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MULTILOOP MODULAR INVARIANCE OF THE COVARIANT LATTICE
CONSTRUCTION OF FERMIONIC STRINGS

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ABSTRACT

A simple proof of multiloop modular invariance is given for all 10- and lower-dimensional string theories that can be obtained with the covariant lattice construction.

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In several recent papers [1-4] a lattice construction of two-dimensional superconformal field theories, yielding fermionic string theories in $d = 10-2n$ dimensions, has been proposed. The construction uses odd Lorentzian self-dual lattices [5], on which the momenta of all bosons (except those with space-time indices), all bosonized world-sheet fermions and a bosonized supergravity ghost are quantized. It has already been shown that self-duality of the lattice guarantees one-loop modular invariance. There are several reasons to expect this to extend to higher loops. First of all, other lattice formulations have that property, and it is in fact one of their advantages over fermionic constructions. This is clearly demonstrated by the ten-dimensional supersymmetric heterotic strings, where one-loop modular invariance in the bosonic formulations selects immediately the consistent $E_8 \times E_8$ and $SO(32)$ theories, whereas fermionic formulations allow several other possibilities, (by splitting fermions in groups of eight) eliminated only by imposing two-loop modular invariance.

The second reason for expecting higher-loop modular invariance for lattice theories is the conjectured equivalence of one-loop modular invariance plus a correct particle interpretation and spin statistics relation, and higher-loop modular invariance [7]. This equivalence has been proved for a class of ten-dimensional strings [8]. The two approaches to fermionic constructions start from different sides of this equivalence: in Ref. [9] spin-statistics is imposed (and this has been claimed to imply two-loop modular invariance in Ref. [10]), while in Refs. [11,12] two-loop modular invariance is imposed, and was shown to lead to a correct spin-statistics relation [12].

In the covariant lattice construction, a correct spin-statistics relation is automatic, and, as we will show, factorization and higher-loop modular invariance are nearly obvious. The lattices describing $10-2n$ -dimensional strings^{*)} are of the form $\Gamma_{16+2n;2n+5,1}$, where "16+2n" refers to the "compactified" left-moving bosons, "2n+5" to $2n$ right-moving bosons plus five bosonized NSR fermions, and "1" represents the ϕ ghost of the bosonized β - γ superghost system. The lattice must be self-dual with respect to a metric $(-)^{16+2n}(+)^{2n+5}(-)$, and the last $6-n$ components of every vector must belong to one of the conjugacy classes (o), (v), (s) or (c) of the lattice $D_{5-n,1}$, [which is the same lattice as D_{6-n} , but with a metric $(+)^{5-n}(-)$]. Finally, the $(\text{length})^2$ of vectors ending in (o) and (v) must be even, while for vectors ending in (s) and (c) it must be odd. The one-loop partition function obtained from the lattice differs in two crucial ways from the usual lattice partition functions: there is a linear term in the mass formula due to the ghost number current anomaly, and one must introduce a relative sign between the

^{*)}We consider heterotic strings here. The generalization to type II strings is straightforward.

contributions of space-time bosons and fermions. As was shown in Ref. [1], both differences are essential for one-loop modular invariance. Since there are no other signs in the partition function - the rest is just a lattice partition function with multiplicities 0 or 1 for the contribution of any vector - the theory has a correct spin-statistics relation.

Although in view of these facts an explicit proof of higher-loop modular invariance seems hardly necessary, we have learned that our lattice intuition is not shared by some sceptics, and feel obliged to dispel any remaining doubts. In this paper, we will not only demonstrate all-loop modular invariance for the covariant lattice theories, but show also that this is as obvious as it ought to be.

The most direct proof would be to extend the one given in Ref. [1] to higher loops. This approach runs quickly into serious technical difficulties, because the light-cone partition function appears as a ratio of ill-defined Lorentzian lattice partition functions already at one loop. If one could write down a well-defined higher-loop partition function, one would also have solved the problem of finding an explicit expression for the Rarita-Schwinger ghost determinant, a notoriously difficult problem, and not one that one should have to solve to prove modular invariance. All one really needs is the transformation properties of this determinant, and this is indeed all that has been used in other proofs.

