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NOWHERE-VANISHING SPINORS AND TOPOLOGICAL OBSTRUCTIONS
TO THE EQUIVALENCE OF THE NSR & GS SUPERSTRINGS

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ABSTRACT

Constructing an equivalence of the NSR and GS superstring formalisms in a general spacetime background, involves relating the δ_v and δ_s representations with the aid of a nowhere-vanishing Majorana-Weyl spinor on the eight-dimensional (transverse) space. We discuss the possibility of a global obstruction to the existence of such spinor fields and show that a necessary and sufficient condition for this to be absent is the vanishing of a certain linear combination of the characteristic classes of the spacetime.

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1. Introduction

The type II and heterotic string theories exhibit supersymmetry both on the two-dimensional world sheet and in the ten-dimensional spacetime. World-sheet supersymmetry is manifest in the Neveu-Schwarz-Ramond (NSR) formalism, in which the fermions transform as vectors in spacetime, while the spacetime symmetry is manifest in the Green-Schwarz (GS) formalism, where the fermions transform as spinors in spacetime. In the light-cone gauge, the fermions of the NSR formalism carry the δ_v vector representation of the SO8 tangent space group of the transverse space, while in the GS formalism they carry the δ_s spinor representation. In certain spacetime backgrounds including, in particular, flat space, the equivalence of the NSR and GS formalisms is easily understood: it follows from the triality properties of SO8, which enable one to use a constant δ_c spinor (with the opposite chirality to that of the δ_s representation) in the transverse space in order to transform between the δ_v and δ_s representations.

For strings propagating in a general curved background spacetime, the relation between the NSR and GS formalisms is considerably more subtle. The essential points can be illustrated by considering the type IIB theory in a curved spacetime and setting the antisymmetric tensor to zero for simplicity. In the light-cone gauge, the NSR action takes the form

$$I = \int d^2\sigma \left\{ \partial_\alpha X'^I \partial_\beta X'^J g_{IJ}(X) + i \lambda_L^a \nabla_\alpha \lambda_L^a + i \lambda_R^a \nabla_\alpha \lambda_R^a + \frac{1}{2} R_{abcd} \lambda_L^a \lambda_L^b \lambda_R^c \lambda_R^d \right\} \quad (1.1)$$

where X'^I ($I=1\dots 8$) are the coordinates of the string map from the two-dimensional world-sheet into the eight-dimensional transverse space M_8 whose Riemannian metric is g_{IJ} . The fermions λ_L^a , λ_R^a are left- and right-moving Majorana-Weyl (one component) spinors on the Lorentzian world-sheet which carry an δ_v tangent-space index a ($a=1\dots 8$).

The action (1.1) is invariant under the world-sheet supersymmetry transformations

$$\begin{aligned} \delta X'^I &= \alpha_L \lambda_L^I + \alpha_R \lambda_R^I \\ \delta \lambda_L^a &= -\omega_I^{ab} \lambda_L^b \delta X'^I + i \alpha_L \partial_- X'^I e_I^a \\ \delta \lambda_R^a &= -\omega_I^{ab} \lambda_R^b \delta X'^I + i \alpha_R \partial_+ X'^I e_I^a \end{aligned} \quad (1.2)$$

where α_L and α_R are left- and right-moving world-sheet supersymmetry parameters and e^a is the vielbein. This (1.1) supersymmetry enlarges to (2,2) if g_{ij} is a Kahler metric, corresponding to the existence of a second, independent pair of supersymmetries of the form (1.2), where λ^a is replaced throughout with $J^a_b \lambda^b$ with J^a_b the complex structure tensor on M_6 . (If g_{ij} is hyperkahler, there is a further enlargement to a (4,4) supersymmetry; one additional pair for each of the three complex structure tensors.)

The situation is much less clear cut when one considers a Green-Schwarz formalism for a curved spacetime background. Unlike the NSR action discussed above, which is world-sheet supersymmetric in an arbitrary spacetime, there is no known GS action that exhibits spacetime supersymmetry in such a background. Indeed, it is not clear that such an action *should* exist, since only certain special spacetime backgrounds will be supersymmetric anyway. Furthermore, the string can only propagate consistently in backgrounds that satisfy the string equations of motion, so for this reason too one should not necessarily expect a GS action to make sense for an arbitrary background. Unfortunately, one does not know explicitly what the conditions are that must be satisfied by a background in order that it be consistent and supersymmetric. It was thought originally that compactifications on compact six-dimensional Ricci-flat Kahler spaces would provide such backgrounds [1], but this turned out not to be the case [2,3,4]; finding the equations of motion that determine the background requires solving the theory to all orders in α' .

For the time being therefore, it seems that there is no way to construct a spacetime supersymmetric light-cone action for a general background manifold. In [5,6], it was proposed that one can construct a GS action for at least a wide class of background spacetimes by generalizing the triality relation that enables the NSR and GS actions to be related when this space is flat. This can be done whenever the transverse space M_8 admits a nowhere-vanishing Majorana-Weyl spinor ξ , which we take to be in the θ_c representation of the SO8 tangent space group. Without loss of generality, such a spinor can be normalized such that

$$\bar{\xi} \xi = 1. \quad (1.3)$$

Using the Dirac matrices γ_a (with $\{\gamma_a, \gamma_b\} = -2\delta_{ab}$) for eight dimensions, we may now define the spinor S , in the θ_s representation of SO8, as

$$S := -\gamma_a \xi \lambda^a \quad (1.4)$$

where λ^a is either λ_L^a or λ_R^a in the NSR action (1.1). Because of (1.3), and the

special properties of the λ -matrices in eight dimensions, equation (1.4) may be inverted to give

$$\lambda^a = \bar{\xi} \gamma^a S \quad (1.5)$$

Hence (1.4) and (1.5) provide an invertible mapping between the θ_v and θ_s representations of SO8. Substituting (1.5) into (1.1) then gives an action [6] that is in GS form and in which ξ appears as part of the background structure. This action is invariant under the set of supersymmetry transformations obtained by substituting (1.5) into (1.2); the new supersymmetry parameters depend on the original ones and the spinor ξ .

Thus the conversion from NSR to GS form can be achieved whenever the transverse space M_8 admits a Majorana-Weyl spinor ξ that is nowhere-vanishing, so that ξ can be normalised to satisfy (1.3). Such spinors play an important role in any discussion of supersymmetry in a curved space (not just in string theory) but then the question arises of when such a spinor field exists. This is the issue which we wish to address in the present paper. One is inevitably reminded of the classical problem of deciding whether or not an orientable differentiable manifold M_n admits a nowhere-vanishing vector field - the famous "hairy ball" problem. The well-known answer is that a necessary and sufficient condition is that the Euler class of M_n must vanish.

This vector field result applies whatever the dimension n of M_n , but we shall show that the situation for spinors is quite different and a significant potential obstruction to a nowhere-vanishing field can only arise when $n=8$, in which case the condition for the existence of a unit-length Majorana-Weyl spinor field is the vanishing of a certain linear combination of the Euler class, the second Pontryagin class and the square of the first Pontryagin class of the manifold. It transpires that the sign of the Euler class appearing in this expression is different for the θ_s and the θ_c spinors. Consequently, if spinors of both types exist, we may subtract the expressions for θ_s and θ_c and deduce that the Euler class itself must vanish. This is consistent with the observation that, in an eight-dimensional space, the Fierz-resuffling properties of the γ -matrices are such that if ξ and ζ are a pair of opposite chirality, nowhere-vanishing, Weyl-spinors then the vector field $V_a = \bar{\xi} \gamma_a \zeta$ is itself nowhere-vanishing.

