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STRING LOCALIZATION AND VACUUM ENERGY FINITENESS

A. Casher^{*)}, E.G. Floratos⁺⁾ and N.C. Tsamis

CERN - Geneva

A B S T R A C T

The vacuum energy density of a general closed string model is calculated by considering the theory in a box and is found to diverge. The clear discrepancy with the finite value of the total vacuum energy is attributed to the inability to maintain conformal invariance while localizing the closed string.

*) On leave from the School of Physics and Astronomy, Tel Aviv University, Tel Aviv, Israel.

+) Permanent address: Department of Physics, University of Crete, Iraklion, Greece.

In a theory with a dynamical gravitational field the vacuum energy density is determined by the vacuum expectation value of the energy-momentum tensor. Being a local quantity it is determined by a local operator measurement. On the other hand, the total vacuum energy is a global quantity which can be independently calculated, for instance, from the effective action the theory generates. If the theory has a local interpretation and a translationally invariant vacuum, the energy density will trivially agree with the total energy divided by the space-time volume.

The closed string can describe gravitational phenomena and, at the same time, seems to be free of ultra-violet divergences. It is tempting to calculate a local physical quantity in such a theory since up to now only S-matrix elements have been calculated. Using Polyakov's path integral [1], Polchinski evaluated the one-loop contribution to the cosmological constant and found a finite result [2]. In doing so, Polchinski first evaluated the total vacuum energy and then separated the space-time volume. One would like to obtain the value of the cosmological constant by directly looking at the vacuum energy density of the system. To do so, we must consider the energy momentum tensor of closed string theory. Although the global quantity can be easily written down, it has not yet been expressed as a meaningful integral over space-time points so that we could identify the proper local tensor.

An alternative method is to put the closed string in a box and calculate the energy within the box. In order to localize a string system within a given spatial volume of linear dimension L , the range of variation of the string co-ordinates $\vec{x}(\sigma)$ must be restricted. More precisely, the wave functionals $\Psi[\vec{x}(\sigma)]$ should have a support only on configurations which lie inside the box. An exact treatment of this restricted string system is difficult since the system is no longer harmonic. However, we shall prove that the cost in energy of confining string states to a box is infinite. In fact, we shall show that in finite mass states the probability of finding any point of the string within a finite spatial distance from the centre-of-mass vanishes.

Consider the transverse co-ordinates of a closed string in the light cone gauge:

$$x^i(\sigma) = x^i + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \left[\alpha_n^i e^{-in\sigma} + \tilde{\alpha}_n^i e^{in\sigma} \right] \quad (1)$$

In a state characterized by occupation numbers $\{N_\eta^i\}$:

$$\alpha_{-\eta}^i \alpha_\eta^i | \{ N_m^j \} \rangle = \eta N_\eta^i | \{ N_m^j \} \rangle \quad (2)$$

and mass:

$$M^2 = 4 \sum_{n=1}^{\infty} \sum_{i=1}^{24} n (N_n^i + \tilde{N}_n^i) + \text{constant} \quad (3)$$

we find for the mean square fluctuation:

$$\langle \{N_n^j\} | (x^i(\sigma) - x^i)^2 | \{N_n^j\} \rangle = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} [N_n^j + \tilde{N}_n^j + 1] \geq \sum_{n=1}^{\infty} \frac{1}{2n} \quad (4)$$

Thus, as claimed, the probability distribution of the string co-ordinates relative to the centre-of-mass in any finite mass state is a Gaussian with a diverging width.

The above result may be strengthened by attempting to force the wave functional to have finite support in space. Define:

$$x^i(\sigma) - x^i \equiv \sum_{n \neq 0} x_n^i e^{-in\sigma} \quad (5)$$

and construct a wave functional (A is a normalization constant):

$$\Psi[x_n^i] = A \exp \left\{ - \sum_{n=1}^{\infty} \frac{1}{2} \Omega_n |x_n^i|^2 \right\} \quad (6)$$

such that:

$$\begin{aligned} \langle \Psi | \int_0^\pi \frac{d\sigma}{\pi} (x^i(\sigma))^2 | \Psi \rangle &= \sum_{n \neq 0} \langle \Psi | |x_n^i|^2 | \Psi \rangle \\ &= \sum_{n=1}^{\infty} \frac{1}{2\Omega_n} < \infty \end{aligned} \quad (7)$$

On the other hand, the mass operator is:

$$M^2 = \frac{1}{2} \sum_{n=1}^{\infty} \left[- \frac{\partial^2}{\partial |x_n|^2} + n^2 |x_n|^2 - n \right] + \text{constant} \quad (8)$$

We easily find:

$$\langle \Psi | M^2 | \Psi \rangle = \frac{1}{2} \sum_{n=1}^{\infty} n \left[\sqrt{\frac{n}{\Omega_n}} - \sqrt{\frac{\Omega_n}{n}} \right]^2 + \text{constant} \quad (9)$$

Comparing Eqs. (7) and (9) it is clear that their convergence is mutually exclusive: if M^2 is finite then $\langle x^2 \rangle$ is infinite and vice versa.

The issue raised above is not special to the light cone formulation. In the Euclidean path integral over closed surfaces, confinement to a box implies a limitation on the range of the functionally integrated variables $x^\mu(\sigma_1, \sigma_2)$.

