✓ CERN - TH.4829/87

ABSTRACT

We calculate the parity-conserving one-loop string corrections to the bosonic part of the 10-dimensional effective field theory for the heterotic string. There is no renormalization of the Chamseddine-Chapline-Manton supergravity action, but terms of fourth order in the curvature tensors are generated. Most of these are of the same forms as those found at the tree level of the string topological expansion, such as $(T_\gamma R^2)^2$, $(T_\gamma R^2)(T_\gamma F^2)$ and $(T_\gamma F^2)^2$. However, we also find a non-zero $T_\gamma(F^4)$ coupling which was not present at the tree level. We comment on the phenomenological implications of these results, which include absences of renormalization of the Kähler metric and of the gauge kinetic function of the effective field theory in 4 dimensions after compactification, and a modification of the classical equations to be obeyed by the manifold of compactification.

ONE-LOOP STRING CORRECTIONS TO THE EFFECTIVE FIELD THEORY

John Ellis
CERN, Geneva

Philippe Jetzer * and Leah Mirzachi
Département de Physique Théorique,
Université de Genève,
1211 Genève 4, Switzerland

✓ UGVA-DPT 1987/08-544

CERN - TH.4829/87
August 1987

* Supported by the Swiss National Science Foundation.
Address after October 1, 1987: Theoretical Astrophysics Group, Fermilab.,
P.O. Box 500, Batavia, Illinois 60510 USA.

I. Introduction

Even if one accepts that the Theory of Everything (TOE) will turn out to be some form of string theory, it seems that for the foreseeable future physics at accessible energies will be governed by a low-energy effective field theory valid at energies $E \ll m_p$. Considerable effort has been devoted to determining the appropriate form of this low-energy effective field theory in string models. Initial calculations started from the tree level of the string topological expansion. First it was shown how to recover from the heterotic string [1] the Chamseddine-Chapline-Manton [2] form of the $N = 1, D = 10$ supergravity action. Then came corrections which were of higher order in α' and in space-time derivatives of fields [3-7]. Some of these were crucial to the consistency of favoured compactification schemes, such as an $\alpha'/(Tr R^2)$ term [4], and some showed that the classical equations for the manifold of compactification needed to be modified, for example an $\alpha'^{3/4}$ term [5,8]. (See also Ref. [9] for a more complete discussion of this issue). More recently, some preliminary results on one-loop string corrections to the low-energy effective field theory action have been reported. These have included one-loop corrections to the effective action of the massless modes in the open bosonic string [10] and in type II superstring [11], and the Green and Schwarz anomaly cancelling terms [12] resulting from the one-loop parity-violating five-point amplitudes [13]. (See also Refs [12,14] for a general discussion of these terms in n -dimensions).

The main purpose of this paper is to present calculations of the one-loop corrections to the 2-, 3- and 4-point functions of the effective field theory derived from the heterotic string. Partial results have been presented previously [13]. Here we present more details of the calculations, including overall numerical factors, and extend the results to include all terms of fourth order in the curvature tensors of the graviton and gauge fields. As has been reported elsewhere [15], there are no one-loop contributions to the 2- and 3-point functions. These results can be seen by simple counting of fermionic zero modes in vertex operators. They can be extended to show that there is no renormalization of any term in the Chamseddine-Chapline-Manton $N = 1, D = 10$ supergravity lagrangian [2]. This result follows from the vanishing of two integrals [11] which can be expressed as integrations over derivatives of world-sheet Green functions [15]. The first non-vanishing terms in the effective actions are of fourth order in the field curvatures. Most of them have forms similar to those found at the string tree-level : $(Tr R^2)^2$, $(Tr R^2)(Tr F^2)$ and $(Tr F^2)^2$. However, we also find a term of the form $Tr(F^4)$ which did not appear at the string tree-level.

We also discuss some phenomenological consequences of these results. The absence of one-loop corrections to the 2- and 3-point field theory Green functions and to the Chamseddine-Chapline-Manton action tells us that there is no renormalization of the Kähler metric or of the gauge kinetic function characterizing the $N = 1, d = 4$ supergravity theory obtained after compactification. The presence of new terms at fourth order in the field momenta implies that one must modify the classical equations to be satisfied by the manifold of compactification.

The structure of the paper is as follows. We present in Section II the arguments for an absence of renormalization of the 2- and 3-point functions and of the Chamseddine-Chapline-Manton action, and in Section III the one-loop correction to the 4-graviton coupling, which is at most fourth order in the curvature tensors, is calculated. Section IV contains analogous results for the 2-graviton/2-gauge boson couplings, and Section V for the 4-gauge boson couplings. Comments on the phenomenological implications of these results are made in Section VI. Technical details are relegated to 5 Appendices : A to introduce notation, B to classify integrals, C to gather relevant results from Ref. [12], D to prove two important identities, and E to elaborate on some useful elliptic functions.

II. No renormalization of the Chamseddine-Chapline-Manton action

We present in this section the arguments for the absence of renormalization of the Chamseddine-Chapline-Manton action [2] by one-loop string effects. These arguments were reported before [15] and they are repeated here for completeness with some additional technical details.

The completed Chamseddine-Chapline-Manton action describing $N = 1$ supergravity and super-Yang-Mills theory in $D=10$ is given by :

$$I = \int d^{10}x \, e \left[-\frac{R}{2k^2} - \frac{1}{8} e^{-K D/\sqrt{2}} \left(tr F_{\mu\nu} F^{\mu\nu} - \frac{1}{g_{10}^2} tr R_{\mu\nu} R^{\mu\nu} \right) - \frac{1}{2} \partial_\mu D \partial^\mu D \right. \\ \left. - \frac{3}{2} e^{-K D/\sqrt{2}} H_{\mu\nu\gamma} H^{\mu\nu\gamma} + \frac{g^2 k^2}{g_{10}^2} R^{\mu\nu\sigma\tau} \bar{H}_{\gamma\rho\alpha} \tilde{H}^\alpha{}_{\sigma\tau} \right], \quad (2.1)$$

where

$$H_{\mu\nu\gamma} = \frac{1}{3} \left[\partial_\mu B_{\nu\gamma} - \frac{K}{4} (\omega_{3\gamma})_{\mu\nu\gamma} + \frac{K}{4 g_{10}^2} (\omega_{3L})_{\mu\nu\gamma} + c y d. \text{ permutations} \right] \quad (2.2a)$$

$$(\omega_{3\gamma})_{\mu\nu\gamma} = tr (A_\mu F_{\nu\gamma} - \frac{2}{3} \partial_{10} A_\mu A_\nu A_\gamma) \quad (2.2b)$$

$$(\omega_{3L})_{\mu\nu\gamma} = tr (\omega_\mu R_{\nu\gamma} - \frac{2}{3} \omega_\mu \omega_\nu \omega_\gamma) \quad (2.2c)$$

$$\bar{H}_{\mu\nu\varrho} = \frac{1}{3} \partial_\mu B_{\nu\varrho} \quad (2.3)$$

Here $A_\mu, F_{\mu\nu}$ are the gauge potential and field strength, $\omega_\mu, R_{\mu\nu}$ are the connection and curvature of the gravitational field, $B_{\mu\nu}$ is the antisymmetric tensor field and D is the dilaton field.

The vertex operator for the emission of a graviton (and its partners $B_{\mu\nu}$ and D) is presented in (A.1). In calculating the amplitudes, traces over both bosonic and fermionic degrees of freedom have to be carried out. In particular, the trace over the fermionic zero modes S_0 has to be calculated. These zero modes appear in the supersymmetrized right moving momentum operator B_μ appearing in the vertex operator (see (A.2)). However, it is well-known that

$$\text{tr}(R_0^n) = 0 \quad \text{for } n \leq 3, \quad (2.4)$$

where

$$R_0^{ij} = \frac{1}{8} \delta^i_j \gamma^{ij-} S_0. \quad (2.5)$$

and γ^{ij-} is the antisymmetric product of γ^i, γ^j and $\gamma^- = \frac{\gamma_0 - \gamma_3}{\sqrt{2}}$. Therefore, there are no contributions to one-, two- or three-point functions at the one-loop level and contributions to higher point functions contain at least four powers of momenta, resulting from the $\text{Tr}(\frac{1}{2}kR)^4$ factor implied by (A.2) and (2.4). It follows immediately that there can be no renormalization of the $R, \partial_\mu D \partial^\mu D, \partial_\mu B_{\nu\rho} [\partial^\mu B^{\nu\rho} - \omega_{3Y}^{\mu\nu\rho}]$ or $\omega_{3Y} \omega_{3L}$ terms in the action. We would therefore expect that there is no renormalization of any term in the Chamseddine-Chapline-Manton action, as we now check explicitly.

To analyze the one-loop effects on $R_{\mu\nu} R^{\mu\nu} e^{-\kappa D/\sqrt{2}}, \partial_\mu B_{\nu\rho} \omega_{3L}^{\mu\nu\rho} e^{-\kappa D/\sqrt{2}}$ and $\omega_{3L} \omega_{3L} e^{-\kappa D/\sqrt{2}}$ we need a more detailed analysis of one-loop corrections to the 4-graviton interaction. These are in fact of order k^8 in the limit $\alpha' \rightarrow 0$ as we analyse below, whilst $R_{\mu\nu} R^{\mu\nu} \sim k^4, \partial_\mu B_{\nu\rho} \omega_{3L}^{\mu\nu\rho} \sim k^4$ and $\omega_{3L} \omega_{3L} \sim k^6$. Therefore, neither are any of these interactions renormalized at the one-loop level. Finally, the $\omega_{3Y} \omega_{3L}$ interaction does not get renormalized because it is $\sim k^4$, whilst the one-loop interaction between the two gravitons and two gauge bosons $\sim k^6$ when $\alpha' \rightarrow 0$.

To see how potential one-loop contributions to the 4-graviton and 2-graviton/2-gauge boson interactions which appear to be of lower order in k in fact vanish, we discuss now the 4-graviton amplitude in some detail. The one-loop contribution to this amplitude in the heterotic string was calculated in Refs. [11, 16, 17] to be (see Appendix A for notations and conventions):

$$A_4 \mathcal{G} = - \frac{4(2\alpha')^4}{\pi^2} \left(\frac{g}{8\pi} \right)^4 \int \frac{d^2\tau}{(Im\tau)^5} \prod_{a=1}^3 d^2v_a [\eta(\bar{\tau})]^{-24} \prod_{1 \leq b < c \leq 4} [\chi(v_{cb}, \tau)]^{k_b k_c} \\ \times K_{i_1 i_2 i_3 i_4} T^{j_1 j_2 j_3 j_4} \sum_{L \in \Lambda} e^{-\pi i L^2 \bar{\tau}}. \quad (2.6)$$

Here

$$\eta(\bar{\tau}) = e^{-\pi i \bar{\tau}/12} \prod_{n=1}^{\infty} (1 - e^{-2\pi i n \bar{\tau}}) \quad (2.7)$$

$$K_{i_1 i_2 i_3 i_4} = \prod_{a=1}^4 k_a \delta_{i_a j_a} T^{j_1 j_2 j_3 j_4} \quad (2.8a)$$

$$T^{j_1 j_2 j_3 j_4} = \text{tr} \left(\prod_{a=1}^4 R_a^{j_a} \right) \quad (2.8b)$$

$$T_{i_1 i_2 i_3 i_4} = A_{i_1} A_{i_2} A_{i_3} A_{i_4} + \frac{1}{2\alpha'} R_{12} \delta_{i_1 i_2} A_{i_3} A_{i_4} + \\ + \frac{1}{(2\alpha')^2} R_{12} R_{34} \delta_{i_1 i_2} \delta_{i_3 i_4} + \text{permutations} \quad (2.9)$$

and

$$A_{i_a} = - \frac{1}{2\pi i} \sum_{b=1}^4 k_b \partial_{\bar{v}} G_{i_a b_a}(v_{b_a}, \tau) \quad (2.10)$$

$$R_{ab} = \frac{1}{2\pi^2} \partial_{\bar{v}}^2 G_{ba}(v_{ba}, \tau) \quad (2.11)$$

with

$$G_{ba} = \ln \chi(v_{ba}, \tau) = \ln \left[2\pi e^{-\pi(Im v_{ba})^2 / Im \tau} \left| \frac{\vartheta_1'(v_{ba}, \tau)}{\vartheta_1'(0, \tau)} \right| \right] \quad (2.12)$$

being the Green's function on the torus and $v_{ba} = v_b - v_a$.

In order to study the four-graviton interaction in the limit $\alpha' \rightarrow 0$, we perform an expansion of (2.6) in powers of $|\alpha' k_a \cdot k_b| \ll 1$. For generic values of the integrand we can expand

$$\prod_{a < b} [\chi(\nu_{ba}, \tau)]^{\alpha' k_a \cdot k_b} = 1 + \alpha' \sum_{a < b} k_a \cdot k_b G_{ab} + \frac{1}{2} (\alpha' \sum_{a < b} k_a \cdot k_b G_{ab})^2 + \dots \quad (2.13)$$

However, this expansion of the integrand is valid only as long as $|\alpha' k_a \cdot k_b G_{ab}(\nu_{ba})| \ll 1$, because $G_{ab}(\nu_{ba})$ has a singularity as $|\nu_{ba}| \rightarrow 0$. This singular region must be treated by considering the correct order of limits $\lim_{\alpha' k_a \cdot k_b \rightarrow 0} \prod_{a=1}^3 \int d^2 \nu_a$ instead of using the expansion (2.13). It should be noticed that in this region of small ν_{ba} , $[\chi(\nu_{ba})]^{\alpha' k_a \cdot k_b} \sim (\nu_{ba})^{\alpha' k_a \cdot k_b}$ is not an analytic function for arbitrary values of $k_a \cdot k_b$. This is reflected in the fact that the one-loop amplitude has a branch cut generated by unitarity. The above expansion (2.13) is valid as long as we are not near the branch cut, which is what we want for our calculation of the effective action.