At one-loop order there is a way to circumvent these problems [6] and this is the route we will follow. Even without knowing its partition function explicitly, we know precisely what the covariant lattice (together with the auxiliary χ field and the oscillators) is supposed to reproduce in fermionic language. Using the notation of Ref. [13], we represent a fermion spin structure on a genus g Riemann surface with period matrix Ω as $\alpha = \begin{bmatrix} a_1 & \dots & a_g \\ b_1 & \dots & b_g \end{bmatrix}$ (with $a_i, b_i = 0$ or $\frac{1}{2}$). The contribution of the $10-2n$ -dimensional NSR fermions and their ghosts to the path integral for a fixed spin structure is

$$\left(\det_{\alpha} \nabla_{\vec{x}} \right)^{5-n} \left(\det_{\alpha} \nabla_{-\vec{x}} \right)^{-1} \equiv C_{\alpha}(\bar{\Omega}) \quad (1)$$

We use the conventions of Ref. [14] for the operators $\nabla_{1-\lambda}$, appearing in the action for a system of b, c fields of conformal dimension $\lambda, 1-\lambda$. The NSR fermions are right-movers with antiholomorphic determinants. It is useful to write (1) as follows [8]

$$C_\alpha(\bar{\Omega}) = (\det_\alpha \nabla_{1/2})^{4-\alpha} \gamma_\alpha(\bar{\Omega}) \quad (2)$$

$$\gamma_\alpha(\bar{\Omega}) = \frac{(\det_\alpha \nabla_{1/2})}{(\det_\alpha \nabla_{1/2}')} \quad (3)$$

In covariant lattice theories, the NSR spin structures are correlated with lattice conjugacy classes in a rather complicated way. To discuss this, we have to introduce some notations. Consider p left-moving and q right-moving bosons with momenta on a lattice $\Gamma_{p,q}$. The multiloop partition function contribution due to such a lattice has been constructed in Refs. [15,16]. Denoting the lattice vectors $\vec{v}$ as $(\vec{v}_L, \vec{v}_R)$ we get

$$\rho(\Omega, \bar{\Omega}) = \sum_{\vec{v} \in \Gamma_{p,q}} e^{i\pi \vec{v}_L' \Omega_{ij} \vec{v}_L^j - i\pi \vec{v}_R' \bar{\Omega}_{ij} \vec{v}_R^j}$$

where the sum is over g copies of the lattice, and v_L^i, v_R^i are associated with the i^{th} homology cycles a^i, b^i . In the special case $q = 0$, $\Gamma_p = D_p$ (for definiteness we will take the complete weight lattice here) the sum splits into 4^g terms corresponding to the 4^g ways of assigning g vectors v_L^i to D_p conjugacy classes. We denote the terms in this sum as follows

$$\mathcal{J}^{(p)}[q_1 \dots q_g] = \sum_{v^k \in (q_k)} e^{i\pi v_L^i \Omega_{ij} v_L^j} \quad (5)$$

where (q_k) is (o), (v), (s) or (c). These conjugacy class θ -functions have a simple relation to the more familiar spin structure θ -functions

$$\mathcal{J}\left[\begin{smallmatrix} \vec{a} \\ \vec{b} \end{smallmatrix}\right] = \sum_{\vec{n} \in \mathbb{Z}^g} \exp[i\pi (\vec{n} + \vec{a}) \cdot \Omega \cdot (\vec{n} + \vec{a}) + 2i\pi (\vec{n} + \vec{a}) \cdot \vec{b}] \quad (6)$$

For genus 1 one finds

$$\begin{aligned} \mathcal{J}^{(p)}[o] &= \frac{1}{2} (\mathcal{J}^p[o] + \mathcal{J}^p[\begin{smallmatrix} o \\ i \end{smallmatrix}]) \\ \mathcal{J}^{(p)}[v] &= \frac{1}{2} (\mathcal{J}^p[o] - \mathcal{J}^p[\begin{smallmatrix} o \\ i \end{smallmatrix}]) \\ \mathcal{J}^{(p)}[s] &= \frac{1}{2} (\mathcal{J}^p[\begin{smallmatrix} \frac{1}{2} \\ o \end{smallmatrix}] + i^p \mathcal{J}^p[\begin{smallmatrix} \frac{1}{2} \\ i \end{smallmatrix}])) \end{aligned} \quad (7)$$

$$g^{(p)}[c] = \frac{1}{2} (g^{(p)}[\frac{1}{2}] - i^p g^{(p)}[\frac{1}{2}]) \quad (7)$$

cont.

For higher genus one uses the same transformations in the obvious way (i.e., as if the multiloop θ -functions were factorized in terms of one-loop functions).

This basis transformation allows us to split off the partition function contribution of D_{4-n} , the lattice components yielding the transverse Lorentz group $SO(8-2n)$, and write it in terms of spin structures. We will then use (2) to put in the longitudinal and ghost contributions. To do so, write the lattice as $\Gamma_{16+2n;3n} \times D_{5-n,1}$. In terms of $D_{5-n,1}$ conjugacy classes, the lattice decomposes as

$$(\Delta_o, (o)) + (\Delta_v, (v)) + (\Delta_c, (c)) + (\Delta_s, (s)) \quad (8)$$

where Δ_q is a set of vectors, not necessarily forming a lattice. The "transverse" part of the partition function is

$$\sum_{\{q^i\}} g^{(4-n)}[q_1 \dots q_g] P(\Delta_{q_1} \dots \Delta_{q_g}) \quad (9)$$

where the sum is over all 4^g assignments of D_{4-n} conjugacy classes to the a^i cycles, and P is defined as

$$P(\Delta_{q_1} \dots \Delta_{q_g}) = \sum_{v^i \in \Delta_{q_i}} e^{i\pi v^i(q_i) \Omega_{ij} v^j(q_j) - i\pi \bar{v}^i(q_i) \bar{\Omega}_{ij} \bar{v}^j(q_j)} \quad (10)$$