Generally a cut-off functional has to be inserted into the sum over surfaces, thus spoiling the conformal invariance. An example is a Gaussian cut-off:

$$Z = \int [dg_{ab}(\sigma)] [dx^\mu(\sigma)] \exp \left\{ -S - \int d^2\sigma \sqrt{g(\sigma)} \frac{x^\mu(\sigma) x_\mu(\sigma)}{R^2} \right\} \quad (10)$$

where S is the standard two-dimensional conformally invariant action and R the radius of the space-time box. Clearly, Eq. (10) violates the two-dimensional conformal symmetry and the functional integral depends explicitly on the Liouville mode $\ln g(\sigma)$. As a result, a quantity like the cosmological constant will diverge when evaluated from (10).

So far we have been unable to extract a finite answer for the vacuum energy density; this seems to happen only for the total vacuum energy. The key ingredient which makes the latter calculation converge is the modular invariance [3] of the expression. In our attempt to confine the string in a box we explicitly broke this symmetry and, therefore, lost all of its consequences. It is not entirely clear whether a string in a finite-size box and modular invariance are mutually exclusive properties but it appears that this is the case. One could, nevertheless, proceed to study the total vacuum energy in a finite-size Universe. This is not a study of a local measurement since now the box is the Universe itself but it can provide some insight.

As a simple example, consider a torus Universe $M = R \times T^{25}$. The time co-ordinate runs over the real line R while all the space components take on values on a 25-dimensional torus $T^{25} = (S^1)^{25}$. Therefore, the space components of the centre-of-mass momentum of the closed string are quantized:

$$p^j = \frac{M^j}{L}, \quad M^j \in \mathbb{Z} \quad (11)$$

where L is the radius of S^1 . Moreover, the space-time centre-of-mass co-ordinates are:

$$x^\mu = q^\mu + p^\mu \tau + 2N^\mu L \sigma \quad (12)$$

with $q^\mu \in R \times T^{25}$ and soliton winding numbers $N^0 = 0$, $N^j \in \mathbb{Z}$.

In order to calculate the one-loop vacuum energy in such a Universe we have to sum over all closed surfaces with the topology of a torus. This can be accomplished in a variety of ways all of which lead to the following answer [4]:

$$E(L) = L^{25} \int_F \frac{d^2\tau}{4\pi(\text{Im}\tau)^2} e^{4\pi\text{Im}\tau} (2\pi\text{Im}\tau)^{-12} |f(\tau)|^{-48} f_2^{25}(\tau) \quad (13)$$

The integration over the complex modular parameter τ has range:

$$F \equiv \left\{ (\text{Re}\tau, \text{Im}\tau) \text{ s.t. } |\tau| \geq 1, \text{Im}\tau \geq 0, |\text{Re}\tau| \leq \frac{1}{2} \right\} \quad (14)$$

The partition function $f(\tau)$ is given by:

$$f(\tau) = \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) \quad (15)$$

and the finite-radius correction factor is:

$$f_2(\tau) = \frac{1}{L} \left(\frac{\text{Im}\tau}{2} \right)^{1/2} \sum_{M,N} \exp \left[-2\pi i M N \text{Re}\tau - \pi \left(\frac{M^2}{2L} + 2LN^2 \right) \text{Im}\tau \right] \quad (16)$$

Due to the modular invariance of the integrand of the final answer (13), there are no ultra-violet divergences present. In the large radius limit, $L \rightarrow \infty$, the function $f_2(\tau)$, which receives contributions both from the oscillator and the soliton modes, approaches unity exponentially fast and we recover the 26-dimensional flat space result.

The preceding calculations imply that the identification of the string vacuum loop with the cosmological constant is not completely justified. An energy density operator is unavailable within the string formalism for the moment, while any attempt to use the infinite volume limit of a finite system leads to divergences due to a manifest clash between localization and conformal invariance. The latter is clearly exhibited when a torus Universe is examined: modular invariance is maintained by taking into account soliton excitations which wrap around the Universe. The contribution of these non-local and global vacuum fluctuations does not vanish in the infinite radius limit and is responsible for the finiteness of the energy per unit volume.

Finally, we should emphasize that these conclusions hold true for superstrings as well. The localization procedure breaks supersymmetry and induces a splitting of the boson-fermion degeneracy in each supermultiplet. When the length of the box goes to infinity, this splitting tends to zero and supersymmetry becomes exact. Since all superstring theories contain an infinite number of supermultiplets, their finite volume vacuum energy would be expected to diverge. In the case of a torus Universe, however, supersymmetry and modular invariance can be preserved. As a result, the total vacuum energy vanishes due to the presence of global soliton modes. It seems that we have reached the same conclusion as with the bosonic

string: the cosmological constant diverges when computed locally and converges when computed globally in a Universe consistent with conformal invariance.

REFERENCES

- [1] A.M. Polyakov - Phys.Lett. 103B (1981) 207.
- [2] J. Polchinski - Comm.Math.Phys. 104 (1986) 37.
- [3] J.A. Shapiro - Phys.Rev. D5 (1972) 1945.
- [4] J. Schwarz - Physics Reports 89 (1982) 223;
P. Ditsas and E.G. Floratos - Crete-TH-87/2 (1987).