Multiplying (2.9) by (2.13) we find that the coefficient $(1/2\alpha')^2$ has typically the form $R_{12} R_{34} \delta_{j_1 j_2} \delta_{j_3 j_4}$, whereas the coefficient of $\frac{1}{2\alpha'}$ has typically the form $R_{12} \delta_{j_1 j_2} A_{j_3} A_{j_4}$ or $R_{12} R_{34} \delta_{j_1 j_2} \delta_{j_3 j_4} \sum_{a < b} k_a \cdot k_b G_{ab}$. We integrate now over the variables ν_a in the regions

$$F_\tau = \left\{ -\frac{1}{2} \leq \text{Re } \nu_a \leq \frac{1}{2}, \quad 0 \leq \text{Im } \nu_a \leq \text{Im } \tau \right\} \quad (2.14)$$

and we find that all these coefficients vanish [15] as results of the following identities [11,15]

$$\int_{F_\tau} d^2 \nu_a \partial_{\bar{\nu}_a} \partial_{\bar{\nu}_b} G_{ab} = 0 \quad (2.15a)$$

$$\int_{F_\tau} d^2 \nu_a \partial_{\bar{\nu}_a}^2 G_{ab} = 0 \quad (2.15b)$$

Both integrals vanish because they are surface integrals and the integrands are periodic on the boundary of F_τ . This follows from the periodicity of the Green function :

$$G(\nu + 4, \tau) = G(\nu + \tau, \tau) = G(\nu, \tau). \quad (2.16)$$

This periodicity, however, is not immediately sufficient to prove (2.15) because there are poles both in $\partial_\nu G$ and $\partial_{\bar{\nu}}^2 G$. From the expression (2.12) for G_{ab} and from the fact that $\theta_1(\bar{\nu}_{ab}, \bar{\tau}) = 0$ for $\bar{\nu}_{ab} = 0, \bar{\tau}$, we find that $\partial_{\bar{\nu}_a} G_{ab}$ has two poles at $\bar{\nu}_{ab} \simeq 0, \bar{\tau}$. We have now two possibilities : If one of the poles is inside F_τ the other would be outside and then integrating in the vicinity of the pole (say $\bar{\nu}_a = \bar{\nu}_b$), we find that the pole residue vanishes ($\int_{\bar{\nu}_a \simeq 0} d^2 \bar{\nu}_a = 0$). If on the other hand one pole is on the boundary (when $\nu_b = 0$), there would be two poles at $\bar{\nu}_a = 0, \bar{\tau}$. We now approximate the function near the poles by $\partial_{\bar{\nu}_a} G_{ab} \simeq \frac{c}{\bar{\nu}_a}$, where both constants are equal as implied by the periodicity, (2.16),

and we can integrate in the vicinity of the poles to get zero. It is the sum over both contributions which gives zero, as each pole is encircled by half a circle but in opposite senses. Thus we have established (2.15a). For (2.15b) the proof is a bit simpler because the poles are of second order and the integral around each pole would vanish independently, whether it is on the boundary or not.

Thus we conclude that the leading contribution to (2.6) in the limit $\alpha' \rightarrow 0$ is of order 1 (modulo the overall factor $(2\alpha')^4$). Its coefficients are order 4 in the momentum and they have typically the form :

$$A_{j_1} A_{j_2} A_{j_3} A_{j_4} \quad (2.17a)$$

$$R_{12} \delta_{j_1 j_2} A_{j_3} A_{j_4} \sum_{a < b} k_a \cdot k_b G_{ab} \quad (2.17b)$$

$$R_{12} R_{34} \delta_{j_1 j_2} \delta_{j_3 j_4} \left(\sum_{a < b} k_a \cdot k_b G_{ab} \right)^2. \quad (2.17c)$$

This means that the contribution of A_{4G} to the effective Lagrangian in the limit $\alpha' \rightarrow 0$ contains at least 8 derivatives (recall that $K_{j_1 j_2 j_3 j_4}$ in (2.8) contains 4 powers of momenta), and hence cannot renormalize the term $e^{-\kappa D/\sqrt{2}} T_{\mu\nu} R^{\mu\nu}$ in eq. (2.1). For the same reason $e^{-\kappa D\sqrt{2}} \omega_{3L} \omega_{3L} \sim k^8$ cannot be renormalized by the 4-graviton interaction as derived from the heterotic string one-loop amplitude.

We study the amplitude for the emission of 2 gravitons and 2 gauge bosons in a similar way. It can be written as in (2.6) but with $T_{j_1 j_2 j_3 j_4}$ being replaced by $T_{j_1 j_2}$ where

$$T_{j_1 j_2} = \frac{1}{2\alpha'} R_{12} \delta_{j_1 j_2} + A_{j_1} A_{j_2} \quad (2.18)$$

and the contribution of the left-moving sector is appropriately changed. (For details see Eq. (4.4).) Multiplying (2.17) by the expansion (2.13) and using (2.15) we find that the term $\frac{1}{2\alpha'} R_{12} \delta_{j_1 j_2}$ vanishes upon integration over ν_1 . Thus we are left with terms of order 1 in the product of (2.17) with (2.13) : $\frac{1}{2} R_{12} \delta_{j_1 j_2} \sum_{a < b} k_a \cdot k_b G_{ab}$ and $A_{j_1} A_{j_2}$ which are quadratic in the momenta. If in addition we take account of the 4 powers of momenta appearing in the kinematical factor (2.8a), we find that this amplitude induces a coupling

involving 6 derivatives in the limit $\alpha' \rightarrow 0$. The interaction term $\omega_{3L}\omega_{3Y}e^{-\kappa D\sqrt{2}}$, on the other hand, is of order k^4 and therefore cannot be renormalized by one-loop string effects.

To complete our non-renormalization theorem we need to study the limit $\alpha' \rightarrow 0$ in higher point functions. We do not give the details here, but arguments similar to those given above show that these do not renormalize the Chamseddine-Chapline-Manton action either. Thus we conclude that none of the terms in the Chamseddine-Chapline-Manton action is renormalized by one-loop string effects. However, there are one-loop corrections to the 4-point couplings of higher orders in the field curvatures, which we now evaluate explicitly.

III. Four-graviton couplings

As argued in the previous section, the one-loop amplitudes for the emission of gravitons do not renormalize the Chamseddine-Chapline-Manton action. In this section we derive the one-loop corrections of higher order in α' to the $N = 1$ supergravity theory resulting from the emission of four gravitons. We will see in the following that these are generally of the same type as the corrections found from tree level amplitudes [6, 7].

With the 4-graviton amplitude as given in (2.6) and the expansion (2.13) in the limit $\alpha' k_a \cdot k_b \ll 1$ we find several $O(1)$ terms (modulo the prefactor $(2\alpha')^4$) which will now be studied systematically.

(i) Using (2.10) we get for the first $O(1)$ term (2.17a)

$$\begin{aligned} \int_{\mathbb{F}_{a=1}}^3 d^2v_a \frac{A_i A_j A_k A_l}{d_1 d_2 d_3 d_4} &= \frac{A}{16\pi^4} \left(k_2^{d_1} k_1^{d_2} k_3^{d_3} k_4^{d_4} + k_1^{d_1} k_2^{d_2} k_3^{d_3} k_4^{d_4} + k_3^{d_1} k_2^{d_2} k_1^{d_3} k_4^{d_4} + k_4^{d_1} k_2^{d_2} k_3^{d_3} k_1^{d_4} \right. \\ &\quad \left. + k_3^{d_1} k_4^{d_2} k_2^{d_3} k_1^{d_4} + k_4^{d_1} k_3^{d_2} k_1^{d_3} k_2^{d_4} + k_1^{d_1} k_4^{d_2} k_3^{d_3} k_2^{d_4} + k_2^{d_1} k_3^{d_2} k_4^{d_3} k_1^{d_4} \right), \end{aligned} \quad (3.1)$$

where

$$A = \int_{\mathbb{F}_{\tau}} d^2v_1 d^2v_2 d^2v_3 \left[\partial_{v_1} G(v_1, \tau) \right]^2 \left[\partial_{v_2} G(v_2, \tau) \right]^2 \left[\partial_{v_3} G(v_3, \tau) \right]^2 \quad (3.2a)$$

$$B = \int_{\mathbb{F}_{\tau}} d^2v_1 d^2v_2 d^2v_3 \partial_{v_1} G(v_1, \tau) \partial_{v_2} G(v_2, \tau) \partial_{v_3} G(v_3, \tau) \partial_{v_4} G(v_4, \tau). \quad (3.2b)$$

The rest of the terms which are found in the product $A_{j_1} A_{j_2} A_{j_3} A_{j_4}$ vanish upon integration over v_a in the fundamental region $\mathbb{F}_{\tau}$ (2.14), as is easily seen from (2.15). In Appendix B we classify the various integrals appearing in this product and demonstrate that they all vanish apart from the terms in (3.1). Using now conservation of momentum $\sum_{a=1}^4 k_a = 0$ and ignoring terms containing k_a^2 which would vanish upon contraction with $\rho_{i_a j_a}$ we find:

$$\int_{\mathbb{F}_{a=1}}^3 d^2v_a \frac{A_i A_j A_k A_l}{d_1 d_2 d_3 d_4} = \frac{A+B}{16\pi^4} \left(k_1^{d_1} k_2^{d_2} k_3^{d_3} k_4^{d_4} + k_1^{d_1} k_3^{d_2} k_2^{d_3} k_4^{d_4} + k_1^{d_1} k_4^{d_2} k_3^{d_3} k_2^{d_4} + k_2^{d_1} k_3^{d_2} k_4^{d_3} k_1^{d_4} \right) \quad (3.3)$$

(ii) Similarly, for the second $O(1)$ contribution (2.17b), we have

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{F}_{a=1}}^3 d^2v_a \frac{A_i A_j A_k A_l}{d_1 d_2 d_3 d_4} \sum_{a < b} k_a \cdot k_b G_{ab} &= \frac{A}{16\pi^4} \left(k_1^{d_1} k_2^{d_2} k_3^{d_3} k_4^{d_4} + k_1^{d_1} k_3^{d_2} k_2^{d_3} k_4^{d_4} + k_1^{d_1} k_4^{d_2} k_3^{d_3} k_2^{d_4} + k_2^{d_1} k_3^{d_2} k_4^{d_3} k_1^{d_4} \right) \\ &\quad + \frac{B}{16\pi^4} \left[\frac{u}{2} \left(k_4^{d_1} k_2^{d_2} + k_1^{d_1} k_3^{d_2} \right) + \frac{t}{2} \left(k_2^{d_1} k_3^{d_2} + k_4^{d_1} k_1^{d_2} \right) - \frac{S}{2} \left(k_1^{d_1} k_2^{d_2} + k_3^{d_1} k_4^{d_2} \right) \right], \end{aligned} \quad (3.4)$$

where we use

$$S = - (k_1 + k_2)^2, \quad t = - (k_1 + k_4)^2, \quad u = - (k_1 + k_3)^2 \quad (3.5a)$$

$$S + t + u = 0. \quad (3.5b)$$

To get (3.4) we integrate by parts and use (2.15) to eliminate the rest of the terms. There is a problem, though, with pole contributions when $\nu_a - \nu_b \approx 0$ which do not cancel in the left-hand-side of (3.4) in the presence of the logarithmic function G_{ab} (see Eq. (2.12)). As was mentioned above, the expansion (2.13) is not valid for $\nu_a - \nu_b \sim 0$. To analyze the integral in this neighbourhood one must revert to the full form $\chi^{\alpha' k_a \cdot k_b} \sim \nu_{ba}^{\alpha' k_a \cdot k_b}$ rather than the expansion (2.13). For $0 < \alpha' k_a \cdot k_b \ll 1$ one finds that the integration gives no contribution, corresponding to our not being on the unitarity cut, and showing that the apparent pole was in fact an artefact of the interchange of order of two operations: the limit $\alpha' \rightarrow 0$ and the integration. Therefore, we can safely perform the integration by parts which led to (2.15). Having thus disposed of the apparent pole contribution, the remaining

integrals A and B in (3.2) are finite. Using now (3.5b) and momentum conservation we find

$$\begin{aligned} & \frac{1}{2} \int \prod_{a=1}^3 d^4 \nu_a R_{12} \nu_a A_i \delta_{i_1 i_2} \nu_a \sum_{a < b} k_a \cdot k_b \mathcal{G}_{ab} = \\ & = \frac{A+B}{16\pi^4} \delta_{i_1 i_2} \delta_{i_3 i_4} k_3^i k_4^j \frac{\mathcal{G}}{2} + \frac{B}{8\pi^4} \delta_{i_1 i_2} \left(\frac{u}{2} k_2^i k_1^j + \frac{t}{2} k_2^i k_1^j \right), \end{aligned} \quad (3.6)$$

where we have ignored terms containing $k_a^{j_a}$ because they vanish upon contraction with $\rho_{i_1 j_1}$. To get the full contribution of (ii) we still need to sum over all the permutations $(R_{34} A_{31} A_{32} \delta_{33 i_4} \text{ etc.})$.

(iii) To find the third $\mathcal{O}(1)$ contribution (2.17c), we once again use (2.10), (2.11), (2.15), (3.2) and (3.5) after integration by parts. Thus

$$\begin{aligned} & \frac{1}{8} \int \prod_{a=1}^3 d^4 \nu_a (R_{12} R_{34} \delta_{i_1 i_2} \delta_{i_3 i_4} + R_{12} R_{34} \delta_{i_1 i_3} \delta_{i_2 i_4} + R_{12} R_{34} \delta_{i_1 i_4} \delta_{i_2 i_3}) (\sum_{a < b} k_a \cdot k_b \mathcal{G}_{ab})^2 \\ & = \frac{A+B}{16\pi^4} (\delta_{i_1 i_2} \delta_{i_3 i_4} \frac{s^2}{4} + \delta_{i_1 i_3} \delta_{i_2 i_4} \frac{t^2}{4} + \delta_{i_1 i_4} \delta_{i_2 i_3} \frac{u^2}{4}) \\ & - \frac{B}{8\pi^4} (\delta_{i_1 i_2} \delta_{i_3 i_4} \frac{t^2}{4} + \delta_{i_1 i_3} \delta_{i_2 i_4} \frac{u^2}{4} + \delta_{i_1 i_4} \delta_{i_2 i_3} \frac{s^2}{4}). \end{aligned} \quad (3.7)$$

Here again the left-hand-side of (3.7) contains poles at $\nu_a - \nu_b \simeq 0$ which do not cancel in the presence of the logarithmic function G_{ab} . These apparent pole contributions in fact vanish for the same reason as those discussed previously. Note that as before the terms not appearing in (3.7) vanish because of (2.15). (See Appendix B.)