2. The Universal Obstruction

2.1

Let us consider first the problem of the existence of a nowhere-vanishing vector field V^i ($i=1..n$) on an orientable, n -dimensional manifold M . This defines a nowhere-vanishing cross-section of the tangent bundle TM of M but, if g_{ij} is some Riemannian metric on M , V can be normalised (i.e. $V^i V^j g_{ij} = 1$) and hence defines a cross-section of a bundle over M whose fibre is an $(n-1)$ -sphere. Similarly, a nowhere-vanishing Majorana spinor can be deformed into one that is of unit length with respect to an invariant inner product on the real vector space carrying the spin representation of $SO(n)$. In both cases therefore, we need to study the potential obstructions to the existence of cross-sections of sphere-bundles.

More generally, let $\beta = F \rightarrow E \rightarrow M$ denote a bundle over M with total space E and fibre F . A standard way [7] of attempting to find a cross-section for β is to triangulate the base manifold M with a triangulation whose mesh is sufficiently fine to ensure that the restriction of β to any n -simplex Δ_n is a trivial bundle. Now construct an arbitrary cross-section S_0 on the 0-simplices of the triangulation and try to extend this to a continuous cross-section S_1 on the 1-simplices that link together the 0-simplices. This is always possible if F is connected (which we shall assume), so now we can consider the problem of extending this section to the 2-skeleton K_2 . Since the bundle is trivial over each Δ_n , S_1 can be represented as a collection of F -valued functions defined on the 1-simplices lying in the boundaries of the n -simplices. Every 2-simplex in a Δ_n is bounded by a specific subset of the 1-simplices and the extension to K_2 involves extending (for each Δ_n) the function on these 1-simplices to the interior of the 2-simplex which they bound. However, an F -valued function on the boundary $\partial\Delta_2$ of a 2-simplex Δ_2 is a map into F from a space with the topology of a circle and, as such, it defines an element of the first homotopy group $\pi_1(F)$. Furthermore, this function can be extended to Δ_2 if and only if this group element is trivial; if it is not, it constitutes an obstruction to extending the cross-section from K_1 to K_2 .

It follows from the above that an obstruction to extending S_1 to a cross-section on the 2-skeleton can be represented by a collection of maps from the 2-simplices into $\pi_1(F)$, and it can be shown that this defines an element of the cohomology group $H^2(M; \pi_1(F))$. If there is no obstruction at this level we can try to extend the section S_2 on the 2-skeleton to a section S_3 on the 3-skeleton, and then stepwise until we come to the final extension from the $(n-1)$ -skeleton to the

Thus the critical cohomology class lies in the group $H^2(M_8)$, which is trivial when M_8 is non-compact. In particular, this means that there are no obstructions to nowhere vanishing spinors when $M_8 \approx \mathbb{R}^2 \times M_6$, and hence there is no problem in the case in which the theory is dimensionally reduced with a compact M_6 and with the effective four-dimensional spacetime being taken to be Minkowski space. Our results are only relevant in cases where the full ten-dimensional space is assigned non-trivial topological structure. Such a situation can arise from cosmological considerations, or in a quantum scheme which incorporates some idea of "spacetime foam" in all ten dimensions, or simply via an effective compactification that is induced by the imposition of vanishing boundary conditions on the fields in the theory (as, for example, in the case of instanton studies on a euclideanised space whose topology is that of S^4). Our results are also of course applicable to any theory in which compactification is performed directly from a ten- to a two-dimensional space, or in any other situation in which one is interested in spinors on a compact eight-dimensional manifold.

The spinorial results are more complicated than those for the vector field and, to derive them, we have delved into the theory of obstructions in universal bundles. The general mathematical theory of such bundles is described in Section 2 and the explicit calculations for our case are given in Section 3. A key computation is the embedding in $Spin(8)$ of the $Spin(7)$ subgroup that leaves invariant a specific unit-length Majorana-Weyl spinor. The results of this calculation are used in Section 3 but the details of the embedding are relegated to the Appendix.

n -dimensional manifold M itself. Thus the potential obstructions to building a cross-section for the bundle β can be represented by elements of the cohomology groups

$$H^2(M; \pi_1(F)), H^3(M; \pi_2(F)), \dots, H^n(M; \pi_{n-1}(F)) \quad (2.1)$$

(higher groups cannot occur since $H^q(M) \approx 0$ if $q > \dim(M)$). If the first non-vanishing homotopy group of F is $\pi_p(F)$, the first obstruction that could be encountered will lie in the cohomology group $H^{p+1}(M; \pi_p(F))$ and is known as the *primary* obstruction. This cohomology element is intrinsic to the bundle itself (it is one of the major characteristic classes) and its vanishing is both a necessary and a sufficient condition for the existence of a cross-section on the $(p+1)$ -skeleton of M .

These ideas are especially fruitful when the fibre F is a sphere S^k since the first $k-1$ homotopy groups then vanish. We also know that $\pi_k(S^k) \approx \mathbb{Z}$ and hence the primary obstruction lies in $H^{k+1}(M; \mathbb{Z})$. In particular, if k is greater than or equal to the dimension of M , there will be no obstructions to the existence of a cross-section. Note that the problem of finding a nowhere-vanishing vector field on the n -dimensional manifold M involves a bundle whose fibre is S^{n-1} ; hence the primary obstruction lies in $H^n(M; \mathbb{Z})$ and it is the *only* obstruction that can occur. For any compact, orientable manifold, $H^n(M; \mathbb{Z}) \approx \mathbb{Z}$ ($H^n(M; \mathbb{Z}) \approx 0$ if M is non-compact) and the obstruction is, of course, just the Euler class.

The situation vis-à-vis real spinor fields is similar to the above except that the sphere is now an S^{s-1} lying in the real vector space $\mathbb{R}^s$ carrying the spinor representation of interest. A crucial property of spinor representations however [8,9] is that their dimension rises rapidly with the dimension n of the manifold M . In particular, for $n > 8$, the value of s is sufficiently large that there are no obstructions at all (for example, the real spinor representation of SO_9 has dimension 16). Note that this does not contradict the existence of a possible obstruction to a nowhere-vanishing vector field since the Fierz-resuffling needed to show that the vector field $\bar{\xi} \gamma_a \zeta$ is nowhere-vanishing (see Introduction) does not work for $n > 8$.

However, the real spinor representations of SO_8 are eight-dimensional (ie. 8_s and 8_c) and in this case there is a possible obstruction, which is primary and lies in $H^8(M; \pi_1(S^7)) \approx H^8(M; \mathbb{Z})$. On the other hand, the real spinor representations for SO_5, SO_6, SO_7 also have dimension eight (being respectively $4 \oplus 4, 4 \oplus \bar{4}, 8$), which is too big to give any obstructions on manifolds of dimensions five, six or seven. If M has dimension four, obstructions may exist but these are

rather trivial since $Spin_4 \approx SU_2 \times SU_2$ and the existence of nowhere-vanishing Weyl spinors of the two types simply corresponds to the vanishing of the left- or right- second Chern classes of the tangent bundle TM . Thus it appears that the only manifolds M for which there may be interesting obstructions to finding nowhere vanishing real spinor fields are those of dimension eight - which is precisely the case that arises in showing the equivalence of the NSR and GS superstrings in a general background spacetime!