Using the basis transformation defined above, we can rewrite (9) as

$$\sum_{\alpha} g^{(4-n)}[\alpha] P_{\alpha}(\Omega, \bar{\Omega}) \quad (11)$$

where the sum is over all spin structures, and P_{α} is a linear combination of the 4^g functions defined in (10). Because the Rarita-Schwinger determinant has the same spin structure as the NSR fermions, the complete genus g partition function is obtained by simply replacing $\theta^{4-n}[\alpha]$ by C_{α} :

$$P_g(\Omega, \bar{\Omega}) = N(\Omega, \bar{\Omega}) \sum_{\alpha} C_{\alpha}(\bar{\Omega}) P_{\alpha}(\Omega, \bar{\Omega}) \quad (12)$$

where $N(\Omega, \bar{\Omega})$ denotes spin structure independent factors (measure factors and oscillator contributions), which play no essential role in the argument.

We will now show that (12) satisfies the requirements of factorization into one-loop contributions in the limit where the surface degenerates into g tori, and of multiloop modular invariance. We will have to assume factorization of the Rarita-Schwinger determinant, since we do not have an explicit expression. The rest is then obvious: the lattice partition function satisfies factorization explicitly, and the spin structure sum in (12) does not include any spin structure dependent coefficients which could upset factorization. This simple factorization property is a typical advantage of a lattice construction, because the lattice determines at once the periodicities of the bosons along the homology cycles of an arbitrary genus Riemann surface, so that the result cannot fail to factorize. By contrast, spin structure summation in fermionic constructions has to be done for every genus separately, and to respect unitarity one has to impose factorization by hand.

In principle, we could have introduced phases in "glueing" Y_α to the lattice partition function. This remaining phase choice is a consequence of the fact that for pragmatic reasons we have not directly used the lattice to obtain $\det_\alpha V_{-\frac{1}{2}}$. As stated above, the choice (12) respects factorization. We will now show that it is also required to obtain multiloop modular invariance.

The modular transformation properties of chiral determinants have recently been calculated in Ref. [14]. The results are as follows. Let $\Lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z})$ represent a symplectic modular transformation, so that Ω transforms into $\tilde{\Omega} = (A\Omega + B)(C\Omega + D)^{-1}$. The θ -function transformation is

$$\mathcal{J} \left[\begin{smallmatrix} \vec{a}' + \vec{e} \\ \vec{b}', \vec{f} \end{smallmatrix} \right] / (\tilde{\Omega}) = \varepsilon(\Lambda) e^{i\pi\phi(a, b, \Lambda)} \det(C\Omega + D)^{-\frac{1}{2}} \mathcal{J} \left[\begin{smallmatrix} \vec{a} \\ \vec{b} \end{smallmatrix} \right] / (\Omega) \quad (13)$$

where

$$\left[\begin{smallmatrix} \vec{a}' \\ \vec{b}' \end{smallmatrix} \right] = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \left[\begin{smallmatrix} \vec{a} \\ \vec{b} \end{smallmatrix} \right] \quad (14)$$

$$\left[\begin{smallmatrix} \vec{e} \\ \vec{f} \end{smallmatrix} \right] = i \left[\begin{smallmatrix} \text{diag}(CD^T) \\ \text{diag}(AB^T) \end{smallmatrix} \right] \quad (15)$$

$$\phi(a, b, \Lambda) = \vec{a}' \cdot \vec{b}' - \vec{a} \cdot \vec{b} + 2\vec{a}' \cdot \vec{f} \quad (16)$$

and $(\varepsilon(\Lambda))^8 = 1$. Then the determinants transform as follows

$$\det_{\alpha} V_{1,2}(\tilde{\Omega}) = \varepsilon^{\frac{1}{6}} e^{i\pi\phi(a,b,\Lambda,\lambda)} [\xi(\Lambda)]^{-12\lambda^2 + 12\lambda - 2} \\ \times \det_{\alpha} V_{1,2}(\Omega) \quad (17)$$

where

$$\phi(a,b,\Lambda,\lambda) = \phi(a,b,\Lambda) + 2(2\lambda-1)(\vec{a}'\vec{f} + \vec{b}'\vec{e}) \quad (18)$$