Define now [6]

$$\mathcal{L}'_{\text{SFT}} = (2g)^4 M^{\mu_1 \mu_2 \mu_3 \mu_4} \nu_1 \nu_2 \nu_3 \nu_4 \prod_{a=1}^4 g_{\mu_a \nu_a} \quad (3.8a)$$

$$\mathcal{L}'_{\text{HET}} = (2g)^4 M^{\mu_1 \mu_2 \mu_3 \mu_4} \overline{M}^{\nu_1 \nu_2 \nu_3 \nu_4} \prod_{a=1}^4 g_{\mu_a \nu_a} \quad (3.8b)$$

which are the typical corrections to the Lagrangian in a superstring model and in the heterotic string as found in Refs. [5-7]. In (3.8) we have (see also Appendix A) :

$$\begin{aligned} M^{\mu_1 \mu_2 \mu_3 \mu_4} &= t^{\mu_1} \tau_1^{\mu_2} \tau_2^{\mu_3} \tau_3^{\mu_4} \tau_4^{\mu_5} \prod_{a=1}^4 k_{\tau_a} \\ &= \frac{1}{4} (ut \delta_{\mu_1 \mu_2} \delta_{\mu_3 \mu_4} + st \delta_{\mu_1 \mu_3} \delta_{\mu_2 \mu_4} + su \delta_{\mu_1 \mu_4} \delta_{\mu_2 \mu_3}) \\ &- \frac{s}{2} (\delta_{\mu_1 \mu_4} k_4^{\mu_2} k_1^{\mu_3} + \delta_{\mu_1 \mu_3} k_3^{\mu_2} k_2^{\mu_4} + \delta_{\mu_1 \mu_2} k_2^{\mu_3} k_3^{\mu_4} + \delta_{\mu_1 \mu_4} k_4^{\mu_3} k_3^{\mu_2}) \\ &- \frac{t}{2} (\delta_{\mu_1 \mu_4} k_4^{\mu_2} k_3^{\mu_3} + \delta_{\mu_1 \mu_3} k_3^{\mu_2} k_2^{\mu_4} + \delta_{\mu_1 \mu_2} k_2^{\mu_3} k_3^{\mu_4} + \delta_{\mu_1 \mu_4} k_4^{\mu_3} k_3^{\mu_2}) \\ &- \frac{u}{2} (\delta_{\mu_1 \mu_2} k_1^{\mu_3} k_4^{\mu_4} + \delta_{\mu_1 \mu_3} k_3^{\mu_2} k_2^{\mu_4} + \delta_{\mu_1 \mu_4} k_4^{\mu_3} k_3^{\mu_2} + \delta_{\mu_1 \mu_2} k_2^{\mu_3} k_3^{\mu_4}) \\ \overline{M}^{\mu_1 \mu_2 \mu_3 \mu_4} &= \overline{t}^{\mu_1} \tau_1^{\mu_2} \tau_2^{\mu_3} \tau_3^{\mu_4} \tau_4^{\mu_5} \prod_{a=1}^4 k_{\tau_a} \\ &= \frac{1}{4} (s^2 \delta_{\mu_1 \mu_2} \delta_{\mu_3 \mu_4} + t^2 \delta_{\mu_1 \mu_3} \delta_{\mu_2 \mu_4} + u^2 \delta_{\mu_1 \mu_4} \delta_{\mu_2 \mu_3}) + k_1^{\mu_1} k_2^{\mu_2} k_3^{\mu_3} k_4^{\mu_4} \\ &+ k_1^{\mu_3} k_2^{\mu_4} k_3^{\mu_2} k_4^{\mu_1} + k_1^{\mu_4} k_2^{\mu_3} k_3^{\mu_1} k_4^{\mu_2} + \frac{s}{2} (\delta_{\mu_1 \mu_2} k_2^{\mu_3} k_3^{\mu_4} + \delta_{\mu_1 \mu_4} k_4^{\mu_3} k_3^{\mu_2}) \\ &+ \frac{t}{2} (\delta_{\mu_1 \mu_3} k_3^{\mu_2} k_2^{\mu_4} + \delta_{\mu_1 \mu_4} k_4^{\mu_3} k_3^{\mu_1}) + \frac{u}{2} (\delta_{\mu_1 \mu_2} k_2^{\mu_3} k_3^{\mu_4} + \delta_{\mu_1 \mu_4} k_4^{\mu_3} k_3^{\mu_2}). \end{aligned} \quad (3.9a)$$

Thus by comparing with (3.4), (3.6) (with all its permutations) and (3.7), we find that the one-loop correction to the effective Lagrangian resulting from the 4-graviton coupling (2.6) is

$$\mathcal{L}_1 = \frac{1}{2\alpha'} (-2\tilde{B} \mathcal{L}'_{\text{SFT}} + (\tilde{A} + \tilde{B}) \mathcal{L}'_{\text{HET}}), \quad (3.10)$$

where we divide by $2\alpha'$ to get the correct dimensionality for $\mathcal{L}_1$. In (3.10) we have

$$\tilde{A} = -\frac{1}{4\pi^6} \left(\frac{1}{16\pi}\right)^4 \int \frac{d^4 t}{(2m\tau)^5} [h(\bar{t})]^{-24} A \sum_{L \in A} e^{-\pi i L^2 \bar{\tau}} \quad (3.11a)$$

$$\tilde{B} = -\frac{1}{4T^2} \left(\frac{1}{4T} \right)^4 \int \frac{d^2z}{(Imz)^5} [h(z)]^{-24} \sum_{\nu \in A} e^{-\pi i \nu^2 z} \quad (3.11b)$$

and A , B are defined in (3.2). To perform the integrations appearing in (3.2) and (3.11) we use the results of Ref. [12], where it was shown that the ν integrations in (3.2) can be performed if the integrands are represented in terms of the Fourier transform of the Eisenstein representation [18]. Using the notation of Ref. [12] we find that (see also Appendix C)

$$A = \frac{(I_{mt})^3}{2} G_2^2 \quad (3.12a)$$

$$B = \frac{(Im \tau)^3}{2} G_4 \quad (3.12b)$$

where $\hat{G}_2$ and G_4 are modular functions of weight 2 and 4 respectively (the second being an antiholomorphic function of τ). Inserting (3.11) in (3.12) and using once again the results of Ref. [12] (and Appendix C) we find

$$\tilde{A} = \frac{-1}{72\pi^2} \left(\frac{1}{16\pi} \right)^4 \int \frac{d^4x}{(2\pi)^4} [\chi(\bar{x})]^{-24} E_2^4 E_4^2 = -\frac{1}{12\pi} \left(\frac{1}{8\pi} \right)^4 \quad (3.13a)$$

$$\tilde{B} = \frac{-1}{36\pi^2} \left(\frac{1}{16\pi} \right)^4 \int \frac{d^2 \bar{z}}{(Im \bar{z})^2} [h(\bar{z})]^{-24} E_4^3 = \frac{-1}{12\pi} \left(\frac{1}{8\pi} \right)^4. \quad (3.13b)$$

Here

$$\gamma G^2 = \frac{\gamma^2 E^2}{3} ; \quad G^4 = \frac{\gamma^4 E^4}{45} \quad (3.14a)$$

$$E_4^2 = \sum_{L \in \Lambda} e^{-\pi i L^2 \bar{z}} \quad (3.14b)$$

We would like to express now (3.8) in terms of the curvature tensor $R_{\mu\nu}^{\alpha\beta}$. Defining

$$R_{\mu\nu}^{\alpha\beta} = -\frac{2k}{(\alpha\alpha')^2} [\mu^{\alpha} \nu^{\beta}] = g_{[\mu}^{\beta} k_{\nu]}^{\alpha} \quad (3.15)$$

where

$$2k = g(2\alpha')^2 \quad (3.16)$$

we then find

$$Z_1 = \frac{(\alpha')^3 g^2}{9.2 \cdot 10^{25}} (Z_{SST} - 12 Z_{HET}) \quad (3.17)$$

where

$$\mathcal{L}_{SPT} = t^{j_1} \varepsilon_{j_1 j_2} \varepsilon_{j_2 j_3} \varepsilon_{j_3 j_4} \varepsilon_{j_4} t^{\frac{j_4}{4}} \prod R_{\mu_r} v_a^{\lambda_a} \quad (3.18a)$$

$$\mathcal{L}_{\text{HET}} = \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \text{Tr} (P_{\mu_1 \mu_2} P_{\mu_3 \mu_4}) \text{Tr} (P_{\mu_1 \mu_3} P_{\mu_2 \mu_4}), \quad (3.186)$$

and we included $\frac{1}{4!}$ in (3.17) to account for all the permutations. (The factor 12 in (3.17) results from the contraction $\bar{f}_{1\lambda_1\lambda_2\lambda_3\lambda_4}\lambda_4\prod_{\alpha=1}^4 R_{\mu_\alpha\nu_\alpha\tau_\alpha\lambda_\alpha}$.)

Comparing now (3.17) with the tree level correction to the effective Lagrangian as found in Ref. [6] (see also Refs. [5,7]) which is

$$\mathcal{L}_0 = \frac{(\alpha')^3}{3 \cdot 4\pi^2} \left(25(3) \mathcal{L}_{ST} - 12 \mathcal{L}_{HET} \right), \quad (3.19)$$

we conclude that indeed g in (3.17) plays the role of the loop expansion parameter. Moreover, the one-loop corrections to the effective Lagrangian simply renormalize the coefficients of the corresponding interaction terms of the tree-level corrections. In view of the

complexity of (3.9a) and (3.9b), it is surprising to find only two coefficients : one for $\mathcal{L}_{SST}$ and one for $\mathcal{L}_{HET}$, both in (3.17) and in (3.19). It is therefore suggestive that some underlying symmetry is hidden in this theory and which is preserved at least up to one-loop corrections.

The fact that one-loop effects only renormalize the coefficients of pre-existing tree level interaction terms is largely a consequence of the common kinematical factors premultiplying all string amplitudes for arbitrary genus. This is so because the two dimensional conformal field theory from which the amplitudes are derived in the Polyakov approach is a free field theory, and the difference between the tree level and higher genus surfaces is in the boundary conditions imposed on the various Green functions. The dependence of the amplitudes on the external momenta results from the coupling to external sources before traces (or functional integrations) are performed [19]. Thus, for the bosonic string, for example, it is certainly true that the tensorial structure of all the higher-loop amplitudes is the same as that of the tree level ones [20]. In superstring models the ghost contributions may spoil this property. Here, though, we find that for the heterotic string the one-loop amplitude does lead to the same effective Lagrangian as the tree-level amplitude. It is then very suggestive that it would also be true for higher genus contributions as well.

IV. 2-graviton / 2-gauge-boson couplings

We now study the low-energy limit of the 2-graviton/2-gauge-boson one-loop amplitude. This amplitude was calculated in Ref. [16], using the cocycle representation for the vertex operators. This, however, is difficult to compare with existing results on tree-level string corrections to the effective Lagrangian [6,7] which involve traces over the products of the generators of the group. We therefore repeat the calculation here using the current algebra representation of the vertex operators [1]. In practice, it only means that we modify the contribution of the internal left-moving bosonic fields by including instead the contribution of the appropriate fermionic fields summed over all spin structures. We will perform this calculation explicitly for $SO(32)$ and comment later on about other models.

Consider then the set of 32 left-moving real anticommuting coordinates $\psi^I(I = 1, \dots, 32)$ transforming under the fundamental representation of $SO(32)$. The vertex operators for the massless gauge bosons can then be written as

$$V = \sqrt{2\alpha'} \frac{1}{2} \psi^\mu \gamma^\mu \psi^\nu \gamma^\nu \gamma^A \gamma^B \gamma^C \gamma^D e^{ik \cdot x} \quad (4.1)$$

where B_μ is the supersymmetrized right-moving momentum operator (A.2), T^A are the generators of the group and $k^\mu \rho_\mu^A = k^2 = 0$. The modification for the left-moving sector involves the Green function of the fermionic fields, which for a spin structure α is given by [21] :

$$\langle \psi^I(\bar{y}) \psi^J(\bar{y}) \rangle_\alpha = -\frac{i}{2\pi} \frac{\psi_\alpha^I(\bar{y}, \bar{z}) \psi_\alpha^J(\bar{y}, \bar{z})}{\psi_\alpha^I(\bar{y}, \bar{z}) \psi_\alpha^J(\bar{y}, \bar{z})} \delta^{IJ} \quad (4.2)$$

whereas the partition function in this sector is

$$\sum_{L \in \Lambda} e^{-\pi i L \bar{z}} = \frac{1}{2} \sum_{\alpha=1}^4 \psi_\alpha^I(\bar{y}, \bar{z}) = \mathcal{E}_4 \quad (4.3)$$

The amplitude for the emission of 2 gravitons and 2 gauge bosons is then given by (see Ref. [16] for the contribution of the right-moving sector):

$$A_{2g, 2g} = \frac{-4(2\alpha')^3}{\pi^2} \left(\frac{g}{2\pi} \right)^4 \int \frac{d^2 z}{(Im z)^5} \frac{1}{\alpha} d^2 y \left[\chi(\bar{z}) \right]^{-24} \prod_{a,b,c,d=1}^4 \chi^a(\bar{y}_a, \bar{z}) \quad (4.4)$$

$$\times K_{ij_2}^{A_3 A_4} \left(\frac{1}{2\alpha'} R_{12} \delta_{ij_2} + A_{ij_2} \right) \bar{\mathcal{E}} \text{Tr} (T^{A_3} T^{A_4}) ,$$

where $K_{ij_2}^{A_3 A_4}$ is defined as in (2.8) with particles 1,2 being the gravitons having wave functions $\rho_{i,j_2}^{A_a}$ ($a = 1, 2$) and particles 3,4 being the gauge bosons having wave functions $\rho_{i,j_2}^{A_a}$ ($a = 3, 4$). $\bar{\mathcal{E}}$ is the contribution of the left moving internal fermions ψ^I

$$\bar{\mathcal{E}} = \frac{-1}{16\pi^2} \sum_{\alpha=1}^4 \psi_\alpha^I(\bar{y}, \bar{z}) \left[\frac{\psi_\alpha^I(\bar{y}, \bar{z}) \psi_\alpha^I(\bar{y}, \bar{z})}{\psi_\alpha^I(\bar{y}, \bar{z}) \psi_\alpha^I(\bar{y}, \bar{z})} \right]^2 \quad (4.5)$$

In the limit $\alpha' \rightarrow 0$ we use the expansion (2.13), and multiply it by $\frac{1}{2\alpha'} R_{12} \delta_{ij_2} + A_{ij_2}$. As already mentioned in Section II the term proportional to $\frac{1}{2\alpha'}$ in this product vanishes upon integration over y_1 . The next would be term of $O(1)$:

$$\int_{\mathbb{F}_2} d^2 y \int_{\mathbb{F}_2} d^2 y \left[\frac{1}{2} R_{12} \sum_{a,b} k_a k_b G_{ab} \delta_{ij_2} + A_{ij_2} \right] =$$

$$= \frac{-1}{4\pi^2} \int_{\mathbb{F}_2} d^2 y \int_{\mathbb{F}_2} d^2 y \left(\partial_{\bar{y}_1} G_{12} \right)^2 (\delta_{ij_2} k_1 k_2 - k_1^i k_2^j) , \quad (4.6)$$

where in the right-hand-side of (4.6) we use integration by parts and (2.10), (2.11) and (2.15) to eliminate the rest of the terms. Once again we have pole contributions on the

left-hand-side of (4.6) which appear not to cancel out, whereas the right-hand-side of (4.6) is finite. The apparent pole contributions in fact disappear in a manner analogous to those discussed previously.