2.2

Our major task now is to express the spinor field obstruction in $H^8(M; \mathbb{Z})$ as a specific combination of the standard characteristic classes of the tangent bundle TM . Taking again the vector field problem as an example, we start by observing that TM can be viewed as a vector bundle associated with the principal $SO(n)$ -bundle $\mathcal{D}(M)$ of oriented orthonormal frames (ie. n -beins) $TM \approx \mathcal{D}(M) \times_{SO(n)} \mathbb{R}^n$. Specifically, the equivalence class of the n -bein $e \in \mathcal{D}(M)$ and vector $\hat{r} \in \mathbb{R}^n$ is identified with the tangent vector $\sum_{a=1}^n e_a \cdot r^a$. In n -bein components, the normalization condition on a vector field V is simply $V^a V^a = 1$, which shows that V lies on a S^{n-1} orbit of $SO(n)$ on $\mathbb{R}^n$. This orbit is diffeomorphic to the coset space $SO(n)/SO(n-1)$ where $SO(n-1)$ is the stability group of some chosen fiducial unit vector in $\mathbb{R}^n$, and hence it follows that the nowhere-vanishing vector field V defines a cross-section of the associated bundle $\mathcal{D}(M) \times_{SO(n)} SO(n)/SO(n-1)$. However, the existence of such a section is precisely equivalent to the statement that the structure group of the bundle can be reduced [8] to the subgroup $SO(n-1)$. This restatement of the problem is useful since the reduction of structure groups is a subject that is amenable to treatment with standard methods in algebraic topology.

For an eight-dimensional manifold M , a similar argument shows that the existence of a nowhere-vanishing Weyl-Majorana spinor (in, say, the 8_s representation) is equivalent to the reducibility of the structure group of the spin bundle from $Spin_8$ to $Spin_7$, where $Spin_7$ is the isotropy group of a reference unit 8_s spinor. In the vector field case, the Lie algebra of the SO_7 group can be readily computed in terms of the usual generators T_{ij} ($i, j = 1..8$) of SO_8 and shown to be the set $\{T_{ij} \mid i, j = 1..7\}$. The relation between the $Spin_7$ group and $Spin_8$ (which of course has the same Lie algebra as $SO(8)$) is somewhat more complicated and will be discussed in detail in the next Section. For the moment it suffices to remark that it is the difference in the embeddings of these SO_7 Lie sub-algebras of SO_8 that is ultimately responsible for the difference between the topological obstructions to nowhere-vanishing vector and spinor fields.

We will now discuss briefly the use of universal bundles to study characteristic classes and the existence of topological obstructions to the reduction of the structure group of a principal G -bundle to some subgroup $H \subset G$. The importance of universal bundles [10] can be motivated by noting that if $\beta = G \rightarrow E \xrightarrow{\pi} M$ is a principal G -bundle over M , then any smooth map φ from another manifold N into M induces a "pull-back" G -bundle $\varphi^*(\beta)$ over N whose total space is the set $\{(x, p) \in N \times E \mid \varphi(x) = \pi(p)\}$. This pull-back operation has the important property that if φ_1 and φ_2 are two such maps which are homotopic (denoted $\varphi_1 \sim \varphi_2$) then the induced bundles are isomorphic (denoted $\varphi_1^*(\beta) \approx \varphi_2^*(\beta)$). A central theorem in the subject is the proof [11] that, for any Lie group G , there exists a universal principal G -bundle $\mathcal{U}_G = G \rightarrow EG \rightarrow BG$ with the properties that

- (a) Given any principal G -bundle β over any manifold M , there exists a map $\varphi: M \rightarrow BG$ such that $\beta \approx \varphi^*(\mathcal{U}_G)$.
- (b) If φ_1 and φ_2 are a pair of such maps, then $\varphi_1^*(\mathcal{U}_G) \approx \varphi_2^*(\mathcal{U}_G)$ implies $\varphi_1 \sim \varphi_2$.

It follows that the set of all inequivalent principal G -bundles over M is in bijective correspondence with the set of homotopy classes of maps from M into BG (this is well-defined since BG is unique up to homotopy). Another (and essentially characteristic) property of universal bundles is that the bundle space EG is contractible. For example, a universal $\mathbb{Z}$ -bundle is $\mathbb{Z} \rightarrow \mathbb{R} \rightarrow S^1$ and a universal $U(1)$ -bundle is $U(1) \rightarrow S^\infty \rightarrow \mathbb{C}P^\infty$ where S^∞ and $\mathbb{C}P^\infty$ are respectively the infinite-sphere and the infinite-dimensional complex projective space.

A universal characteristic class for G is defined to be any element $c_{\mathcal{U}}$ in a cohomology group $H^k(BG; \pi)$ where k is any positive integer and π is any abelian group. If β is a G -bundle over a manifold M , the characteristic classes for β are defined as the elements $\varphi^*(c_{\mathcal{U}}) \in H^k(M; \pi)$ where $\varphi: M \rightarrow BG$ is such that $\beta \approx \varphi^*(\mathcal{U}_G)$ (it follows from the remarks above that these "pulled-back" cohomology elements are independent of the choice of φ which induces β). It is clear that any generic relations among characteristic classes need to be computed only once within the cohomological structure of the universal base-space BG (which is why the method is so useful) and, in particular, it should be possible to obtain the obstruction results we seek in this way.

However, this requires a knowledge of the cohomology groups of the relevant BG , and this is not a trivial matter. Fortunately, with the aid of a standard trick described below, it is often possible to restrict one's attention to the case where G

is a torus, for which the necessary information is known. More precisely, if T_n is a n -torus (the Cartesian product of n circles), a key cohomological group is $H^1(T_n; \mathbb{Z})$ which is just a direct sum of n copies of the integers $\mathbb{Z}$ (ie. it has n generators with no relations between them). The space BT_n is homotopic to a Cartesian product of n copies of $BU(1) \sim \mathbb{C}P^\infty$ and the integral cohomology of the latter is well known to be the algebra generated by a single element in $H^2(BU(1); \mathbb{Z}) \approx \mathbb{Z}$. Thus $H^2(BT_n; \mathbb{Z})$ is also isomorphic to n copies of $\mathbb{Z}$, and this information is more or less sufficient for our needs.

Suppose then that $j: H \rightarrow G$ is a homomorphism of a group H into a group G . Any H -bundle can be extended by j to form a principal G -bundle and, in particular, there must exist a map $Bj: BH \rightarrow BG$ which extends the structure group of $\mathcal{U}_H$ from H to G . These maps Bj are uniquely determined up to homotopy and, most importantly, any commutative diagram of groups and homomorphisms yields an associated "BG-diagram" (which commutes up to homotopy) in which every group G and homomorphism j is replaced by BG and Bj respectively.

It follows from the above that if α is any H -bundle over M , induced by a map $\psi: M \rightarrow BH$, then the corresponding G -bundle associated with the homomorphism $j: H \rightarrow G$ is induced by the composite map $Bj \circ \psi: M \rightarrow BG$. Conversely, the structure group of a G -bundle (induced by $\varphi: M \rightarrow BG$) is reducible to a subgroup H (with injection $i: H \rightarrow G$) if and only if there exists $\psi: M \rightarrow BH$ such that the following diagram is homotopy commutative

$$\begin{array}{ccc} & BH & \\ \psi \nearrow & & \downarrow Bj \\ M & \xrightarrow{\varphi} & BG \end{array} \quad (2.2)$$

The relevance of this for us is as follows. Suppose that $c_{\mathcal{U}}$ is any universal characteristic class for G with the property that $Bj^*(c_{\mathcal{U}}) = 0$ and let β be a G -bundle whose group is reducible to H . Then the defining map $\varphi: M \rightarrow BG$ must factor through some $\psi: M \rightarrow BH$ as above and hence $\varphi^*(c_{\mathcal{U}}) = \psi^*(Bj^*(c_{\mathcal{U}})) = 0$. Thus we arrive at the crucial conclusion that a necessary condition for the structure group of a G -bundle β over M to be reducible to a subgroup H is that all characteristic classes of β corresponding to elements in the kernel of Bj^* should vanish. The question of sufficiency is more complicated but it can be studied successfully with the aid of, for example, a Postnikov factorization [10,12] of the fibration $Bj: BH \rightarrow BG$ (whose fibre is the coset space G/H). We shall not go into the details here except to remark that such an investigation shows that the potential obstruction classes lie

In the cohomology groups $H^{k+1}(BG; \pi_k(G/H))$. It also reaffirms the comment made earlier that if there is just one single primary obstruction then its vanishing provides both a necessary and a sufficient condition for reduction.

