to convert the old idea into a computational scheme. One gained the insight that the central sectors discovered via the global decomposition theory of the 70's are generated by primary fields. Some of the detailed calculations, in particular the determination of dimensional spectra without the use of the Kac determinant but only using reality of 3 point functions of quasi-primary fields, will be dealt with elsewhere. Also a systematic investigation of the model dependent "exchange algebra" for light cone fields has been deferred to a future publication. In this paper I was able to bypass the issue of positive definite representations of this algebra by using a generalization of the result of Luther and Peschel for the Ising field theory. In the first section I derived this generalization to arbitrary mixed (order-disorder) function.

The metric free construction of a correlation function on Riemann surfaces clearly showed the dominating role of the covering of the circular diffeomorphism group. This group is the Wigner-Weyl symmetry group of conformal quantum field theory. Its simplest description can be given in terms of Möbius-like subgroups obtained by "integrating up" the $L_{\pm n}$ L_0 of all quasisymmetric diffeomorphism $\tilde{Q}(S^1)$. The normalized quasisymmetric diffeomorphism $Q_{norm.}$ form a contractible subspace and constitutes the universal Teichmüller-space used in the work of Bers⁴⁶. I find it somewhat confusing that some of my colleagues introduced their own universal Teichmüller-spaces without telling why they did not use the Bers space. With this universal space one also obtains

a universal modular group. The finite-dimensional Teichmüller-spaces of Riemann surfaces are certain subsets (not subgroups) of normalized quasisymmetric transformation on which a finite modular group acts. As a result of the quantum field theoretical covering aspect, the “quantum” modular group will be a covering of the usual classical one. This explains the phases in the modular transformation properties of the “light cone partition functions” which are related to the characters of the Virasoro algebra. By emphasizing these aspects I want to stress that there is no new physical invariance called “modular invariance”. Everything is already present in the good old flat conformal quantum field theory. Modular invariance is of course a useful consistency requirement in any phenomenological approach and is helpful in classifying models.

String theories are conceptually different from conformal field theories. They are unitary theories with a vanishing Virasoro constant. The covering aspect of the diffeomorphism group becomes irrelevant and the modular forms lose the “quantum phases”. String amplitudes are in some sense true (non-local) diffeomorphism invariants. They have been analyzed in terms of Riemann surfaces with a fixed genus. The universal Teichmüller-space of Bers which has no direct reference to Riemann surfaces is presumably the better conceptual framework. It appears that Mandelstam’s version of string physics⁴⁸ is closely related.

The unitarity situation on Riemann surfaces observed in this paper does not seem to be a prerequisite for string physics since, technically speaking, one only expects unitarity after integration over the Teichmüller deformations. The positivity of correlation functions on subsets and the possibility of having even some sort of Osterwalder-Schrader positivity by attaching a “mirror” Riemann-surface is interesting for the understanding of quantum relativity. It seems that two-dimensional quantum gravity with conformal matter is a solvable problem.

The apparent mathematical richness of the heterotic string⁴⁹ and its recent derivatives⁵⁰ result (in the "bosonic language") from playing quite sophisticated games with quantum mechanical zero modes of a free scalar Bose field. It would be absolutely amazing if free field theory, albeit in the veil of string theory becomes physically relevant at energies of 10^{19} GeV.

A physically more realistic picture has been painted by Polyakov⁵¹. A deeper understanding of the extremely evasive "Liouville field theory" may allow to solve the 3-dimensional Ising model at the critical point in terms of a genuinely interacting string theory with a 3-dimensional target space. Although one does not expect the classically conformal invariant Liouville field theory to have a discrete or even rational L_0 spectrum for generic coupling constants, there may exist quantized values for which just this happens. The proposed values⁵² allow for a 3-dimensional target space. The similarity with the still somewhat mysterious "special quantization" formula (IV.1) for the Sine-Gordon equation may be more than superficial.

Extending "Polyakov's dream" one may speculate that string theories are essential for a non-perturbative understanding of certain (critical) higher dimensional quantum field theories. Looking at the "experimental" data for such models obtained via Monte Carlo computer simulation there is at least some guidance for future analytic research. The hope to find a physical application of string theories in this direction is admittedly far-fetched, but perhaps to a lesser degree than the hope that free fields may explain the physics at 10^{19} GeV.

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References

- [1] A. Luther and I. Peschel, Phys. Rev. **B12**, 3908 (1975)
- [2] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov Nucl. Phys. **B241**, 333 (1984)
- [3] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975)
W.G. Unruh, Phys. Rev. **D14**, 870 (1976)
- [4] B. Schroer, Nucl. Phys. **B210**, 103 (1982)
B. Schroer "Functional Integration for Kinks and Disorder-Variables" in Structural Elements in Particle Physics and Statistical Mechanics, Ed. J. Honerkamp. K. Pohlmeier and H. Römer, Plenum Publishing Col. 1983
- [5] H. Lehmann and J. Stehr, DESY preprint 29 (1976) unpublished
M. Sato, T. Miwa and M. Jimbo, Publication of the Research Institute for Mathematical Sciences, Kyoto University, vol. 15, 871 (1979)
B. Schroer and T. T. Truong, Nucl. Phys. **B144**, 80 (1978), in particular page 102 and Nucl. Phys. **B154**, 125 (1979)
- [6] S. Mandelstam, Phys. Rev. **D11**, 3026 (1975)
- [7] S. Coleman, Phys. Rev. **D11**, 2088 (1975)
- [8] B. Schroer and T. T. Truong, Phys. Rev. **D15**, 1684 (1977)
- [9] B. Schroer and T. T. Truong, Phys. Lett. **72B**, 371 (1978)
- [10] L. P. Kadanoff, J. Phys. **A11**, 1399 (1978)ysics, Boulder Lectures (1967) Gordon and Breach
- [11] E. C. Marino and J. A. Swieca, Nucl. Phys. **B170**, 175 (1980)
- [12] B. Klaiber, Lecture Notes in Theoretical Physics, Boulder Lectures (1967) Gordon and Breach N.Y.
- [13] N. K. Nielson, K. D. Rothe and B. Schroer, Nucl. Phys. **B160**, 330 (1979)
- [14] A. M. Polyakov and P. B. Wiegman, Phys. Lett **141B**, 223 (1984)
- [15] D. Quillen, Funct. Anal. Appl. **19**, 32 (1986)