As for the integrals in (3.2), we can perform the ν integrations in (4.6) using the results of Ref. [12] (see Appendix C) to get :

$$\int_{\vec{F}_c} d^4 \nu_1 \int_{\vec{F}_c} d^4 \nu_2 \left(\partial_{\nu_1 \nu_2} \mathcal{G}_2 \right)^2 = -(\text{Im } \tau)^2 \mathcal{G}_2^2(\tau) \quad , \quad (4.7)$$

where the minus sign results from the fact that $\partial_{\nu_1 \nu_2} G_{12} = -\partial_{\nu_2 \nu_1} G_{21}$. For the rest of the integrals in (4.4) we use the Weierstrass p-function [22] which can be written as (see Appendix E)

$$p(\vec{\nu}, \vec{\tau}) = -2\partial_{\vec{\nu}}^2 \mathcal{G}(\nu, \tau) - \mathcal{G}_2^2(\tau) = e_{\alpha-1}(\tau) + \left(\frac{\nu_1^4(\tau, \vec{\tau}) \partial_{\alpha}^4(\vec{\nu}, \vec{\tau})}{\nu_{\alpha}^4(\vec{\nu}, \vec{\tau}) \partial_{\alpha}^4(\tau, \vec{\tau})} \right)^2 ; \quad \alpha = 2, 3, 4 \quad , \quad (4.8)$$

where

$$e_1(\vec{\tau}) = \frac{\pi^2}{3} \left[\nu_3^4(\tau, \vec{\tau}) + \nu_4^4(\tau, \vec{\tau}) \right] \quad (4.9a)$$

$$e_2(\vec{\tau}) = \frac{\pi^2}{3} \left[\nu_2^4(\tau, \vec{\tau}) - \nu_4^4(\tau, \vec{\tau}) \right] \quad (4.9b)$$

$$e_3(\vec{\tau}) = -\frac{\pi^2}{3} \left[\nu_2^4(\tau, \vec{\tau}) + \nu_3^4(\tau, \vec{\tau}) \right] \quad (4.9c)$$

Then

$$\int_{\vec{F}_c} d^2 \nu_3 \vec{\mathcal{L}} = \frac{\text{Im } \tau}{8\pi^2} \sum_{\alpha=2}^4 \nu_{\alpha}^4(\tau, \vec{\tau})^{16} \left[e_{\alpha-1}(\tau) + \mathcal{G}_2^2 \right] \quad , \quad (4.10)$$

where we use (4.5), (4.8) and (2.15b). In Appendix D we prove the identity

$$\frac{1}{2} \sum_{\alpha=2}^4 \nu_{\alpha}^4(\tau, \vec{\tau})^{16} e_{\alpha-1}(\tau) = -\frac{\pi^2}{3} \mathcal{E}_4 \mathcal{E}_6 \quad , \quad (4.11)$$

where E_6 is a modular function of weight 6 given by

$$\left(\frac{\text{Im } \tau}{3} \right)^3 E_6 = \frac{1}{2} e_1(\vec{\tau}) e_2(\vec{\tau}) e_3(\vec{\tau}) \quad . \quad (4.12)$$

Inserting (3.14), (4.3), (4.6), (4.7), (4.10), and (4.11) in (4.4) we get

$$A_{2g, 2G} = \bar{A} \frac{(2\pi i)^3}{36\pi^2} \left(\frac{g}{\partial \pi} \right)^4 K_{i_1 i_2}^{A_3 A_4} T_r \left(T^{A_3 A_4} \right) (\delta_{i_1 i_2} k_1 \cdot k_2 - k_1^i k_2^i) \quad (4.13)$$

where

$$\bar{A} = \int \frac{d^2 \tau}{(\text{Im } \tau)^2} \left[\eta(\vec{\tau}) \right]^{-24} (\bar{E}_2 E_4 E_6 - E_2^2 E_6^2) = -192\pi \quad . \quad (4.14)$$

Here we have used once again the results of Ref. [12] for the integral in (4.14). (See also Appendix C).

To convert (4.13) to an expression in terms of the curvature tensors $F_{\mu\nu}$ and $R_{\mu\nu}^{\alpha\beta}$ we use (2.8) and then

$$K_{i_1 i_2}^{A_3 A_4} T_r \left(T^{A_3 A_4} \right) (\delta_{i_1 i_2} k_1 \cdot k_2 - k_1^i k_2^i) = -\left(\frac{1}{2} \right)^5 t^{i_1 i_2 i_3 i_4 i_5 i_6} \times \frac{2}{\prod_{a=1}^6 k_{[i_a} p_{j_a]} \delta_{i_1 i_2} \delta_{j_2 j_4} \prod_{b=3}^4 k_{[i_b} p_{j_b]} T_r \left(T^{A_3 A_4} \right) \quad . \quad (4.15)$$

In (4.15) we exploited the fact that the tensor t is antisymmetric in every pair of indices. (See (A.4)). Using now (3.15) and the definition of the gauge field strength (up to this order in g)

$$F_{\mu\nu} = \frac{1}{(2\alpha')^2} \partial_{[\mu} g_{\nu]} \quad (4.16)$$

we find

$$\mathcal{L}'_1 = \frac{g_{10}^2 (\alpha')^3}{3\pi^5 14 k^2} g^2 t^{i_1 i_2 i_3 i_4 i_5 i_6} \nu_3^4 \nu_4^4 T_r \left(R_{\mu_1 \mu_2} R_{\mu_3 \mu_4} \right) T_r \left(F_{\mu_3 \mu_4}^T F_{\mu_1 \mu_2}^T \right) \quad , \quad (4.17)$$

where

$$g_{10} = g (2\alpha')^{3/2} \quad (4.18)$$

Comparing (4.17) with the results of Refs. [6,7] for the tree-level correction to the 2-graviton/2-gauge-boson coupling in the effective Lagrangian

$$\mathcal{L}'_0 = \frac{g_{10}^2 (\alpha')^3}{2 k^2} \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon^{\nu_1 \nu_2 \nu_3 \nu_4} \text{Tr}(\mathcal{B}_{\mu_1 \nu_1} \mathcal{B}_{\mu_2 \nu_2}) \text{Tr}(\mathcal{F}_{\mu_3 \nu_3} \mathcal{F}_{\mu_4 \nu_4}) \quad (4.19)$$

we see again that g plays the role of a loop expansion parameter, and that one-loop corrections merely renormalize the coefficient of a pre-existing term.

V. 4-gauge-boson coupling

The amplitude for the emission of 4-gauge bosons was also calculated in Refs. [1,16,23], using the cocycle representation of the vertex operators. Here again we repeat the calculation using the fermionized vertex which is easier to compare with known results of the tree-level effective Lagrangian. Using (4.1)-(4.3) and the contribution from the right-moving sector as given in Ref. [16] we find

$$A_{4g} = \frac{-4 (2\alpha')^2}{\pi^2} \left(\frac{g}{g\pi} \right)^4 \int \frac{d^2 z}{(Im z)^5} \prod_{a=1}^3 d^2 v_a [\eta(\bar{z})]^{-24} \prod_{1 \leq b < c \leq 4} \pi \left[\chi(v_b, z) \right]^{\alpha' k_c \cdot k_b} \times K^{A_1 A_2 A_3 A_4} T_{A_1 A_2 A_3 A_4} \quad (5.1)$$

where $K^{A_1 A_2 A_3 A_4}$ is defined as in (2.8) with $\rho_{j_a}^{A_a}$ ($a = 1, \dots, 4$) being the wave functions of the gauge bosons and

$$T^{A_1 A_2 A_3 A_4} = \left[A_1 \text{Tr}(T^{A_1} T^{A_2}) \text{Tr}(T^{A_3} T^{A_4}) + \text{permutations} \right] + \left[B_1 \text{Tr}(T^{A_1} T^{A_2} T^{A_3} T^{A_4}) + \text{permutations} \right]. \quad (5.2)$$

Here

$$A_1 = \frac{1}{512\pi^4} \sum_{a=2}^4 \mathcal{V}_a'(0, \bar{z}) \left(\frac{\mathcal{V}_a'(\bar{v}_2, \bar{z})}{\mathcal{V}_a'(\bar{v}_1, \bar{z})} \mathcal{V}_a'(\bar{v}_3, \bar{z}) \right)^2 \left(\frac{\mathcal{V}_a'(\bar{v}_4, \bar{z})}{\mathcal{V}_a'(0, \bar{z})} \right)^4 \quad (5.3a)$$

$$B_1 = \frac{-1}{512\pi^4} \sum_{a=2}^4 \mathcal{V}_a'(0, \bar{z}) \frac{\mathcal{V}_a'(\bar{v}_2, \bar{z}) \mathcal{V}_a'(\bar{v}_3, \bar{z}) \mathcal{V}_a'(\bar{v}_4, \bar{z})}{\mathcal{V}_a'(\bar{v}_1, \bar{z}) \mathcal{V}_a'(\bar{v}_2, \bar{z}) \mathcal{V}_a'(\bar{v}_3, \bar{z}) \mathcal{V}_a'(\bar{v}_4, \bar{z})} \left(\frac{\mathcal{V}_a'(0, \bar{z})}{\mathcal{V}_a'(0, \bar{z})} \right)^4 \quad (5.3b)$$

with the appropriate permutations of the indices $\bar{v}_{ab}$ ($a, b = 1, \dots, 4$) according to the permutations in (5.2). However, it should be noted that when integrated over ν_a ($a=1,2,3$) all the coefficients of the type A_1 (found by permuting the indices) are equal just as all the terms of type B_1 are equal. Therefore, it will be sufficient to integrate over A_1 and B_1 only. In the limit $\alpha' \rightarrow 0$, we may approximate $[\chi(\nu_b, \tau)]^{\alpha' k_c \cdot k_b} \simeq 1$ and we need to integrate over A_1, B_1 multiplied by $|\eta(\bar{z})|^{-24}$ only. Inserting (4.8) in A_1 (5.3a) we find that the integrations over ν_3 and ν_1, ν_2 can be separated. We can thus write

$$\int_{\mathbb{C}} \prod_{a=1}^3 d^2 v_a \left(\frac{\mathcal{V}_a'(0, \bar{z})}{\mathcal{V}_a'(0, \bar{z})} \right)^4 \left(\frac{\mathcal{V}_a'(\bar{v}_1, \bar{z})}{\mathcal{V}_a'(\bar{v}_2, \bar{z})} \mathcal{V}_a'(\bar{v}_3, \bar{z}) \right)^2 = (2Im z)^3 (e_{a-1} + \hat{G}_2)^2 \quad (5.4)$$

and therefore

$$\int_{\mathbb{C}} \prod_{a=1}^3 d^2 v_a A_1 = \frac{(2Im z)^3}{512\pi^4} \sum_{a=2}^4 \mathcal{V}_a'(0, \bar{z})^{16} (e_{a-1}^2 + 2\hat{G}_2 e_{a-1} + \hat{G}_2^2). \quad (5.5)$$

In Appendix D we prove

$$\frac{1}{2} \sum_{a=2}^4 \mathcal{V}_a'(0, \bar{z})^{16} e_{a-1}^2 = \frac{\pi^4}{9} (E_4^3 - 2 \cdot 3^2 \eta^{24}) \quad (5.6)$$

and using also (3.14), (4.3) and (4.11) we get by inserting (5.5) in (5.1)

$$\int_{\mathbb{C}} \prod_{a=1}^3 d^2 v_a A_1 = \frac{(Im z)^3}{2 \cdot 5 \cdot 3^2} (E_4^3 - 2 \cdot 3^2 E_4 E_6 + \frac{1}{2} E_4^2 E_8 - 2 \cdot 3^2 \eta^{24}) \quad (5.7)$$

Inserting now (5.7) in (5.1) and integrating over τ we find that the integral vanishes. Here again we use the results of Ref. [12] (see (C.13)) :

$$\int \frac{d^2\bar{z}}{(Im\bar{z})^5} \prod_{a=1}^3 d^2\nu_a [\eta(\bar{z})]^{-24} A_1 = 0. \quad (5.8)$$

To calculate the contribution of B_1 to (5.1) we use the Jacobi elliptic functions [22] :

$$cs(zk\bar{\nu}) = \frac{\nu_2(\bar{\nu}, \bar{z})}{\nu_1(\bar{\nu}, \bar{z})} \frac{\nu_4(o, \bar{z})}{\nu_3(o, \bar{z})} \quad (5.9a)$$

$$ds(zk\bar{\nu}) = \frac{\nu_3(\bar{\nu}, \bar{z})}{\nu_1(\bar{\nu}, \bar{z})} \frac{\nu_4(o, \bar{z})}{\nu_2(o, \bar{z})} \frac{\nu_2(o, \bar{z})}{\nu_3(o, \bar{z})^2} \quad (5.9b)$$

$$hs(zk\bar{\nu}) = \frac{\nu_4(\bar{\nu}, \bar{z})}{\nu_1(\bar{\nu}, \bar{z})} \frac{\nu_2(o, \bar{z})}{\nu_3(o, \bar{z})} \quad , \quad (5.9c)$$

where

$$zk = \pi \nu_3(o, \bar{z})^2 \quad (5.10)$$

Inserting (5.9) in (5.3b) and performing the integrations over ν_a ($a=1,2,3$) while using the expansions (E.9) and (E.10) as explained in Appendix E we find (see (E.18)) :

$$\int_{\mathbb{F}} \prod_{a=1}^3 d^2\nu_a B_1 = \frac{(Im\bar{z})^3}{4} \sum_{\alpha=2}^4 \nu_\alpha(o, \bar{z})^{16} f'_\alpha(\bar{z}) \quad , \quad (5.11)$$

where the functions $F_\alpha(\bar{\tau})$ ($\alpha = 2, 3, 4$) are defined in (E.20). For the integration over τ we use the fact that (5.11) (without the prefactor $(Im\tau)^3$) multiplied by $[\eta(\bar{\tau})]^{-24}$ is an antiholomorphic function of τ and we can therefore write the integrand as a total derivative thus picking up the contribution from the boundary of the integration region (C.9). Due to the modular invariance of the integrand the result of the integration is then the coefficient

of $\bar{q}^0$ in the expansion of the integrand after multiplying it by the function $G_2(\bar{\tau})$. (See Appendix C.) Using now (C.10), (5.11), (E.20), (2.7), the expansion of the θ -functions [22]

$$\nu_2(o, \bar{z}) = 2\bar{q}^{-1/4} \sum_{n=0}^{\infty} \bar{q}^{n(n+1)} \quad (5.12a)$$

$$\nu_3(o, \bar{z}) = 1 + 2 \sum_{n=1}^{\infty} \bar{q}^{n^2} \quad (5.12b)$$

$$\nu_4(o, \bar{z}) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n \bar{q}^{n^2} \quad , \quad (5.12c)$$

and the expansion of $G_2(\bar{\tau})$ [18] (see also (E.6))

$$G_2(\bar{z}) = \frac{\pi^2}{3} \left[1 - 24 \sum_{n=1}^{\infty} \frac{\bar{q}^{2n}}{(1 - \bar{q}^{2n})^2} \right] \quad , \quad (5.13)$$

we find

$$\int_{\mathbb{F}} \frac{d^2\bar{z}}{(Im\bar{z})^5} [\eta(\bar{z})]^{-24} \prod_{a=1}^3 d^2\nu_a B_1 = \frac{8\pi}{3} \quad . \quad (5.14)$$

In Appendix E we prove the identity (E.25), from which (5.14) immediately follows. (See also (C.13a).) We thus note that B_1 is the contribution of the zero modes only [12,14].