2.3

The problem of constructing nowhere-vanishing vector or spinor fields can be tackled successfully using these methods and, as explained already, the potential obstruction is just a single primary cohomology class. In the case of vector fields on a $2n$ -dimensional manifold (the odd-dimensional case is trivial) we are confronted with the embedding $i: SO(2n-1) \rightarrow SO(2n)$ and it is a classic result that the kernel of $B_i^*: H^{2n}(BSO(2n); \mathbb{Z}) \rightarrow H^{2n}(BSO(2n-1); \mathbb{Z})$ is just the universal Euler class. Thus we recover the well-known result for the existence of a nowhere-vanishing vector field.

The task now is to consider the embedding $i_s: Spin(7) \rightarrow Spin(8)$ and to find the kernel of the associated homomorphism $B_i^*: H^8(BSpin(8); \mathbb{Z}) \rightarrow H^8(BSpin(7)_s; \mathbb{Z})$. We shall approach this by considering the effects of the embedding on the maximal tori of the two groups, a problem that is relatively straightforward because of the simply cohomological structure of a space BT_n . The effectiveness of this method [13] stems from the following striking fact. Let T denote a maximal torus for the compact Lie group G with an associated embedding $\rho: T \rightarrow G$. Then the associated cohomology map $B\rho^*: H^k(BG; \pi) \rightarrow H^k(BT; \pi)$ is frequently *injective*, and this allows relations between the cohomology elements of BG to be studied via their images in the considerably simpler cohomological structure of BT . In addition, the universal characteristic classes in $H^*(BG; \pi)$ are frequently defined with the aid of the map $B\rho^*$. For example, if T_n is the maximal torus of the unitary group $U(n)$, the universal Chern classes $c_k \in H^{2k}(BU(n); \mathbb{Z})$ are uniquely defined by the relation [13]

$$B\rho^*\left(1 + \sum_{k=1}^n c_k\right) = \prod_{k=1}^n (1 + \gamma_k) \quad (2.3)$$

where γ_k ($k = 1 \dots n$) are the generators of $H^2(BT_n; \mathbb{Z})$.

Suppose now that $i: H \rightarrow G$ is a subgroup embedding and that T_H and T_G are the maximal tori of H and G respectively. We shall assume that i is a regular embedding in the sense that i maps T_H into T_G (all the examples of interest to us are of this type) so that we get the pair of commutative diagrams

$$\begin{array}{ccc} T_H & \xrightarrow{\rho_H} & H \\ \downarrow i & & \downarrow i \\ T_G & \xrightarrow{\rho_G} & G \end{array} \quad \begin{array}{ccc} BT_H & \xrightarrow{B\rho_H} & BH \\ \downarrow B_i & & \downarrow B_i \\ BT_G & \xrightarrow{B\rho_G} & BG \end{array} \quad (2.4)$$

where i is the restriction of i to T_H . Now suppose there is some universal characteristic class c for G such that $B_i^*(B\rho_G^*(c)) = 0$, i.e. $B\rho_G^*(c)$ lies in the kernel of B_i^* . Then, by commutativity, we have $B\rho_H^*(B_i^*(c)) = 0$ which, in those cases where $B\rho_H^*$ is injective, means that $B_i^*(c) = 0$. Thus information on the kernel of B_i^* can be extracted from the intersection of the kernel of B_i^* with the image under $B\rho_G^*$ of the cohomology of BG . We will employ this powerful method in Section 3 to compute the obstructions to nowhere-vanishing spinor fields.

Finally, let us recall for later reference that the maximal torus for the group $SO(2n)$ consists of all block diagonal matrices of the form $\text{diag}(D_1, D_2, \dots, D_n)$ where D_j ($j = 1 \dots n$) are a set of 2×2 rotation matrices parametrized by angles θ_j lying in the range 0 to 2π . For the group $SO(2n+1)$, the relevant matrices are of the form $\text{diag}(D_1, D_2, \dots, D_n, 1)$.

3. The Spin 8 to Spin 7 Reduction

3.1

We must turn now from the general theory to the specific case of interest and hence to the problem of finding the explicit embedding of the maximal torus of $Spin 7_s$ in the maximal torus of $Spin 8$. In effect, we need to know the sums of $Spin 8$ generators $\lambda = \sum_{i,j=1}^8 \lambda_{ij} \Gamma_{ij}$ such that $\lambda \eta = 0$ where η is a reference unit-length, Majorana-Weyl spinor. This in turn involves computing a concrete form for this spinor representation, which we will do in terms of ideals in the Clifford algebra.

The Clifford algebra $[8,9] C_{2n}$ over $\mathbb{R}^{2n}$ is the real algebra with generators $\gamma_1, \dots, \gamma_n$ satisfying the relations $\gamma_a \gamma_b + \gamma_b \gamma_a = -2\delta_{ab}$ ($a, b = 1, \dots, n$). A natural conjugation operation $\alpha \rightarrow \alpha^*$ is defined in C_{2n} by the act of reversing the order of all strings of γ -generators. In the complexified algebra $C_{2n} \otimes \mathbb{C}$ we can define the elements $Z_a = \gamma_1 \gamma_2 \dots \gamma_{2a-1} \gamma_{2a}$ which satisfy the commutation relations $[Z_a, Z_b] = 0$ and hence can be diagonalised simultaneously. Thus it is meaningful to define

$$\Delta_{2n} := \{ \alpha \in C_{2n} \otimes \mathbb{C} \mid \alpha Z_a = -\alpha \text{ for } a = 1, \dots, n \} \quad (3.1)$$

and this is clearly mapped into itself (ie. it is an ideal) under the action of Clifford multiplication from the left. If we define $\alpha_a := \gamma_{2a-1} + i \gamma_{2a}$ and $\beta_a := (1 - Z_a)$, $a = 1, \dots, n$, then it is a standard result that a basis for Δ_{2n} can be formed by taking all the 2^n possible products of n elements of the form α or β : for example $\alpha_1 \alpha_2 \dots \alpha_n$, $\beta_1 \alpha_2 \dots \alpha_n$ etc.

The group $Spin(2n)$ is defined to be the subset of even, invertible elements of C_{2n} and, under its operation, the vector space Δ_{2n} splits into a direct sum $\Delta_{2n} \approx \Delta_{2n}^s \oplus \Delta_{2n}^c$ of two invariant vector spaces Δ_{2n}^s and Δ_{2n}^c whose dimension is 2^{n-1} and which are generated by basis elements with even and odd numbers of β -elements respectively. In particular, the ideals Δ_{2n}^s and Δ_{2n}^c are generated by the sets $\{\alpha\alpha\alpha, \beta\beta\beta, \alpha\alpha\beta\beta, \beta\beta\alpha\alpha, \alpha\beta\alpha\beta, \beta\alpha\beta\alpha, \alpha\beta\beta\alpha, \beta\alpha\alpha\beta\}$ and $\{\beta\alpha\alpha\alpha, \alpha\beta\beta\beta, \beta\alpha\beta\beta, \alpha\beta\alpha\alpha, \alpha\alpha\beta\alpha, \alpha\alpha\alpha\beta\}$ in which, for example, $\alpha\alpha\alpha\alpha$ means $\alpha_1 \alpha_2 \alpha_3 \alpha_4$. The sets Δ_{2n}^s and Δ_{2n}^c of Weyl spinors are complex vector spaces but, when n is a multiple of 4 (and especially therefore for the case of interest, $2n=8$), they are the complexification of real, $Spin(2n)$ -invariant, spaces and thus correspond to Weyl-Majorana spinors; in particular Δ_8^s and Δ_8^c carry the 8_s and 8_c representations of $Spin 8$. Note that the analogue of the " γ_5 " matrix is defined as $\Gamma := \gamma_1 \gamma_2 \dots \gamma_{2n}$ and that $(1 - \Gamma)$ and $(1 + \Gamma)$ vanish identically on Δ_{2n}^s and Δ_{2n}^c respectively.