The phase factor $\xi(\Lambda)$ was not explicitly written down in Ref. [14], because it cancels in conformally invariant combinations of determinants, and does not depend on the spin structure. We can now derive the transformation of Y_{α}

$$Y_{\alpha}(\tilde{\Omega}) = \xi^{12} e^{-i\pi(\vec{a}'\vec{f} + \vec{b}'\vec{e})} Y_{\alpha}(\Omega) \quad (19)$$

The crucial point is now that we can write down a combination of determinants which transforms exactly like Y_{α} as long as only periodic and antiperiodic boundary conditions are considered (which is all that is needed for NSR fermions), namely

$$X_{\alpha}(\Omega) = [\det_{\alpha} V_{1,2}(\Omega)]^{12} \quad (20)$$

The equivalence of the modular transformation properties of X_{α} and Y_{α} is not manifest, because it holds only for half-integral characteristics, but it is easy to demonstrate by checking the generators of all modular transformations, namely the Dehn twists along the a_i and b_i cycles, and the cycles linking two holes. Therefore, (12) is modular invariant if and only if the following function is modular invariant

$$P_g^B(\Omega, \bar{\Omega}) = N(\Omega, \bar{\Omega}) \sum_{\alpha} [\det_{\alpha} V_{1,2}(\tilde{\Omega})]^{16-n} P_{\alpha}(\Omega, \bar{\Omega}) \quad (21)$$

where we have simply replaced Y_{α} by X_{α} . By expressing the determinants in terms of $\wp$ -functions, absorbing spin structure independent factors into N , and undoing the transformation leading to (11), we get

$$P_g^B(\Omega, \bar{\Omega}) = N'(\Omega, \bar{\Omega}) \sum_{\{g_i\}} \mathcal{J}[\varrho_1, \dots, \varrho_g] P(\Delta_{\varrho_1}, \dots, \Delta_{\varrho_g}) \quad (22)$$

But this is nothing but the partition function of the bosonic string compactified on a lattice $\Gamma_{16+2n;16+2n}$, obtained from the lattice $\Gamma_{16+2n;3n} \times D_{5-n,1}$ by simply replacing all conjugacy classes of $D_{5-n,1}$ by the same classes of the lattice D_{16-n} .

It is easy to see that this map preserves self-duality, and changes all odd $(\text{length})^2$ vectors into even ones. Hence $\Gamma_{16+2n;16+2n}$ is even and self-dual, and (22) is all-loop modular invariant, as was shown in Ref. [15]. Therefore, (12) must be all-loop modular invariant as well.

Although Y_α transforms exactly as X_α , these functions are of course not the same. At one loop, Y_α is known to be a constant. From its modular transformations (including the phase ξ , which is easy to determine at one loop), we find that we can consistently choose

$$Y\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] = -Y\left[\begin{smallmatrix} i \\ 0 \end{smallmatrix}\right] = -Y\left[\begin{smallmatrix} 0 \\ i \end{smallmatrix}\right] = / \quad (23)$$

which yields the spin-statistics sign and the GSO projection rules. At higher loops, the transformation properties of Y_α are inconsistent with it being a constant.

The function X_α is not unique. Another choice is

$$X'_\alpha = \left(\det_\alpha \nabla_{\frac{1}{2}} \right)^4 \sum_\beta \left(\det_\beta \nabla_{\frac{1}{2}} \right)^8 \quad (24)$$

This corresponds to replacing $D_{5-n,1}$ by $D_{8-n} \times E_8$ instead of D_{16-n} . The resulting lattice is the one used in Refs. [17,18] where fermionic strings were constructed in terms of the bosonic string. The replacement of X'_α by Y_α corresponds at one loop, in combination with (23), precisely to the truncation rules proposed in Ref. [18]. Although both X_α and X'_α are equally suitable for proving modular invariance, the lattice with the E_8 factor has the interesting property that the masses of the fermionic string states are equal to those of the bosonic string states. Although it is tempting to speculate about deeper connections between X'_α and Y_α [6], it must be emphasized that in this paper we have only used the fact that they transform in the same way.

Instead of using the explicit transformation properties of the determinants, we could also have relied upon Witten's proof of modular invariance of the ten-dimensional heterotic strings [17]. In ten dimensions, the map $Y_\alpha \rightarrow X_\alpha$ must preserve modular invariance, because we know from Refs. [15] and [17] that both the initial and final theories are modular invariant. The same map is used in four dimensions, and results also in a bosonic string compactified on an even self-dual lattice. Hence the fermionic strings constructed from the odd self-dual lattice must be modular invariant as well. This is the clearest way of understanding the result.

To summarize, the spin-statistics relation, factorization and multiloop modular invariance are (nearly) manifest in the covariant lattice construction,

giving it a clear conceptual advantage over fermionic constructions for those string theories that can be obtained with both methods.

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