In order to express the effective Lagrangian in terms of the field strength we use (2.8)

$$K_{A_1 A_2 A_3 A_4} T_{A_1 A_2 A_3 A_4} = \left(\frac{1}{2}\right)^4 \prod_{a=1}^4 k_{[i_a} s_{\nu_a]} t^{i_{\nu_1} i_{\nu_2} i_{\nu_3} i_{\nu_4}} T_{\nu_1 \nu_2 \nu_3 \nu_4} T_{\nu_1 \nu_2 \nu_3 \nu_4} \quad (5.15)$$

and with (4.16), (5.8), (5.14) and (5.15) we get

$$\mathcal{L}_1'' = \frac{-(\alpha')^3 g_{10}^4}{3 \cdot 2^{10} \pi^5 K^2} \oint t^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3 \mu_4 \nu_4} T_{\nu_1 \nu_2 \nu_3 \nu_4} \left(\prod_{a=1}^4 F_{\mu_a \nu_a} \right) \quad . \quad (5.16)$$

It is interesting to compare (5.16) with the results found in Refs. [6,7] for the tree-level correction to the 4-gauge-boson coupling in the effective Lagrangian :

$$\mathcal{L}_0'' = \frac{-(\alpha')^3 g_0^4}{2^9 K^2} t^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3 \mu_4 \nu_4} \text{Tr} \left(F_{\mu_1 \nu_1} F_{\mu_2 \nu_2} \right) \text{Tr} \left(F_{\mu_3 \nu_3} F_{\mu_4 \nu_4} \right) \quad (5.17)$$

We notice an important difference between (5.16) and (5.17). Due to the vanishing of the integral in (5.8), the coupling $t^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3 \mu_4 \nu_4} \text{Tr}(F_{\mu_1 \nu_1} F_{\mu_2 \nu_2}) \text{Tr}(F_{\mu_3 \nu_3} F_{\mu_4 \nu_4})$ is missing in the one-loop level for the gauge group $SO(32)$, but will show up for $E_8 \times E_8$ as we will argue later. On the other hand the coupling $t^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3 \mu_4 \nu_4} \text{Tr}(\Pi_{a=1}^4 F_{\mu_a \nu_a})$ is missing at the tree level, for some reason which is not obviously profound, but shows up at the one-loop level. It is important to note that the gauge boson couplings at the tree level are the same whether the gauge group is $SO(32)$ or $E_8 \times E_8$, whereas this is not the case at the one-loop level. We will later on comment on a much simpler way of getting the gauge couplings at the one-loop level which helps to explain the origin of the $\text{Tr}(F^4)$ coupling.

VI. Discussion and Phenomenological comments

We conclude with a discussion and some phenomenological comments on the implications of our results.

The vanishing of one-loop corrections to the 2- and 3-point functions, as well as the absence of terms in the 4-point functions of low order in the field curvatures have told us that the Chamseddine-Chapline-Manton $N = 1, D = 10$ supergravity action is not renormalized by one-loop string corrections. Therefore, any previous result which was deduced from this action, and does not depend on higher derivative terms, will not be renormalized by one-loop string corrections. This means in particular that aspects of the low-energy $N = 1, d = 4$ supergravity theory obtained by dimensional reduction of the $N = 1, D = 10$ supergravity theory will be unchanged. This includes all terms up to second order in derivatives of the bosonic fields, which are characterized by the Kähler potential $\mathcal{G}$ and the gauge kinetic function f_{ab} . In particular, the vanishing of one-loop corrections to the bosonic kinetic terms tells us that

$$\delta_{1\text{-loop}} \mathcal{G}_i^i = 0 \quad , \quad (6.1a)$$

$$\delta_{1\text{-loop}} f_{ab} = 0 \quad (6.1b)$$

where $\mathcal{G}_i^i = \frac{\partial^2 \mathcal{G}}{\partial \Phi^i \partial \bar{\Phi}^i}$ (the Φ are generic chiral supermultiplet fields) is the Kähler metric. The only modification to $\mathcal{G}$ which is allowed by (6.1a) is a superpotential term

$$\delta_{1\text{-loop}} \mathcal{G} = \mathcal{G}(\phi) + \text{herm. conj.} \quad (6.2)$$

which has been specifically excluded by arguments given elsewhere [24]. We conclude that there is no renormalization of the Kähler potential [15]. The result (6.1b) means that there is no coupling of the dilaton/axion supermultiplet T (in the notation of Ref. [25]) to pairs of gauge fields, i.e. no $(\text{Re } T) \text{Tr}(FF)$ or $(\text{Im } T) \text{Tr}(F\bar{F})$ coupling. Arguments for and against the possible existence of such a term have been given in the literature. We believe this no-renormalization theorem settles the issue.

The first one-loop corrections to the effective $N = 1, D = 10$ action that we find, namely those of fourth order in the field curvatures are summarized by

$$\begin{aligned}
\Delta \mathcal{L}_1 &= \frac{(\alpha')^3 g^2}{3 \cdot 2^5 k^2 \pi^5} t^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3 \mu_4 \nu_4} \left\{ \frac{1}{3 \cdot 2^5} t_{\zeta_1 \lambda_1 \zeta_2 \lambda_2 \zeta_3 \lambda_3 \zeta_4 \lambda_4} \prod_{a=1}^4 \mathcal{R}_{\mu_a \nu_a} \zeta_a \lambda_a \right. \\
&\quad \left. - \frac{1}{2} \text{Tr}(\mathcal{R}_{\mu_1 \nu_1} \mathcal{R}_{\mu_2 \nu_2}) [\text{Tr}(\mathcal{R}_{\mu_3 \nu_3} \mathcal{R}_{\mu_4 \nu_4}) - g_{\mu_0 \nu_0}^2 \text{Tr}(\mathcal{F}_{\mu_3 \nu_3} \mathcal{F}_{\mu_4 \nu_4})] + g_{\mu_0 \nu_0}^2 \text{Tr}(\mathcal{F}_{\mu_3 \nu_3} \mathcal{F}_{\mu_4 \nu_4}) \right\} \\
&\quad (6.3) \\
\Delta \mathcal{L}_0 &= \frac{(\alpha')^3}{2^5 k^2} t^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3 \mu_4 \nu_4} \left\{ \frac{5(3)}{6} t_{\zeta_1 \lambda_1 \zeta_2 \lambda_2 \zeta_3 \lambda_3 \zeta_4 \lambda_4} \prod_{a=1}^4 \mathcal{R}_{\mu_a \nu_a} \zeta_a \lambda_a \right. \\
&\quad \left. - [\text{Tr}(\mathcal{R}_{\mu_1 \nu_1} \mathcal{R}_{\mu_2 \nu_2}) - g_{\mu_0 \nu_0}^2 \text{Tr}(\mathcal{F}_{\mu_1 \nu_1} \mathcal{F}_{\mu_2 \nu_2})] [\text{Tr}(\mathcal{R}_{\mu_3 \nu_3} \mathcal{R}_{\mu_4 \nu_4}) - g_{\mu_0 \nu_0}^2 \text{Tr}(\mathcal{F}_{\mu_3 \nu_3} \mathcal{F}_{\mu_4 \nu_4})] \right\} \\
&\quad (6.4)
\end{aligned}$$

and should be compared to the tree level correction up to this order [6,7]

Thus the one-loop corrections are of phenomenological significance for compactification schemes, as they modify the classical equations that should be satisfied by any proposed manifold of compactification. These fourth-order one-loop terms have two novel features. One is that a $\text{Tr}(F^4)$ term is present in (6.3), which is absent in the tree level correction (6.4). The second feature is that the one-loop corrections are not identical for $\text{SO}(32)$ and $E_8 \times E_8$ heterotic strings, as (6.3) is valid for $\text{SO}(32)$ only, whereas the tree-level correction (6.4) is valid for both gauge groups. In particular, (6.3) does not have $(\text{Tr} F^2)^2$ coupling, whereas we would have such an interaction term for $E_8 \times E_8$. To see that this is indeed the case, we note that by replacing $t^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3 \mu_4 \nu_4}$ in the 4-point functions by $\frac{-1}{2} g_{\mu_2 \nu_2}^{\mu_1 \nu_1 \mu_3 \nu_3 \mu_4 \nu_4}$ we would recover the 5-point parity violating amplitudes in the limit $\alpha' \rightarrow 0$. (See Ref. [13] for explicit expressions of these amplitudes. For comparison, it would be useful to remember the new conventions as explained in Appendix A.) As the parity-violating amplitudes yield the anomaly cancelling terms [28] in the limit $\alpha' \rightarrow 0$, we can read the effective Lagrangian for the fourth order couplings directly from these anomaly cancelling terms [28] (remembering to add the appropriate permutation factors). Indeed, by comparing the integrals representing the coefficients $\hat{A}$, $\hat{B}$, (3.13), $\hat{A}$ (4.14), A_1 (5.7) and B_1 (5.14) with the appropriate coefficients in Ref. [12] (for $\text{SO}(32)$), we find that they are proportional (various factors of 2 result from contractions of the tensor $\hat{I}$ with the fields). Thus, since the anomaly cancelling term of $E_8 \times E_8$ does contain a $(\text{Tr} F^2)^2$ term [28], we would find such a fourth-order coupling in the one-loop correction to the effective Lagrangian for this gauge group. It is absent, though, for $\text{SO}(32)$ as

we found by explicit calculation, and it is also absent in the anomaly cancelling term of $\text{SO}(32)$ [12,28]. This also suggests an explanation for the origin of the $\text{Tr}(F^4)$ term found at the one-loop level both for $\text{SO}(32)$ (6.3) and $E_8 \times E_8$ [28]. It is an essential coupling for the cancellation of anomalies in both theories. As we remarked in the text, string loop amplitudes have universal kinematic factors which mean that it should be no surprise to see many old friends from the tree level reappearing at the one-loop level. Equally, one should not be surprised at the appearance of a term at the one-loop level which was absent at the tree level, since its coefficient might have been accidentally zero at the tree level. This seems to be the case for the $\text{Tr}(F^4)$ coupling, whose absence at the tree level has no obvious profound origin, whereas its appearance at one-loop level is essential as explained above.

We conclude with two comments about the future of calculations of loop corrections to the effective field theory for the heterotic string. One is that the interpretation of the results would be aided by knowledge of the possible higher-order action terms for $N = 1, D = 10$ supergravity. Many tree and loop calculations might be short-circuited with knowledge of higher-order invariants. The second comment is that the techniques we developed here for calculating one-loop corrections derived from the 4-point functions can be immediately generalized to higher-point functions. In any n-point function we would encounter generalization of A_1, B_1 (in (5.3)) or A, B (in (3.2)). (General expressions for n-point functions at the one-loop level can be found in Yahikozawa, Ref. [21].) Thus, our use of the fermionized vertex operators (4.1) for calculating the amplitudes and the techniques we developed in Appendices D and E for calculating the ν integrations of A_1, B_1 in addition to the techniques developed in Ref. [12] for calculating the ν integrations of A, B and the τ integrations of holomorphic (or antiholomorphic) modular invariant functions should be sufficient for deriving any one-loop correction resulting from n-point functions in the limit $\alpha' \rightarrow 0$. (See Ref. [7] for a discussion of the subtraction of reducible interaction terms.) The calculations may be long but they are straightforward, as they would include the modular invariant functions $\hat{G}_n$ (C.3) or the Jacobi elliptic functions (5.9)(E.9). The identities (2.15) would be useful in eliminating many terms. Furthermore, the techniques developed for one-loop calculations can be used immediately to make statements about multi-loop contributions to the effective field theory action. In particular, the no-renormalization results (6.1) seem to generalize to more loops as we will report elsewhere.

Acknowledgements

We thank A. Casher and W. Lerche for useful discussions. P.J. would like to thank Prof. H. Ruegg for his support and encouragement, and L.M. would like to thank Prof. M. Guenin for his support.

Appendix A

We set up in this Appendix the notation used in the text. We use $i, j, k = 1, \dots, 8$ for space-like indices and $\mu, \nu = 0, \dots, 9$ for space-time indices. $a, b = 1, 2, 3, 4$ are particle indices and k_{j_a} would mean the momentum of particle a with components j_a . Only when the momentum of particle a has components j_a ($b \neq a$) (to be contracted later on with particle b 's wave function) do we use $k_a^{j_b}$. Similarly $\rho_{a, j_a}(\rho_{j_a}^{A_a})$ would mean the graviton (gauge boson) wave function of particle a .

The vertex operator for the graviton in the light-cone gauge is given by [1]

$$V_G = 2\alpha' g \tilde{P}_i B_j \sigma^{ij} e^{ik \cdot x} \quad , \quad (A.1)$$

where $\tilde{P}_\mu$ is the left moving momentum operator and B_μ is the supersymmetrized right-moving momentum operator

$$B_i = P_i + \frac{1}{2} k_j R_{ij} \quad (A.2a)$$

$$R^{ij} = \frac{1}{8} \bar{S} \gamma^{ij} S \quad . \quad (A.2b)$$

Here the S are the Majorana-Weyl fermions in the Green-Schwarz formalism, and γ^{ij} is the antisymmetric product of γ^i, γ^j and $\gamma^\mu = \frac{1}{\sqrt{2}}(\gamma^0 - \gamma^9)$. V_G in (A.1) is constructed to be dimensionless so as to get dimensionless amplitudes later on. The Lagrangian is found by dividing the amplitude by $(2\alpha')^5$ so as to get correctly a dimensionless action.

The string propagator between neighbouring vertices is

$$\Delta_a = \frac{1}{4\pi} \int d^3 z_a \left| \tilde{z}_a \right|^{\frac{1}{2} q_a^2 - 2} \tilde{z}_a^{N - \frac{1}{2} + \frac{1}{2} Q_a^2} \quad , \quad (A.3)$$

where $\tilde{N}$ (N) is the number operator in the left- (right-) moving sector and q_a, Q_a are the momenta in space-time and internal space respectively. The variable z_a is related to the variable ν_a in the text by the relation $\nu_a = \frac{1}{2\pi i} \ln(\prod_{b=1}^a z_b)$ and $\nu_4 = \tau$.

Note that we have scaled the product (A.1), (A.3) by a factor $1/4$ compared to Ref. [16] so as to get the appropriate normalization of the Lagrangian (compared to

the Chamseddine-Chapline-Manton action). We have also used the normalized volume element for the integration over the loop momentum l , thus adding a factor $(2\pi)^{-10}$ compared to the results of Ref. [16]. Finally, it was argued in Ref. [11] that a factor $1/2$ has to be added to the one-loop amplitudes to get correctly the unitarity relations; we have therefore added this factor $1/2$ in the definition of all our one-loop amplitudes.