The generators of the Lie algebra $\mathfrak{g}(Spin 8)$ of $Spin 8$ are the set of commutators $\Gamma_{ij} := (\gamma_i \gamma_j - \gamma_j \gamma_i)/2$ and their actions on the ideals Δ_{2n}^s and Δ_{2n}^c may be computed directly from the Clifford multiplication in C_8 . Considering first the example of Δ_{2n}^s , a real basis is given by the set of eight spinors $\{(\beta^4 + \alpha^4)/i, (\beta^4 - \alpha^4), (\alpha^2 \beta^2 + \beta^2 \alpha^2), (\alpha^2 \beta^2 - \beta^2 \alpha^2)/i, (\alpha \beta \alpha \beta - \beta \alpha \beta \alpha), (\alpha \beta \alpha \beta + \beta \alpha \beta \alpha)/i, (\alpha \beta^2 \alpha + \beta \alpha^2 \beta), (\alpha \beta^2 \alpha - \beta \alpha^2 \beta)/i\}$ where, for example, $\alpha^2 \beta^2$ means $\alpha_1 \alpha_2 \beta_3 \beta_4$. We will choose as a fiducial unit spinor $\eta := (\beta^4 - \alpha^4)$; explicit calculation then yields results such as $\Gamma_{12}(\beta^4 - \alpha^4) = -(\beta^4 + \alpha^4)/i$, from which can be computed the Lie algebra leaving η invariant. The details of these calculations are not particularly interesting and will not be given here. The results are summarised in the Appendix which gives the generators Λ_{ij} ($i, j = 1, \dots, 7$) of the Lie algebra of the $Spin 7_s$ isotropy subgroup in terms of the generators Γ_{ij} ($i, j = 1, \dots, 8$) of the $Spin 8$ group.

The Cartan subalgebra of $\mathfrak{g}(Spin 8)$ (ie. the Lie algebra of the maximal torus of $Spin 8$) is generated by the set $\{\Gamma_{12}, \Gamma_{34}, \Gamma_{56}, \Gamma_{78}\}$ and, for our purposes, the most important feature of this embedding of $Spin 7$ in $Spin 8$ is that the Cartan subalgebra of $Spin 7_s$ is the set of all elements of the form

$$\{r_1 \Gamma_{12} + r_2 \Gamma_{34} + r_3 \Gamma_{56} + r_4 \Gamma_{78} \mid \sum_{i=1}^4 r_i = 1\} \quad (3.2)$$

and that the generators $\{\Lambda_{12}, \Lambda_{34}, \Lambda_{56}\}$ of this Cartan subalgebra are given in terms of the generators of $\mathfrak{g}(Spin 8)$ by the expression

$$\Lambda_{12} = (\Gamma_{12} - \Gamma_{34} - \Gamma_{56} + \Gamma_{78})/2 \quad (3.3)$$

$$\Lambda_{34} = (\Gamma_{34} - \Gamma_{12} - \Gamma_{56} + \Gamma_{78})/2 \quad (3.4)$$

$$\Lambda_{56} = (\Gamma_{56} - \Gamma_{12} - \Gamma_{34} + \Gamma_{78})/2 \quad (3.5)$$

Or, equivalently,

$$\begin{aligned} (r_1 \Lambda_{12} + r_2 \Lambda_{34} + r_3 \Lambda_{56}) = & \{(-r_1 - r_2 - r_3) \Gamma_{12} + (-r_1 + r_2 - r_3) \Gamma_{34} + \\ & (-r_1 - r_2 + r_3) \Gamma_{56} + (r_1 + r_2 + r_3) \Gamma_{78}\}/2 \end{aligned} \quad (3.6)$$

The crucial embedding of the maximal torus of $Spin 7_s$ in that of $Spin 8$ can be computed from these results for the Lie algebras as follows. Let τ_3 denote the 3-torus $\{(\theta_1, \theta_2, \theta_3) \mid 0 \leq \theta_a \leq 2\pi, a = 1, 2, 3\}$ and define the map $\omega: \tau_3 \rightarrow Spin 7$ by

$$\begin{aligned} \omega(\theta_1, \theta_2, \theta_3) := & (\cos \theta_1 + \lambda_1 \lambda_2 \sin \theta_1)(\cos \theta_2 + \lambda_3 \lambda_4 \sin \theta_2)(\cos \theta_3 + \lambda_5 \lambda_6 \sin \theta_3) \\ = & \exp(\theta_1 \lambda_1 \lambda_2 + \theta_2 \lambda_3 \lambda_4 + \theta_3 \lambda_5 \lambda_6) \end{aligned} \quad (3.7)$$

where $\lambda_1, \lambda_2, \dots, \lambda_7$ are the generators of the Clifford algebra C_7 and the exponential map is defined algebraically in C_7 . The image of τ_3 in $Spin 7$, $T'_3 := \omega(\tau_3)$, is a maximal torus for $Spin 7$ [8] and is related to a maximal torus for $SO 7$ via the commutative diagram

$$\begin{array}{ccc} \omega & T'_3 & \xrightarrow{\rho'_1} Spin 7 \\ \tau_3 \downarrow \pi & \downarrow \pi & \\ \rho_1 & T_3 & \xrightarrow{\rho_1} SO 7 \end{array} \quad (3.8)$$

in which the "p" maps are the embeddings of the maximal tori into the appropriate Lie groups. The map π is the usual projection of a spinor group $Spin(n)$ onto $SO(n)$ defined by

$$(\pi(u))(x) := uxu^* \quad (3.9)$$

in which $x = \sum_{i=1}^n r_i \gamma_i$ is the element in the Clifford algebra C_n corresponding to the vector $(r_1, r_2, \dots, r_n)$ in the vector space $\mathbb{R}^n$. Note that the map ω is not one-to-one but has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ kernel made up of the four elements $(0,0,0)$, $(0,\pi,\pi)$, $(\pi,\pi,0)$, $(\pi,0,\pi)$. For this reason it turns out to be convenient to perform our cohomological operations with respect to τ_3 rather than the actual maximal torus T'_3 . It is then relevant to note that the composite map Ω from τ into the maximal torus T_3 of $SO 7$, defined by $\Omega := \pi \circ \omega$, is simply

$$\Omega(\vartheta_1, \vartheta_2, \vartheta_3) = \begin{pmatrix} \cos 2\vartheta_1 & -\sin 2\vartheta_1 & 0 & 0 & 0 & 0 \\ \sin 2\vartheta_1 & \cos 2\vartheta_1 & \cos 2\vartheta_2 & -\sin 2\vartheta_2 & 0 & 0 \\ 0 & 0 & \sin 2\vartheta_2 & \cos 2\vartheta_2 & \cos 2\vartheta_3 & -\sin 2\vartheta_3 \\ 0 & 0 & 0 & 0 & \sin 2\vartheta_3 & \cos 2\vartheta_3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (3.10)$$