- [16] D. Friedan, E. Martinec and S. Shenker, Phys. Lett **160B**, 55 (1985)
- [17] R. F. Streater and A. S. Wightman "PCT, Spin and Statistics, And All That", Mathematical Physics Monograph Series Benjamin Inc. 1964
R. Jost, General Theory of Quantized Fields, Americ. Math. Soc. Publication 1963
- [18] M. Hortacsu, R. Seiler and B. Schroer Phys. Rev. **D2**, 2519 (1972)
J. A. Swieca and A. H. Völkel, Commun. Math. Phys. **29**, 319 (1973)
- [19] B. Schroer and J. A. Swieca, Phys. Rev. **D10**, 480 (1974)
- [20] J. Kupsch, W. Rühl and B. C. Yunn, Ann. Phys. (N.Y.) **89**, 115 (1975)
- [21] M. Lüscher, G. Mack, Commun. Math. Phys. **50**, 23 (1976)
- [22] B. Schroer, J. A. Swieca and A. H. Völkel, Phys. Rev. **D11**, 1509 (1975)
- [23] I. T. Todorov, "Infinite Lie Algebra in D=2 Conformal Field Theories" ISAS 2/85 (ITCP Trieste)
- [24] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. **52**, 1575 (1984)
- [25] P. Goddard, A. Kent and D. Olive, Commun. Math. Phys. **103**, 105 (1986)
- [26] A. Rocha-Caridi, in Vertex operators in mathematics and physics, Proc. Conf. Nov. 10-17 (1983) eds. J. Lepowsky, S. Mandelstam and I. M. Singer (Springer 1984)
- [27] J. Cardy, Nucl. Phys. **B270** [FS 16] 186 (1986)
- [28] A. Capella C. Itzykson and J. B. Zuber, Nucl. Phys. **B280**, 445 (1987)
- [29] Ph. Christe and R. Flume, Nucl. Phys. **B282**, 466 (1987)
- [30] V.J. S. Dotsenko and V. A. Fateev, Nucl. Phys. **B240**, 312 (1984)
- [31] V. G. Knishnik and A. B. Zamolodchikov, Nucl. Phys. **B247**, 83 (1984)
- [32] K. H. Rehren and B. Schroer "Exchange Algebra and Ising n-point Function", FU Berlin preprint June 87
- [33] D. Gepner, Nucl. Phys. **B287**, 111 (1987)
- [34] N. N. Bogoliubov, V. E. Korepin and A. G. Izergin, Phys. Lett **159B**, 345 (1985)

- [35] H. W. Blöte, J. D. Cardy and M. P. Nightingale, Phys. Rev. Lett. **56**, 742 (1985)
- [36] I. Affleck, Phys. Rev. Lett. **53**, 742 (1986)
- [37] M. Karowski and H. de Vega, to be published in Nucl. Phys. B.
- [38] M. Karowski "Conformal Properties of Integrable Models" FUB-HEP/87 3
- [39] B. Schroer, T. T. Truong and P. Weisz, Nucl. Phys. **B154**, 140 (1979)
- [40] E. C. Marino, Nucl. Phys. **B251** [FS 13], 227 (1983)
- [41] C. Itzykson and B. Saleur, Saclay PhT/87.01 (1987)
- [42] A. M. Polyakov, Phys. Lett. **103B**, 207 (1981)
- [43] L. Alvarez Gaumé, J. B. Bost, G. Moore, Ph. Nelson and C. Vafa, CERN Th 4660/87 (1987)
E. Verlinde and H. Verlinde "Chiral Bosonization, Determinants and the String Partition
Function" Utrecht preprint 1986
- [44] R. Haag, N. M. Hugenholtz and M. Winnink, Commun. Math. Phys. **5**, 214 (1967)
- [45] H. M. Farkas and J. Kra "Riemann Surfaces", Springer 1980
- [46] L. Bers, "Riemann Surfaces", Lecture Notes, New York University, Inst. of
Inst. of Math. Sciences Lecture Notes 1957
- [47] K. Fredenhagen, Commun. Math. Phys. **97**
- [48] S. Mandelstam, Phys. Rep. **136**, 259 (1974)
- [49] D. Gross, J. Harvey, E. Martinec and R. Rohm, Nucl. Phys. **B256**, 253 (1985)
- [50] H. Kawai, D. Lewellen and S. Tyë, Cornell preprint CLNS 86/751 (1986)
W. Lerche, D. Lüst and A. N. Schellekens, CERN-TH 4590 (1986)
- [51] A.M. Polyakov, Phys. Letters **B103**, 211 (1981)
- [52] J. L. Gervais and A. Neveu, Nucl. Phys. **B147**, 59 (1985)
D. Förster, Nucl. Phys. **B282** 699 (1987)