The tensor $t^{ijklmnopq}$ appearing in the text is given by [26] :

$$t^{ijklmnopq} = -\frac{1}{2} \varepsilon^{ijklmnopq} - \frac{1}{2} \tilde{\varepsilon}^{ijklmnopq} + \frac{1}{2} (\delta^{jk} \delta^{lm} \delta^{pn} \delta^{qi} + \delta^{jm} \delta^{nk} \delta^{lp} \delta^{qi} + \delta^{jn} \delta^{mk} \delta^{lp} \delta^{qi} + \text{permutations}) \quad (A.4)$$

whereas the tensor $\tilde{t}^{ijklmnopq}$ is given by [6] :

$$\begin{aligned} \tilde{t}^{ijklmnopq} = & (\delta^{ik} \delta^{jl} - \delta^{il} \delta^{jk}) (\delta^{mp} \delta^{nq} - \delta^{mq} \delta^{np}) \\ & + (\delta^{kn} \delta^{lm} - \delta^{ln} \delta^{km}) (\delta^{pi} \delta^{qj} - \delta^{pj} \delta^{qi}) + (\delta^{im} \delta^{jn} - \delta^{in} \delta^{jm}) (\delta^{kp} \delta^{lq} - \delta^{lp} \delta^{kq}) . \end{aligned} \quad (A.5)$$

Note that both t and $\tilde{t}$ are antisymmetric in each pair of indices (ij) , (kl) , (mn) , and (pq) .

Appendix B

We classify in this Appendix the various integrals appearing in the product (2.17a), $(A_{j_1} A_{j_2} A_{j_3} A_{j_4})$.

$$(i) \quad A = \int_{F_C} \prod_{a=4}^3 d^2 v_a (\partial_{\tilde{v}_{12}} \mathcal{G}_{12})^2 (\partial_{\tilde{v}_{34}} \mathcal{G}_{34})^2 \quad , \quad (B.1)$$

where the region F_τ is defined in (2.14). Define

$$A(v_2, \tau) = \int_{F_\tau} d^2 v_1 (\partial_{\tilde{v}_{12}} \mathcal{G}_{12})^2 \quad , \quad (B.2)$$

then using the periodicity of the Green function on the boundary of F_τ (Eq. (2.16)) it can easily be proven that

$$\partial_{\tilde{v}_2} A(v_2, \tau) = \partial_{v_2} A(v_2, \tau) = 0 \quad . \quad (B.3)$$

Note that the integral in (B.2) involves pole contributions at $\nu_{12} = 0$ and $\nu_{12} = \tau$ on the boundary of F_τ . However, as the pole residues are equal and opposite in sign they cancel. (When the pole is inside the region F_τ , there is one pole only, however the integration around it is zero as explained in Section II.) Using now (B.3) we can write

$$A = \frac{1}{2\text{Im}\tau} \int_{F_\tau} \prod_{a=1}^4 d\nu_a (\partial_{\bar{\nu}_a} G_a)^2 (\partial_{\bar{\nu}_{34}} G_{34})^2, \quad (B.4)$$

where we have added the integral over ν_4 and divided by the volume of F_τ

$$\int_{F_\tau} d^2\nu_4 = 2 \text{Im}\tau. \quad (B.5)$$

$$(ii) \quad B = \int_{F_\tau} \prod_{a=1}^3 d\nu_a \partial_{\bar{\nu}_a} G_a \partial_{\bar{\nu}_{12}} G_{12} \partial_{\bar{\nu}_{23}} G_{23} \partial_{\bar{\nu}_{34}} G_{34} \partial_{\bar{\nu}_{41}} G_{41}. \quad (B.6)$$

B as defined in (B.6) is a function of ν_4 and τ . However, using the periodicity of the Green functions on the boundary of F_τ it can be shown by a repeated use of partial integrations and the relation: $\partial_{\nu_a} G_{ab} = -\partial_{\nu_b} G_{ab} = \partial_{\nu_{ab}} G_{ab}$, that

$$\partial_{\bar{\nu}_4} B(\nu_4, \tau) = \partial_{\bar{\nu}_4} B(\nu_4, \tau) = 0. \quad (B.7)$$

Once again the pole contributions at $\nu_{ab} = 0, \tau$ cancel either because they have equal residues with opposite signs when they are on the boundary or because the integral around the pole is zero when it is inside the region F_τ . Thus we can add the integral over ν_4 while dividing by the volume (B.5)

$$B = \frac{1}{2\text{Im}\tau} \int_{F_\tau} \prod_{a=1}^4 d\nu_a \partial_{\bar{\nu}_a} G_a \partial_{\bar{\nu}_{12}} G_{12} \partial_{\bar{\nu}_{23}} G_{23} \partial_{\bar{\nu}_{34}} G_{34} \partial_{\bar{\nu}_{41}} G_{41}. \quad (B.8)$$

Note that here ν_4 is taken as an independent variable in the region F_τ , whereas it is in fact equal to τ . This trick is used in order to utilize later on (C.3). (The fact that $\nu_4 = \tau$ means that B does not depend on ν_4 as is inferred from the periodicity of the integrand.) What is shown here though is that even if $\nu_4 \neq \tau$ still B does not depend on ν_4 and we can integrate it as in (B.8), while dividing by the volume of F_τ .

$$(iii) \quad C = \int_{F_\tau} \prod_{a=1}^3 d\nu_a (\partial_{\bar{\nu}_a} G_a)^2 \partial_{\bar{\nu}_{42}} G_{42} \partial_{\bar{\nu}_{43}} G_{43} \partial_{\bar{\nu}_{41}} G_{41}. \quad (B.9)$$

Using (B.3) we find that C vanishes

$$C = - \int_{F_\tau} d^2\nu (\partial_{\bar{\nu}} G)^2 \int_{F_\tau} d^2\nu_3 \partial_{\bar{\nu}_{43}} G_{43} \int_{F_\tau} d^2\nu_1 \partial_{\bar{\nu}_{41}} G_{41} = 0. \quad (B.10)$$

This integral vanishes because it involves an integration of a total divergence of a periodic function. (This is the identity (2.15a).)

$$(iv) \quad D = \int_{F_\tau} \prod_{a=1}^3 d\nu_a \partial_{\bar{\nu}_a} G_a \partial_{\bar{\nu}_{12}} G_{12} \partial_{\bar{\nu}_{23}} G_{23} \partial_{\bar{\nu}_{34}} G_{34} \partial_{\bar{\nu}_{41}} G_{41}. \quad (B.11)$$

By exchanging the variables $\nu_2 \leftrightarrow \nu_3$ and using the fact $\partial_{\nu_a} G_{ab} = -\partial_{\nu_b} G_{ab}$ we can show that D=0, and thus D=0.

(v) Integrals involving at least one variable only once. For example

$$E = \int_{F_\tau} \prod_{a=1}^3 d\nu_a \partial_{\bar{\nu}_a} G_a \partial_{\bar{\nu}_{12}} G_{12} (\partial_{\bar{\nu}_{13}} G_{13})^2 \partial_{\bar{\nu}_{14}} G_{14} = 0. \quad (B.12)$$

This integral is zero because $\partial_{\nu_{12}} G_{12} = -\partial_{\nu_{21}} G_{12}$ and the integration over ν_2 can be isolated. (See Eq. (2.15a)).

In summary all the integrals appearing in the product (2.17a) can be classified as integrals of type A-E in various permutations of ν_a (a=1,2,3). Out of all these only A and B are non-vanishing.

In the products $A_{j_1 j_2} R_{34} \sum_{a < b} k_a \cdot k_b G_{ab}$ (2.17b) or $R_{12} R_{34} (\sum_{a < b} k_a \cdot k_b G_{ab})^2$ (2.17c) we can bring all the integrals to the form A-E by repeated use of integration by parts and cancellation of contributions on the boundaries due to the periodicity of the integrands. As already mentioned in Section II, pole contributions which do not cancel in the presence of the logarithmic function G_{ab} can be avoided by taking the correct order of limit ($\alpha' \rightarrow 0$) and integration with momenta being out of the unitarity cut.

Appendix C

To find the integrals over ν_a ($a=1,2,3$) and τ of the various functions appearing in the text we use the results of Ref. [12]. For convenience we repeat briefly the arguments yielding the various integrals.

As the function $\partial_\nu G(\nu, \tau)$ is a modular function of weight 1, i.e. under the modular transformation $\tau \rightarrow \frac{a\tau+b}{c\tau+d}$, $\nu \rightarrow \frac{a\nu+b}{c\nu+d}$

$$\partial_\nu G(\nu, \tau) \longrightarrow -\tau \partial_\nu G(\nu, \tau), \quad (C.1)$$

we can therefore express it in terms of an Eisenstein series [18]

$$\partial_\nu G(\nu, \tau) = \lim_{s \rightarrow 0} \frac{1}{2} \sum_{k, m} \frac{1}{k\tau - m} \frac{1}{|k\tau - m|^s} e^{2\pi i k(\text{Re } \nu - \text{Re } \tau) \frac{\text{Im } \nu}{\text{Im } \tau}} e^{2\pi i m \frac{\text{Im } \nu}{\text{Im } \tau}}, \quad (C.2)$$

where the sum is over all the integers except for $k=m=0$, and we take a regularized sum (with $\lim_{s \rightarrow 0}$) as otherwise it would not converge. With this representation the integrals over ν_a can be easily performed and we find [12]

$$\int_{F_\tau} \prod_{a=1}^n d^2 \nu_a \partial_{\nu_1} G_{\nu_2} \dots \partial_{\nu_n} G_{\nu_{n-1}} = (\text{Im } \tau)^n \hat{G}_n(\tau), \quad (C.3)$$

where F_τ is the region in the complex plane defining the torus

$$F_\tau = \left\{ \nu / -\frac{1}{2} \leq \text{Re } \nu \leq \frac{1}{2}, \quad 0 \leq \text{Im } \nu \leq \text{Im } \tau \right\}. \quad (C.4)$$

In (C.3) the $\hat{G}_n(\tau)$ are modular functions of weight n

$$\hat{G}_n(\tau) = G_n(\tau) = \sum_{k, m} \frac{1}{(k\tau + m)^n}, \quad n > 2. \quad (C.5)$$

This sum converges for $n > 2$ and yields a holomorphic function of τ , whereas for $n=2$ we need to regularize, thus getting

$$\hat{G}_2(\tau) = \lim_{s \rightarrow 0} \sum_{k, m} \frac{1}{(k\tau + m)^2} \frac{1}{|k\tau + m|^s}. \quad (C.6)$$

Hence for $G_2(\tau)$ defined as in (C.5) (for $n=2$) we have

$$\hat{G}_2(\tau) = G_2(\tau) - \frac{\pi}{\text{Im } \tau}, \quad (C.7)$$

which is a modular function of weight 2, but unlike $G_2(\tau)$ it is not a holomorphic function of τ .

To perform the τ integration of the various functions appearing in the text we use the following property: Given a holomorphic and modular invariant function $F(\tau)$ then [12, 27]

$$\int_F \frac{d^2 \tau}{(\text{Im } \tau)^2} F(\tau) = \int_F d^2 \tau \frac{d}{d\bar{\tau}} \left(\frac{2i}{\text{Im } \tau} \right) F(\tau) = -\frac{2i}{\pi} \int_F d^2 \tau \frac{d}{d\bar{\tau}} [\hat{G}_2(\tau) F(\tau)], \quad (C.8)$$

where F is the fundamental region in the complex plane

$$F = \left\{ \tau / |\tau| \geq 1, \quad -\frac{1}{2} \leq \text{Re } \tau \leq \frac{1}{2}, \quad 0 \leq \text{Im } \tau \right\}. \quad (C.9)$$

Thus the integral in (C.8) has a contribution from the boundary of F only. Since the integrand is modular invariant the contributions from $\text{Re } \tau = \pm 1/2$ cancel each other and the contribution on $|\tau| = 1$, $\text{Im } \tau > 0$, $-1/2 \leq \text{Re } \tau \leq 1/2$ is zero. So we conclude that the only non-zero contribution is from the region $\text{Im } \tau \rightarrow \infty$, $-1/2 \leq \text{Re } \tau \leq 1/2$. For the modular invariant functions in the integrand this is a circle surrounding the point at infinity. Thus the only non-zero contribution to the integral is the coefficient of q^0 ($q = e^{i\pi\tau}$) in the expansion of the integrand when multiplied by $G_2(\tau)$:

$$\int_F \frac{d^2 \tau}{(\text{Im } \tau)^2} F(\tau) = \frac{2}{\pi} G_2(\tau) F(\tau) \Big|_{\text{coefficient of } q^0} \quad (C.10)$$

This property was exploited in Ref. [12] to get the integrals over τ . Define

$$G_n(\tau) = -\frac{(2\pi i)^n}{n!} B_n E_n, \quad (C.11)$$

with B_n being the Bernoulli numbers ($B_2 = \frac{1}{6}$, $B_4 = \frac{-1}{30}$) and

$$I(n, \ell, m) = \int_F \frac{d^2 c}{(Im \bar{c})^2} \bar{E}_2^{1n} \bar{E}_4^\ell \bar{E}_6^m \eta^{-24} \quad (C.12)$$

with $n + 2l + 3m = 6$, then

$$I(0, 0, 0) = \frac{2\pi}{3}$$

$$I(0, 3, 0) = 480\pi$$

$$I(4, 4, 4) = -96\pi$$

$$I(2, 2, 0) = 96\pi$$

Appendix D

We prove in this Appendix two identities which are used in the text. In the following we will use a shorthand notation $\theta_\alpha(0, \bar{\tau}) = \theta_\alpha$.