The embedding (3.6) of the Lie algebra of the maximal torus of $Spin 7$ in that of the group $Spin 8$ can be exponentiated to give the embedding $i_s : T'_3 \rightarrow T'_4$ of the torus itself. It is however convenient to combine this with the map $\omega : \tau_3 \rightarrow T'_3$ and the projection $\pi : T'_4 \rightarrow T_4$ to give the composite map $k := \pi \circ i_s \circ \omega$ from τ_3 into the maximal torus T_4 of $SO 8$. The result is the block diagonal matrix

$$k(\vartheta_1, \vartheta_2, \vartheta_3) = \begin{pmatrix} D_1 & 0 & 0 & 0 & 0 \\ 0 & D_2 & 0 & 0 & 0 \\ 0 & 0 & D_3 & 0 & 0 \\ 0 & 0 & 0 & D_3 & 0 \\ 0 & 0 & 0 & 0 & D_4 \end{pmatrix} \quad (3.11)$$

where

$$\begin{aligned} D_1 &= \begin{pmatrix} \cos(\vartheta_1 - \vartheta_2 - \vartheta_3) & -\sin(\vartheta_1 - \vartheta_2 - \vartheta_3) \\ \sin(\vartheta_1 - \vartheta_2 - \vartheta_3) & \cos(\vartheta_1 - \vartheta_2 - \vartheta_3) \end{pmatrix} & D_2 &= \begin{pmatrix} \cos(-\vartheta_1 + \vartheta_2 - \vartheta_3) & -\sin(-\vartheta_1 + \vartheta_2 - \vartheta_3) \\ \sin(-\vartheta_1 + \vartheta_2 - \vartheta_3) & \cos(-\vartheta_1 + \vartheta_2 - \vartheta_3) \end{pmatrix} \\ D_3 &= \begin{pmatrix} \cos(-\vartheta_1 - \vartheta_2 + \vartheta_3) & -\sin(-\vartheta_1 - \vartheta_2 + \vartheta_3) \\ \sin(-\vartheta_1 - \vartheta_2 + \vartheta_3) & \cos(-\vartheta_1 - \vartheta_2 + \vartheta_3) \end{pmatrix} & D_4 &= \begin{pmatrix} \cos(\vartheta_1 + \vartheta_2 + \vartheta_3) & -\sin(\vartheta_1 + \vartheta_2 + \vartheta_3) \\ \sin(\vartheta_1 + \vartheta_2 + \vartheta_3) & \cos(\vartheta_1 + \vartheta_2 + \vartheta_3) \end{pmatrix} \end{aligned} \quad (3.12)$$

This will play a central role in our cohomological calculations.

3.2

We can now turn to these cohomological considerations with the aid of the maps above which fit together into the commutative diagram

$$\begin{array}{ccccc} SO 7 & \xrightarrow{\pi} & Spin 7 & \xrightarrow{\pi} & SO 8 \\ \downarrow \rho_1 & & \downarrow \rho'_1 & & \downarrow \rho_n \\ T_3 & \xrightarrow{\pi} & T'_3 & \xrightarrow{\pi \circ i_s} & T_4 \\ & \searrow \Omega & \swarrow \omega & \nearrow k & \\ & & \tau_3 & & \end{array} \quad (3.13)$$

in which the "p" maps are the embeddings of the maximal tori into the indicated Lie groups. The map $\pi : Spin 8 \rightarrow SO 8$ is obtained by composing the embedding of $Spin 7$ into $Spin 8$ with the usual projection of $Spin 8$ onto $SO 8$; we shall use this in preference to the former in order that the obstruction results be expressed in terms of the familiar characteristic classes of the orthogonal group $SO 8$.

The computation of these classes involves forming a diagram that is identical to the one above except that every group G is replaced by the universal space BG and every map $\varphi : G_1 \rightarrow G_2$ between any pair of groups G_1 and G_2 is replaced by the induced map $B\varphi : BG_1 \rightarrow BG_2$. We shall not write down this diagram explicitly but, where necessary, we will refer to the one above with the understanding that the change to the universal diagram has been performed. The main results can now be derived in the following sequence of steps in which the cohomology groups are always taken over the field $\mathbb{Q}$ of rational numbers (the real numbers $\mathbb{R}$ can be used equally effectively). Nothing is lost thereby since the obstruction in which we are interested lies in $H^q(M; \mathbb{Z})$ and, for any compact, eight-dimensional orientable manifold M , this is simply the integers $\mathbb{Z}$. Hence there are no torsional elements which might otherwise be missed by using rational (or real), rather than integer, cohomology.

(a) The Pontryagin classes $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathfrak{P}_4$ in $H^*(BSO8)$ can be defined [13] by the relation

$$B\rho_8^*(1 + \mathfrak{P}_1 + \mathfrak{P}_2 + \mathfrak{P}_3 + \mathfrak{P}_4) = \prod_{i=1}^4 (1 + \gamma_i^2) \quad (3.14)$$

where $\gamma_1, \dots, \gamma_4$ are the four elements in $H^2(BT_4; \mathbb{Q})$ that generate this cohomology group. Equation (3.14) is meaningful as a definition of $\mathfrak{P}_1, \dots, \mathfrak{P}_4$ since $B\rho_8^*$ is injective in rational (and real) cohomology. We are concerned with $\mathfrak{P}_1$ and $\mathfrak{P}_2$ only, for which (3.14) gives

$$B\rho_8^*(\mathfrak{P}) = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 + \gamma_4^2 \in H^4(BT^4; \mathbb{Q}) \quad (3.15)$$

$$B\rho_8^*(\mathfrak{P}) = \gamma_1^2 \gamma_2^2 + \gamma_1^2 \gamma_3^2 + \gamma_1^2 \gamma_4^2 + \gamma_2^2 \gamma_3^2 + \gamma_2^2 \gamma_4^2 + \gamma_3^2 \gamma_4^2 \in H^6(BT^4; \mathbb{Q}) \quad (3.16)$$

and, similarly, the Euler class e can be defined by

$$B\rho_8^*(e) = \gamma_1 \gamma_2 \gamma_3 \gamma_4 \in H^8(BT^4; \mathbb{Q}). \quad (3.17)$$

(b) The analogous results for the Pontryagin classes $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3$ of the group $SO7$ are

$$B\rho_7^*(1 + \mathfrak{P}_1 + \mathfrak{P}_2 + \mathfrak{P}_3) = \prod_{i=1}^3 (1 + \delta_i^2) \quad (3.18)$$

so that

$$B\rho_7^*(\mathfrak{P}_1) = \delta_1^2 + \delta_2^2 + \delta_3^2 \in H^4(BT_3; \mathbb{Q}) \quad (3.19)$$

$$B\rho_7^*(\mathfrak{P}_2) = \delta_1^2 \delta_2^2 + \delta_1^2 \delta_3^2 + \delta_2^2 \delta_3^2 \in H^6(BT_3; \mathbb{Q}) \quad (3.20)$$

where $\delta_1, \delta_2, \delta_3$ are the three generators of $H^2(BT_3; \mathbb{Q})$.