(i) We would like to prove

$$\frac{1}{2} \sum_{\alpha=2}^4 \vartheta_\alpha^{16} e_{\alpha-1}(\bar{\tau}) = -\frac{\pi^2}{3} E_4 E_6, \quad (D.1)$$

where the $e_\alpha(\bar{\tau})$ ($\alpha = 1, 2, 3$) are defined in (4.9) and

$$E_4 = \frac{1}{2} \sum_{\alpha=2}^4 \vartheta_\alpha^8 \quad (D.2)$$

$$E_6 = \frac{1}{2} (\vartheta_2^4 + \vartheta_3^4) (\vartheta_3^4 + \vartheta_4^4) / (\vartheta_4^4 - \vartheta_2^4) \quad (D.3)$$

We start with the right-hand-side of (D.1) and show that it is equal to the left-hand-side. Multiplying (D.2) by (D.3) we get

$$\begin{aligned} E_4 E_6 = \frac{1}{4} [& -\vartheta_2^{16} (\vartheta_3^4 + \vartheta_4^4) + \vartheta_2^8 \vartheta_3^8 (\vartheta_4^4 - \vartheta_2^4) + \vartheta_2^8 \vartheta_4^8 (\vartheta_3^4 + \vartheta_2^4) \\ & + \vartheta_3^{16} (\vartheta_4^4 - \vartheta_2^4) + \vartheta_3^8 \vartheta_4^8 (\vartheta_3^4 + \vartheta_2^4) - \vartheta_3^8 \vartheta_2^8 (\vartheta_3^4 + \vartheta_2^4) \\ & + \vartheta_4^{16} (\vartheta_3^4 + \vartheta_2^4) + \vartheta_4^8 \vartheta_3^8 (\vartheta_4^4 - \vartheta_2^4) - \vartheta_4^8 \vartheta_2^8 (\vartheta_3^4 + \vartheta_2^4)]. \end{aligned} \quad (D.4)$$

Using now the Riemann identity [22]

$$\vartheta_3^4 - \vartheta_2^4 - \vartheta_4^4 = 0, \quad (D.5)$$

it is possible to show that

$$E_4^2 = \vartheta_2^8 \vartheta_3^8 + \vartheta_3^8 \vartheta_4^8 + \vartheta_2^8 \vartheta_4^8 \quad (D.6)$$

Inserting (D.6) in (D.4) we may write

$$E_4 E_6 = -\frac{1}{4} \frac{3}{\pi^2} \sum_{\alpha=2}^4 \vartheta_{\alpha}^{16} e_{\alpha-1} + \frac{1}{4} \left[-\vartheta_{\frac{1}{2}}^8 \vartheta_{\frac{1}{4}}^8 (\vartheta_{\frac{1}{2}}^4 - \vartheta_{\frac{1}{4}}^4) + \vartheta_{\frac{1}{2}}^8 \vartheta_{\frac{3}{4}}^8 (\vartheta_{\frac{1}{3}}^4 + \vartheta_{\frac{1}{4}}^4) - \vartheta_{\frac{2}{3}}^8 \vartheta_{\frac{2}{3}}^8 (\vartheta_{\frac{1}{3}}^4 + \vartheta_{\frac{1}{2}}^4) \right], \quad (D.7)$$

and we have used once again (4.9) for the definition of e_{α} . Rearranging the terms in the second bracket of (D.7) and using (D.5) (and the identity $a^8 - b^8 = (a^4 - b^4)(a^4 + b^4)$) we finally get

$$E_4 E_6 = -\frac{1}{2} \frac{3}{\pi^2} \sum_{\alpha=2}^4 \vartheta_{\alpha}^{16} e_{\alpha-1}, \quad (D.8)$$

from which (D.1) follows immediately.

(ii) We would like to prove that

$$\frac{1}{2} \sum_{\alpha=2}^4 \vartheta_{\alpha}^{16} e_{\alpha-1}^2 = \frac{\pi^4}{9} (E_4^3 - \vartheta_{\frac{1}{2}}^3 \vartheta_{\frac{1}{3}}^2 \vartheta_{\frac{1}{4}}^{24}) \quad (D.9)$$

In order to prove this identity we use the identity

$$2\eta^3 = \vartheta_{\frac{1}{2}} \vartheta_{\frac{1}{3}} \vartheta_{\frac{1}{4}} \quad (D.10)$$

and we will therefore show that

$$\frac{1}{2} \sum_{\alpha=2}^4 \vartheta_{\alpha}^{16} e_{\alpha-1}^2 = \frac{\pi^4}{9} (E_4^3 - \frac{9}{2} \vartheta_{\frac{1}{2}}^8 \vartheta_{\frac{1}{3}}^8 \vartheta_{\frac{1}{4}}^8) \quad (D.11)$$

We start once again with the right-hand-side of (D.9) and show its equality to the left-hand-side. Using (D.10) and the product of (D.2) with (D.6) to represent E_4^3 , we find

$$\begin{aligned} E_4^3 - \frac{9}{2} \vartheta_{\frac{1}{2}}^8 \vartheta_{\frac{1}{3}}^8 \vartheta_{\frac{1}{4}}^8 &= \frac{1}{2} \left[\vartheta_{\frac{1}{2}}^{16} (\vartheta_{\frac{1}{3}}^8 + \vartheta_{\frac{1}{4}}^8) - 6 \vartheta_{\frac{1}{2}}^8 \vartheta_{\frac{1}{3}}^8 \vartheta_{\frac{1}{4}}^8 + \vartheta_{\frac{1}{2}}^{16} \vartheta_{\frac{1}{3}}^8 (\vartheta_{\frac{1}{2}}^8 + \vartheta_{\frac{1}{4}}^8) + \vartheta_{\frac{1}{4}}^{16} (\vartheta_{\frac{1}{2}}^8 + \vartheta_{\frac{1}{3}}^8)^2 \right] \\ &= \frac{1}{2} \left[\vartheta_{\frac{1}{2}}^{16} (\vartheta_{\frac{1}{3}}^4 + \vartheta_{\frac{1}{4}}^4)^2 + \vartheta_{\frac{1}{2}}^{16} (\vartheta_{\frac{1}{2}}^4 - \vartheta_{\frac{1}{4}}^4)^2 + \vartheta_{\frac{1}{4}}^{16} (\vartheta_{\frac{1}{2}}^4 + \vartheta_{\frac{1}{3}}^4)^2 \right] \\ &\quad + \vartheta_{\frac{1}{2}}^4 \vartheta_{\frac{1}{3}}^4 \vartheta_{\frac{1}{4}}^4 (\vartheta_{\frac{1}{2}}^{12} - \vartheta_{\frac{1}{4}}^{12} - \vartheta_{\frac{1}{2}}^{12} - 3 \vartheta_{\frac{1}{2}}^8 \vartheta_{\frac{1}{3}}^8 \vartheta_{\frac{1}{4}}^8) \end{aligned} \quad (D.12)$$

where we have completed the squares to get the last equality in (D.12). The last term in (D.12) vanishes as can easily be seen from (D.5)

$$0 = (\vartheta_{\frac{1}{2}}^4 - \vartheta_{\frac{1}{4}}^4) \vartheta_{\frac{1}{2}}^3 = \vartheta_{\frac{1}{2}}^{12} - \vartheta_{\frac{1}{4}}^{12} + 6 \vartheta_{\frac{1}{2}}^4 \vartheta_{\frac{1}{3}}^4 \vartheta_{\frac{1}{4}}^4 - 3 \vartheta_{\frac{1}{2}}^8 (\vartheta_{\frac{1}{2}}^4 + \vartheta_{\frac{1}{4}}^4) + 3 \vartheta_{\frac{1}{2}}^8 (\vartheta_{\frac{1}{3}}^4 - \vartheta_{\frac{1}{4}}^4) + 3 \vartheta_{\frac{1}{4}}^8 (\vartheta_{\frac{1}{3}}^4 - \vartheta_{\frac{1}{2}}^4). \quad (D.13)$$

Using once again (D.5) we find

$$\vartheta_{\frac{1}{2}}^{12} - \vartheta_{\frac{1}{4}}^{12} - \vartheta_{\frac{1}{2}}^{12} = 3 \vartheta_{\frac{1}{2}}^4 \vartheta_{\frac{1}{3}}^4 \vartheta_{\frac{1}{4}}^4 \quad (D.14)$$

Thus with the definition (4.9) for e_{α} we find

$$\frac{\pi^4}{9} [E_4^3 - \frac{9}{2} \vartheta_{\frac{1}{2}}^8 \vartheta_{\frac{1}{3}}^8 \vartheta_{\frac{1}{4}}^8] = \frac{1}{2} \sum_{\alpha=2}^4 \vartheta_{\alpha}^{16} e_{\alpha-1}^2 \quad (D.15)$$

from which (D.9) follows immediately.

Appendix E

We elaborate in this Appendix on some useful elliptic functions.

(i) The Weierstrass p-function is defined as [22] :

$$\wp(\vec{v}, \vec{\tau}) = -2\eta_1(\vec{\tau}) - \vartheta_{\vec{v}}^2 \partial_{\vec{v}} \vartheta_1(\vec{v}, \vec{\tau}) \quad (E.1)$$

where

$$2\eta_1(\vec{\tau}) = \frac{\pi^2}{3} \left[1 + 6 \sum_{n=1}^{\infty} \operatorname{cosec}^2(\pi n \vec{\tau}) \right] \quad (E.2)$$

But

$$\sum_{n=1}^{\infty} \operatorname{cosec}^2(\pi n \vec{\tau}) = -4 \sum_{n=1}^{\infty} \frac{e^{-2\pi i n \vec{\tau}}}{(1 - e^{-2\pi i n \vec{\tau}})^2} \quad (E.3)$$

If we define

$$\bar{q} = e^{-\pi i \bar{\tau}}$$

(E.4)

$$\begin{aligned} \text{then } \sum_{n=1}^{\infty} \text{cosec}^2(\pi n \bar{\tau}) &= -4 \sum_{n=1}^{\infty} \bar{q}^{2n} \frac{d}{d(\bar{q}^{2n})} \sum_{k=0}^{\infty} (\bar{q}^{2n})^k \\ &= -4 \sum_{k=0}^{\infty} k \sum_{n=1}^{\infty} \bar{q}^{2nk} = -4 \sum_{k=0}^{\infty} \frac{k \bar{q}^{2k}}{1 - \bar{q}^{2k}}, \end{aligned}$$

(E.5)

where we have used repeatedly $\frac{1}{1-\bar{q}^{2n}} = \sum_{k=0}^{\infty} \bar{q}^{2nk}$. Therefore

$$2\eta_1(\bar{\tau}) = \frac{\pi^2}{3} \left[1 - 24 \sum_{n=1}^{\infty} \frac{n e^{-2\pi i n \bar{\tau}}}{1 - e^{-2\pi i n \bar{\tau}}} \right] \equiv \mathfrak{E}_2(\bar{\tau}).$$

(E.6)

Using now (2.12) we find

$$2 \partial_{\bar{v}}^2 G(v, \tau) = \partial_{\bar{v}}^2 \ln \vartheta_1(\bar{v}, \bar{\tau}) + \frac{\pi}{I_m \tau},$$

(E.7)

and we can therefore write

$$p(\bar{v}, \bar{\tau}) = -2 \partial_{\bar{v}}^2 G(v, \tau) - \mathfrak{E}_2(\tau)$$

(E.8)

where $\hat{G}_2(\tau)$ is defined in (C.7).

(ii) The Jacobi elliptic functions are defined in Section V ((5.9a)-(5.9c)) and have the following expansion [22]:

$$cS(2k\bar{v}) = \frac{\pi}{2K} \left\{ \cot \pi \bar{v} - 4 \sum_{n=0}^{\infty} \frac{\bar{q}^{2n}}{1 + \bar{q}^{2n}} \sin(2\pi n \bar{v}) \right\} \quad (E.9a)$$

$$dS(2k\bar{v}) = \frac{\pi}{2K} \left\{ \text{cosec}(\pi \bar{v}) - 4 \sum_{n=0}^{\infty} \frac{\bar{q}^{2n+1}}{1 + \bar{q}^{2n+1}} \sin[(2n+1)\pi \bar{v}] \right\} \quad (E.9b)$$

- 35 -

$$hS(2k\bar{v}) = \frac{\pi}{2K} \left\{ \text{cosec}(\pi \bar{v}) + 4 \sum_{n=0}^{\infty} \frac{\bar{q}^{2n+1}}{1 - \bar{q}^{2n+1}} \sin[(2n+1)\pi \bar{v}] \right\}. \quad (E.9c)$$

These can be written in terms of the Fourier expansions

$$cS(2k\bar{v}) = \frac{i\pi}{K} \sum_{n=-\infty}^{\infty} \frac{e^{-2\pi i n \bar{v}}}{1 + \bar{q}^{2n}} \quad (E.10a)$$

$$dS(2k\bar{v}) = \frac{i\pi}{K} \sum_{n=-\infty}^{\infty} \frac{e^{-\pi i (2n+1) \bar{v}}}{1 + \bar{q}^{2n+1}} \quad (E.10b)$$

$$hS(2k\bar{v}) = \frac{i\pi}{K} \sum_{n=-\infty}^{\infty} \frac{e^{-\pi i (2n+1) \bar{v}}}{1 - \bar{q}^{2n+1}}, \quad (E.10c)$$

which are valid for

$$0 \leq \text{Im } v \leq \text{Im } \tau. \quad (E.11)$$

Here $\bar{q}$ is defined in (E.4) and K is defined in (5.10).

As is seen from (5.1), (5.3b) and (5.9) we need to calculate integrals of the type

$$\int_{\bar{\tau}}^{\bar{\tau} + \frac{3}{2}} \frac{d^2 \bar{v}_a}{\bar{\tau}} f(\bar{v}_{1a}, \bar{\tau}) f(\bar{v}_{23}, \bar{\tau}) f(\bar{v}_{34}, \bar{\tau}) f(\bar{v}_{41}, \bar{\tau}), \quad (E.12)$$

where $f(\bar{v}, \bar{\tau})$ is any one of the Jacobi elliptic functions (E.9). Due to the periodicity of the integrand, we can rewrite (E.12) as

$$\begin{aligned} & \int_{\bar{\tau}}^{\bar{\tau} + \frac{3}{2}} \frac{d^2 \bar{v}_a}{\bar{\tau}} f(\bar{v}_1, \bar{\tau}) f(\bar{v}_2, \bar{\tau}) f(\bar{v}_3, \bar{\tau}) f(-\bar{v}_1 - \bar{v}_2 - \bar{v}_3, \bar{\tau}) \\ &= - \int_{\bar{\tau}}^{\bar{\tau} + \frac{3}{2}} \frac{d^2 \bar{v}_a}{\bar{\tau}} f(\bar{v}_1, \bar{\tau}) f(\bar{v}_2, \bar{\tau}) f(\bar{v}_3, \bar{\tau}) f(\bar{v}_1 + \bar{v}_2 + \bar{v}_3, \bar{\tau}) \end{aligned} \quad (E.13)$$