(c) The crucial information in which we are interested is the kernel of the map Bx^* from $H^*(BSO8)$ to $H^*(BSpin_7)$. We shall compute this via the map Bk^* from the cohomology of BT_4 to that of BT_3 , which in turn involves studying the map k^* from the cohomology of T_4 to that of T_3 . The critical map is $k^*: H^1(T_4) \rightarrow H^1(T_3)$ and this is readily calculated from the form of the map $k: T_5 \rightarrow T_4$ given in (3.11-12) and its effects on the fundamental groups of the two tori. Thus if we denote the generators of $H^1(T_4)$ and $H^1(T_3)$ by the sets $\{x_1, x_2, x_3, x_4\}$ and $\{z_1, z_2, z_3\}$ respectively, it follows at once that

$$\begin{aligned} k^*(x_1) &= z_1 - z_2 - z_3 & k^*(x_2) &= -z_1 + z_2 - z_3 \\ k^*(x_3) &= -z_1 - z_2 + z_3 & k^*(x_4) &= z_1 + z_2 + z_3 \end{aligned} \quad (3.21)$$

Similarly, the map Ω^* from $H^1(T_3)$ to $H^1(T_2)$ is given at once from (3.10) as

$$\Omega^*(y_a) = 2z_a, \quad a = 1, 2, 3 \quad (3.22)$$

where $\{y_1, y_2, y_3\}$ are the generators of $H^1(T_3)$. However, the generators γ_i of $H^2(BT_4)$ are related to those of $H^1(T_4)$ by $\gamma_i = -\tau(x_i)$, $i = 1, 2, 3, 4$, where τ is the transgression [13] in the universal bundle $T_4 \rightarrow ET_4 \rightarrow BT_4$. Similarly $\delta_a = -\tau(y_a)$ ($a = 1, 2, 3$) for the generators δ_a of $H^2(BT_3)$ and $\varepsilon_a = -\tau(z_a)$ ($a = 1, 2, 3$) for the generators ε_a of $H^2(BT_2)$. But the transgression homomorphism commutes with the cohomology pull-back operation, and hence (3.22) implies

$$B\Omega^*(\delta_a) = 2\varepsilon_a, \quad a = 1, 2, 3 \quad (3.23)$$

Then (3.21) together with (3.23) gives the crucial result

$$\begin{aligned} Bk^*(\gamma_1) &= B\Omega^*(\delta_1 - \delta_2 - \delta_3)/2 & Bk^*(\gamma_2) &= B\Omega^*(\delta_1 - \delta_2 + \delta_3)/2 \\ Bk^*(\gamma_3) &= B\Omega^*(\delta_1 - \delta_2 + \delta_3)/2 & Bk^*(\gamma_4) &= B\Omega^*(\delta_1 + \delta_2 + \delta_3)/2 \end{aligned} \quad (3.24)$$

(d) We now possess all the information needed to compute the obstruction to the existence of a nowhere-vanishing spinor field. Let us consider first the calculation of the pull-back $Bx^*(e)$ of the Euler class $e \in H^8(BT_4)$. We know from (the BG-version of) the diagram (3.13) that $Bx \circ B\rho_4^* \circ B\omega$ is homotopic to $B\rho_8^* \circ Bk$ and hence

$$\begin{aligned} (B\rho_7^* \circ B\omega)^*(Bx^*(e)) &= Bk^*(B\rho_8^*(e)) \\ &= Bk^*(\gamma_1 \gamma_2 \gamma_3 \gamma_4) \end{aligned} \quad (3.25)$$

where we have used (3.17) to relate $B\rho_8^*(e)$ to the generators γ_i of $H^2(BT_4)$.

Now we use (3.19-20) and then (3.24) to write the right hand side of (3.25) as

$$Bk^*(\gamma_1 \gamma_2 \gamma_3 \gamma_4) = B\Omega^*\{(\delta_1 \delta_2 + \delta_1 \delta_3) - 2(\delta_1^2 \delta_2^2 + \delta_2^2 \delta_3^2 + \delta_1^2 \delta_3^2)\}/16 \quad (3.26)$$

$$= B\Omega^*(B\rho_7^*\{P_1^2 - 4P_2\})/16 \quad (3.27)$$

$$= (B\rho_7^* \circ B\omega)^*(B\pi^*\{P_1^2 - 4P_2\})/16 \quad (3.28)$$

where the last line follows from the commutativity of the "BG-version" of (3.13). But, $(B\rho_7^* \circ B\omega)^* = B\omega^* \circ B\rho_7^*$ and both $B\omega^*$ and $B\rho_7^*$ are injective in rational (and real) cohomology [13]. Thus (3.25) and (3.28) imply that

$$Bx^*(e) = B\pi^*(P_1^2 - 4P_2)/16 \quad (3.29)$$

which is the first of the results we are seeking.

Precisely the same procedure can be applied to the other characteristic classes in $H^*(BSO8)$. Thus,

$$(B\rho_1^* \circ B\omega)^*[Bx^*(\mathfrak{P}_2)] = Bk^*(B\rho_8^*(\mathfrak{P}_2)) \quad (3.30)$$

$$= Bk^*(Y_1^2 Y_2^2 + Y_1^2 Y_3^2 + Y_1^2 Y_4^2 + Y_2^2 Y_3^2 + Y_2^2 Y_4^2 + Y_3^2 Y_4^2) \quad (3.31)$$

After some calculation with (3.24), the right hand side of (3.31) can be written as

$$BO^*[6(\delta_1^4 + \delta_2^4 + \delta_3^4) + 4(\delta_1^4 \delta_2^4 + \delta_1^4 \delta_3^4 + \delta_2^4 \delta_3^4)]/16 \quad (3.32)$$

which can then be expressed in terms of the pull-backs of the characteristic classes of SO8 as

$$Bx^*(\mathfrak{P}_2) = B\pi^*(3P_1^2 - 4P_2)/8 \quad (3.33)$$

and a similar calculation applied to the first Pontryagin class yields the simple result

$$Bx^*(\mathfrak{P}_1) = B\pi^*(P_1). \quad (3.34)$$

The information we seek on the kernel of Bx^* can be extracted from equations (3.29/33/34) by finding a rational combination of the right hand sides of these equations that vanishes. The answer is clearly

$$Bx^*(e - \mathfrak{P}_2/2 + \mathfrak{P}_1^2/B) = 0 \quad (3.35)$$

and hence we conclude that a necessary condition for a nowhere-vanishing Weyl-Majorana spinor (ie of type 8_s) on a eight-dimensional orientable manifold M is that the characteristic classes of the frame bundle of M should satisfy:

$$e - P_2/2 + P_1 \cup P_1/8 = 0 \quad (3.36)$$

where $e \in H^0(M; \mathbb{Q})$, $P_2 \in H^4(M; \mathbb{Q})$ and $P_1 \in H^2(M; \mathbb{Q})$ are respectively the Euler class, the second Pontryagin class and the first Pontryagin class of M . Note that if real, rather than rational, cohomology is used then (3.36) will still apply but with e , P_2 and P_1 being now understood as the usual differential forms and with the cup-product $\cup$ being replaced with the exterior product. That (3.36) is also a

sufficient condition follows from the fact that the obstruction we have computed to the construction of a cross-section of the appropriate 7-sphere bundle is the primary one (see Section 2). Somewhat more sophisticatedly, we could consider a Postnikov decomposition of the bundle $S^7 \rightarrow BSpin7 \rightarrow BSpin8$, but we shall not give the details here.