- 36 -

where the arguments in the first three functions of (E.13) are in the domain (E.11) whereas the argument in the fourth function is in the domain

$$0 \leq \operatorname{Im} z \leq 3\operatorname{Im} \tau \quad (E.14a)$$

with

$$z = z_1 + z_2 + z_3 \quad (E.14b)$$

In order to use the expansion (E.10) for $f(z)$ with z defined as in (E.14), we need to shift the argument z to lie in the domain (E.11). Remembering that the elliptic functions (E.9) are periodic or antiperiodic, we find

$$f(z) = \epsilon f(z - \tau) = f(z - 2\tau) \quad (E.15)$$

where $\epsilon = -1$ for (E.9a), (E.9b) and $\epsilon = 1$ for (E.9c). (Comparing (E.15) with (E.13) we note that in (E.13) we always have two functions involving the same variable z_a ($a = 1, 2, 3$) therefore the phase ϵ in (E.15) disappears when periodicity is used in (E.13)). Using now (E.15) we can write

$$\begin{aligned} f(z) &= f(z) \theta(\operatorname{Im}(z - \tau)) + \epsilon f(z - \tau) \theta(\operatorname{Im}(z - \tau)) \theta(\operatorname{Im}(2\tau - z)) \\ &+ f(z - 2\tau) \theta(\operatorname{Im}(z - 2\tau)) \theta(\operatorname{Im}(3\tau - z)) \quad (E.16) \end{aligned}$$

thus having in each region an argument for the function f which is in the range of the validity of the Fourier expansion (E.10). Using this expansion the integrations in (E.13) can be easily performed and we get ($\epsilon = -1$)

$$\begin{aligned} &\int_{\mathcal{F}} d^2 z_a \operatorname{cs}(2Kz_a) \operatorname{cs}(2Kz) = \\ &= \left(\frac{\pi i}{K}\right)^4 \int_{\mathcal{F}} d^2 z_a \sum_{n_4, n_3 = -\infty}^{\infty} e^{-2\pi i n_4 z_a - 2\pi i n_3 z} \frac{1}{(1 + q^{4n_3})} \frac{1}{(1 + q^{2n_4})} \\ &\times \left[\theta(\operatorname{Im}(z - \tau)) - q^{2n_4} \theta(\operatorname{Im}(z - \tau)) \theta(\operatorname{Im}(2\tau - z)) + q^{4n_4} \theta(\operatorname{Im}(3\tau - z)) \theta(\operatorname{Im}(z - 2\tau)) \right] \end{aligned}$$

$$= \left(\frac{\pi}{K}\right)^4 (2\operatorname{Im} \tau)^3 \sum_{n=-\infty}^{\infty} \left(\frac{1}{1 + q^{2n}}\right)^3 \frac{q^{2n}}{1 + q^{2n}} \left(\frac{1}{6} - \frac{2}{3} q^{2n} + \frac{1}{6} q^{4n}\right) \quad (E.17)$$

In a similar way we can integrate the rest of the elliptic functions in (E.10) (with $\epsilon = -1, +1$ for (E.9b), (E.9c) respectively), and we find

$$\begin{aligned} &\int_{\mathcal{F}} d^2 z_a \frac{\vartheta_4'(\bar{z}_2, \bar{\tau}) \vartheta_4'(\bar{z}_3, \bar{\tau}) \vartheta_4'(\bar{z}_4, \bar{\tau})}{\vartheta_1'(\bar{z}_2, \bar{\tau}) \vartheta_1'(\bar{z}_3, \bar{\tau}) \vartheta_1'(\bar{z}_4, \bar{\tau})} \frac{\vartheta_4'(\bar{z}_2, \bar{\tau})}{\vartheta_1'(\bar{z}_2, \bar{\tau})} \frac{\vartheta_4'(\bar{z}_3, \bar{\tau})}{\vartheta_1'(\bar{z}_3, \bar{\tau})} \frac{\vartheta_4'(\bar{z}_4, \bar{\tau})}{\vartheta_1'(\bar{z}_4, \bar{\tau})} \left(\frac{\vartheta_4'(\bar{z}_2, \bar{\tau})}{\vartheta_1'(\bar{z}_2, \bar{\tau})}\right)^2 \\ &= - (2\pi)^4 \frac{(2\operatorname{Im} \tau)^3}{3} F_\alpha(\bar{\tau}) \quad ; \quad \alpha = 2, 3, 4 \quad (E.18) \end{aligned}$$

where we use

$$\vartheta_1'(\bar{z}_2, \bar{\tau}) = \pi \vartheta_2'(\bar{z}_2, \bar{\tau}) \vartheta_3'(\bar{z}_2, \bar{\tau}) \vartheta_4'(\bar{z}_2, \bar{\tau}) \quad (E.19)$$

The functions $F_\alpha(\bar{\tau})$ ($\alpha = 2, 3, 4$) are given by :

$$F_2(\bar{\tau}) = -\frac{1}{16} + \sum_{n=1}^{\infty} \frac{q^{2n}}{(1 + q^{2n})^4} (1 - 4q^{2n} + q^{4n}) = -\frac{1}{16} - \sum_{n=1}^{\infty} \frac{(-1)^n q^{2n}}{1 - q^{2n}} \quad (E.20a)$$

$$F_3(\bar{\tau}) = \sum_{n=1}^{\infty} \frac{q^{2n-1}}{(1 + q^{2n-1})^4} (1 - 4q^{2n-1} + q^{4n-2}) = - \sum_{n=1}^{\infty} \frac{(-1)^n q^{2n}}{1 - q^{2n}} \quad (E.20b)$$

$$F_4(\bar{\tau}) = - \sum_{n=1}^{\infty} \frac{q^{2n-1}}{(1 - q^{2n-1})^4} (1 + 4q^{2n-1} + q^{4n-2}) = - \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \quad (E.20c)$$

(iii) We prove now the relation between the functions $F_\alpha(\bar{\tau})$ ($\alpha = 2, 3, 4$) in (E.20) and the θ -functions. For that we need some mathematical preliminaries [18].

A modular function $H(\bar{\tau})$ is defined to be entire if it is antiholomorphic (or holomorphic) and if in its expansion near $\bar{q} \simeq 0$ we have

$$H(\bar{\tau}) = \sum_{n \geq n_0} b_n \bar{q}^{2n}, \quad b_{n_0} \neq 0, \quad (E.21)$$

with $n_0 \geq 0$. Moreover, any entire modular function of weight 12 can be written as

$$H_{12}(\bar{\tau}) = c_1 G_4(\bar{\tau})^3 + c_2 \eta(\bar{\tau})^{24}, \quad (E.22)$$

where c_1 and c_2 are some constants, $\eta(\bar{\tau})$ is the Dedekind function defined in (2.7) and $G_4(\bar{\tau})$ is defined in (3.14). Both $[\eta(\bar{\tau})]^{24}$ and $[G_4(\bar{\tau})]^3$ are entire modular functions of weight 12, and $G_4(\bar{\tau})$ can be expanded as [18]:

$$G_4(\bar{\tau}) = \frac{\pi^4}{45} \left[1 + 240 \sum_{n=1}^{\infty} \frac{n^3 \bar{q}^{2n}}{1 - \bar{q}^{2n}} \right]. \quad (E.23)$$

We note that $[\eta(\bar{\tau})]^{24} \simeq \bar{q}^2$ near $\bar{q} \simeq 0$ whereas $G_4(\bar{\tau}) = \frac{\pi^4}{45}$ for $\bar{q} = 0$.

As already seen from (E.18) and (5.11) we need to study the function

$$h(\bar{\tau}) = \frac{1}{12} \sum_{\alpha=2}^4 \mathcal{V}_{\alpha}(\bar{\tau}, \bar{\tau})^{16} F_{\alpha}(\bar{\tau}), \quad (E.24)$$

which when multiplied by $[\eta(\bar{\tau})]^{24}$ yields a modular invariant function. Thus $h(\bar{\tau})$ is a modular function of weight 12. Moreover, it is entire and $h(\bar{\tau}) \simeq \bar{q}^2$ near $\bar{q} \simeq 0$, as is seen from the expansions (5.12) and (E.20). Thus it may be expanded as in (E.22) with $c_1 = 0$. To fix c_2 it is sufficient to find the coefficient of $\bar{q}^2$ in $h(\bar{\tau})$ and to compare it with the coefficient of $\bar{q}^2$ in $[\eta(\bar{\tau})]^{24}$. We thus find from the expansions (5.12), (E.20) and (2.7)

$$\frac{1}{12} \sum_{\alpha=2}^4 \mathcal{V}_{\alpha}(\bar{\tau}, \bar{\tau})^{16} F_{\alpha}(\bar{\tau}) = 4 \eta(\bar{\tau})^{24}. \quad (E.25)$$

On the other hand we know from (D.9) and (4.3) that

$$\eta(\bar{\tau})^{24} = \frac{1}{2^8 3^2} \sum_{\alpha=2}^4 \mathcal{V}_{\alpha}(\bar{\tau}, \bar{\tau})^{16} \left[E_4 - \frac{9}{\pi^4} e_{-1}^2 \right]. \quad (E.26)$$

Using now (D.2), (4.9) and (D.5) we can prove

$$E_4 - \frac{9}{\pi^4} e_{-1}^2 = -3 \mathcal{V}_3^4 \mathcal{V}_4^4 \quad (E.27a)$$

$$E_4 - \frac{9}{\pi^4} e_{-2}^2 = 3 \mathcal{V}_2^4 \mathcal{V}_4^4 \quad (E.27b)$$

$$E_4 - \frac{9}{\pi^4} e_{-3}^2 = -3 \mathcal{V}_2^4 \mathcal{V}_3^4. \quad (E.27c)$$

Thus by comparing (E.25) with (E.26) and (E.27) we find

$$F_2(\bar{\tau}) = -\frac{1}{16} \mathcal{V}_3^4 \mathcal{V}_4^4 \quad (E.28a)$$

$$F_3(\bar{\tau}) = \frac{1}{16} \mathcal{V}_2^4 \mathcal{V}_4^4 \quad (E.28b)$$

$$F_4(\bar{\tau}) = -\frac{1}{16} \mathcal{V}_2^4 \mathcal{V}_3^4. \quad (E.28c)$$

Indeed, by studying a bit the expansion of $F_{\alpha}(\bar{\tau})$ ($\alpha = 2, 3, 4$) in (E.20) and that of $\theta_{\alpha}(0, \bar{\tau})$ ($\alpha = 2, 3, 4$) in (5.12) one finds that the identity (E.28) holds.

References

- [1] D.J. Gross, J.A. Harvey, E. Martinec and R. Rohm, Nucl. Phys. **B265**, 75 (1986); Nucl. Phys. **B256**, 253 (1985).
- [2] A.H. Chamseddine, Nucl. Phys. **B185**, 403 (1981);
E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuisen, Nucl. Phys. **B195**, 97 (1982);
G. F. Chapline and N. S. Manton, Phys. Lett. **120B**, 105 (1983).
- [3] R.I. Nepomechie, Phys. Rev. **D32**, 3201 (1985);
D. Chang and H. Nishino, Phys. Lett. **179B**, 257 (1986).
- [4] C.G. Callan, D. Friedan, E.J. Martinec and M.J. Perry, Nucl. Phys. **B263**, 593 (1985);
C.G. Callan, I.R. Klebanov and M.J. Perry, Nucl. Phys. **B278**, 78 (1986).
- [5] D.J. Gross and E. Witten, Nucl. Phys. **B277**, 1 (1986).
- [6] Y. Cai and C.A. Núñez, Nucl. Phys. **B287**, 279 (1987).
- [7] D.J. Gross and J. Sloan, Nucl. Phys. **B291**, 41 (1987).
- [8] M.T. Grisaru, A.E.M. van de Ven and D. Zanon, Phys. Lett. **173B**, 423 (1986);
M.T. Grisaru, A. van de Ven and D. Zanon, Nucl. Phys. **B277**, 409 (1986).
- [9] M.D. Freeman and C.N. Pope, Phys. Lett. **174B**, 48 (1986);
C.N. Pope, M.F. Sohnius and K.S. Stelle, Nucl. Phys. **B283**, 192 (1987);
D. Nemeschansky and A. Sen, Phys. Lett. **178B**, 365 (1987).
- [10] R.R. Metsaev and A.A. Tseytlin, Preprint Lebedev Physical Institute (1987).
- [11] N. Sakai and Y. Tanii, Nucl. Phys. **B287**, 457 (1987).
- [12] W. Lerche, B.E.W. Nilson, A.N. Schellekens and N.P. Warner, CERN preprint TH.4765/87 (1987).
- [13] J. Ellis and L. Mizrahi, CERN preprint TH.4616/86 (1986), to appear in Nucl. Phys. **B**.
- [14] W. Lerche, B.E.W. Nilson and A.N. Schellekens, Nucl. Phys. **B289**, 609 (1987).
- [15] J. Ellis, P. Jetzer and L. Mizrahi, CERN preprint TH.4739/87 (1987), to appear in Phys. Lett. **B**.
- [16] P. Jetzer and L. Mizrahi, Phys. Rev. Lett. **58**, 89 (1987);
P. Jetzer, J. Lacki and L. Mizrahi, Univ. of Geneva preprint UGVA-DPT 1986/09-517 (1986), to appear in Int. Journal of Mod. Phys. **A**.
- [17] A. Sugamoto and H. Suzuki, KEK preprint TH 137 (1986).
- [18] B. Schoeneberg "Elliptic Modular Functions" (Springer Verlag, Berlin-Heidelberg 1984);
Serge Lang "Elliptic Functions" (Addison-Wesley Publ. Company 1973);
J.P. Serre "Cours d'Arithmétique" (PUF 1970).
- [19] M. Ademollo, A. D'Adda, R. D'Auria, F. Gliozzi, E. Napolitano, S. Sciuto and P. Di Vecchia, Nucl. Phys. **B94**, 221 (1975);
H. Kawai, D.C. Lewellen and S.H.H. Tye, Nucl. Phys. **B269**, 1 (1986);
P. Jetzer and J. Lacki, Nucl. Phys. **B282**, 189 (1987);
J.P. Rodriguez, Phys. Lett. **178B**, 350 (1986).
- [20] S. Randjbar-Daemi, A. Salam and J.A. Strathdee, Int. Journal of Mod. Phys. **A2**, 667 (1987);
H. Oguri and N. Sakai, Tokyo Inst. of Technology preprint TIT/HEP-117, UT-512.
- [21] M.A. Namazie, K.S. Narain and M.H. Sarmadi, Phys. Lett. **177B**, 329 (1986);
S. Yahikozawa, Nucl. Phys. **B291**, 369 (1987);
M. Hayashi, N. Kawamoto, T. Kuramoto and K. Shigemoto, Kyoto Univ. preprint KUNS 856 HE/TH 87/03.
- [22] D. Mumford "Tata Lectures on Theta" (Birkhauser, Basel 1983);
A. Erdélyi et al. "Higher Transcendental Functions" (Mc Graw-Hill 1953);
W. Magnus, F. Oberhettinger and R.P. Soni "Formulas and Theorems for Special Functions of Mathematical Physics" (Springer-Verlag, N.Y. 1966).
- [23] S. Yahikozawa, Phys. Lett. **166B**, 135 (1986).
- [24] E. Witten, Nucl. Phys. **B268**, 79 (1986).
- [25] E. Witten, Phys. Lett. **155B**, 151 (1985).
- [26] J.H. Schwarz, Phys. Rep. **89**, 223 (1982);
M.B. Green, Survey in High Energy Physics **3**, 127 (1983);
M.B. Green, J.H. Schwarz and E. Witten "Superstring Theory" (Cambridge Univ. Press, Cambridge 1987).
- [27] H. Suzuki and A. Sugamoto, Phys. Rev. Lett. **57**, 1665 (1986).

- [28] M.B. Green and J.H. Schwarz, Phys. Lett. **149B**, 117 (1984);
M.B. Green, J.H. Schwarz and P.C. West, Nucl. Phys. **B254**, 327 (1985).