As a consistency check on the results above we note that the existence of a linearly independent pair of nowhere-vanishing Majorana-Weyl spinors of the same chirality corresponds to the reduction of the structure group from $Spin8$ to $Spin6 \simeq SU4$ and it can be shown that this $SU4$ is contained in (a covering) of the particular $U4$ subgroup of $SO8$ that corresponds to placing a complex structure on the eight-dimensional manifold M . If ν denotes the embedding of $U4$ in $SO8$, a well-known analysis (similar in spirit to the one above) shows that the map $B\nu^*$ from the cohomology groups of $BSO8$ to those of $BU4$ satisfies the relations

$$B\nu^*(e) = c_4 \quad (3.37)$$

$$B\nu^*(\mathfrak{P}_2) = c_2^2 - 2c_1 c_3 + 2c_4$$

$$B\nu^*(\mathfrak{P}_1) = c_1^2 - 2c_2$$

where $c_1 \dots c_4$ are the universal Chern classes for the unitary group $U4$. Thus, on the complex manifold M ,

$$(8e - 4P_2 + P_1^2) = c_1(c_1^3 - 4c_1 c_2 + 8c_3). \quad (3.39)$$

However, the condition for the structure group to be further reduced from $U4$ to $SU4$ (as for example in a Calabi-Yau space) is precisely that the first Chern class c_1 should vanish, and hence our condition (3.36) is indeed satisfied.

3.3

Finally, we must consider the problem for spinors of the opposite chirality, ie. belonging to the ideal Δ_8^c rather than Δ_8^s . In this case, a convenient fiducial spinor is $\alpha\beta^3 - \beta\alpha^3$ and an analysis of the corresponding isotropy subgroup $Spin7_c$ shows it to have the same form as that for $Spin7_s$ given in the Appendix except that a minus sign appears in front of a generator f_{ij}^c whenever one of its indices i or j is equal to 2. Thus the analogue of (3.3-5) for the Cartan sub-algebra generators M_{12}, M_{34}, M_{56} of this new subgroup is

$$M_{12} = (-I_{12} - I_{34} - I_{56} + I_{78})/2 \quad (3.40)$$

$$M_{34} = (I_{34} + I_{12} - I_{56} + I_{78})/2 \quad (3.41)$$

$$M_{56} = (I_{56} + I_{12} - I_{34} + I_{78})/2 \quad (3.42)$$

while (3.2) becomes

$$\{-r_1 I_{12} + r_2 I_{34} + r_3 I_{56} + r_4 I_{78} \mid \sum_{i=1}^4 r_i = 1\} \quad (3.43)$$

It is easy to check that the net effect of this sign change is that the analogue of Bk^* acts in the same way as Bk^* itself except that the pull-back of $\gamma_i \in H^2(BT_+)$ appears with an additional minus sign. The pull-backs of the characteristic classes $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are even functions of γ_i and are hence unaffected by this change. However, the Euler class e is an odd function of γ_i and so the final result for the existence of a nowhere-vanishing spinor is the same as (3.36) apart from a minus sign:

$$-e - P_2/2 + P_1 \cup P_1/8 = 0 \quad (3.43)$$

Note that if there exist nowhere-vanishing spinors of both chiralities then both (3.36) and (3.43) must be satisfied. Subtracting these two equations shows that the Euler class of M must then vanish, in accordance with the remarks made in the Introduction to the effect that, in 8 dimensions, a nowhere-vanishing vector field can be constructed from a pair of such spinors.

4. Conclusion

We have seen that the necessary and sufficient conditions for the existence of nowhere-vanishing Majorana-Weyl spinors of type 8_s or 8_c are given by (3.36) and (3.43). There are plenty of examples where one of these conditions is satisfied; for example, any complex manifold M_6 whose first Chern class vanishes (whether this gives a 8_s or a 8_c spinor depends on the orientation of M_6 and its relation to e and c_1).

There are also a number of cases where neither (3.36) or (3.43) are satisfied. To see this it is helpful to recall that, on a eight-dimensional manifold, the real cohomology classes $\mathfrak{P}_1, \mathfrak{P}_2$ are related to the signature $\tau(M_8)$ [14] by

$$\tau(M_8) = \frac{1}{45} (7\mathfrak{P}_2 - \mathfrak{P}_1^2)(M_8) \quad (4.1)$$

where the right hand side refers to the value of the integral of the eight-form over M_8 . Suppose now that M_8 is such that $H^4(M_8; \mathbb{R}) \approx 0$. Then the signature vanishes and hence (4.1) implies that $7\mathfrak{P}_2 = \mathfrak{P}_1^2$. But, since $\mathfrak{P}_1 \in H^4(M_8; \mathbb{R})$, this implies that $\mathfrak{P}_1 = \mathfrak{P}_2 = 0$, and so the conditions (3.36) and (3.43) reduce to $e = 0$, i.e. the same as for a nowhere-vanishing vector field.

This yields a number of examples of manifolds which possess obstructions to nowhere-vanishing real, chiral spinors; for example, S^8 satisfies $H^4(S^8) = 0$ and $e(S^8) = 2$. Other examples can be obtained by noting [14] that $\mathfrak{P}_1$ and $\mathfrak{P}_2$ vanish on a manifold of the form $M_6 = M_2 \times M_4$, and that $e(M_2 \times M_4) = e(M_2)e(M_4)$. A natural question now is how one should proceed if there is a need to construct a nowhere-vanishing spinor field on a manifold for which neither (3.36) or (3.43) are satisfied. This is reminiscent of a problem that arose some time ago in the euclidean approach to conventional quantum gravity in which functional integrals are taken over all compact four-dimensional Riemannian manifolds M , including spaces such as $\mathbb{C}P^2$ whose second Stiefel-Whitney class $w_2 \in H^2(M; \mathbb{Z}_2)$ is non-zero. Spaces of this type admit no globally defined spinors which, for example, makes it impossible to construct supergravity theories. One resolution of the problem [15] is to form a $U(1)$ extension of the spin group from $Spin(4)$ to $Spin^c(4)$ [9] (or to use an extension by an even bigger, non-abelian, group [16, 17, 18]) in which the unwanted w_2 class is "killed" by the mod-2 reduction of the $H^2(M; \mathbb{Z})$ characteristic class associated with the extra $U(1)$ (or larger) group.

The problem of nowhere-vanishing spinors can be tackled in a similar way and there are examples of manifolds which do not satisfy either (3.36) or (3.38) but which do admit nowhere-vanishing complex spinors transforming under the group $Spin^c 8$. A good example is when M_8 is a Kähler space. Even if $c_1 \neq 0$, there still exist [6] so-called "gauge-covariantly" constant Weyl spinors ξ satisfying

$$\nabla_i \xi - \frac{1}{2} V_i \xi = 0 \quad (4.2)$$

where the locally-defined one-form V_i (the $U(1)$ connection) is related to the Ricci two-form R_{ij} by $R_{ij} = \partial_i V_j - \partial_j V_i$. It is then a straight-forward matter to show that (4.2) implies that $\xi \bar{\xi}$ is a constant, and hence that ξ is nowhere-vanishing. Note that this does not contradict our earlier results since $Spin^c 8 \approx Spin 8 \times \mathbb{Z}_2 U(1)$, and the presence of the " $\mathbb{Z}_2$ -twist" means that $Spin 8$ is not a subgroup of the extended group $Spin^c 8$ and hence does not act on ξ . The "physical" cost of this extension is the need to introduce a background "gauge field" for the extra $U(1)$ (or larger) group, which must of course be capable of being coupled consistently to the existing fields in the theory.

However, it is interesting to note that, at least for a wide class of string compactifications, the string equations of motion imply that the conditions derived in this paper for the existence of a nowhere-vanishing spinor are satisfied automatically. If we consider configurations $M_2 \times K_6$ for the ten-dimensional target space in which K_6 is Kähler, then to lowest order in α' the string equations of motion imply that the metric on K_6 must be Ricci-flat [1]. Taking into account the higher-order corrections [2,3,4,6], one finds that the Ricci form must satisfy

$$R_{a\bar{b}} = \partial_a \partial_{\bar{b}} S \quad (4.3)$$

where $a, \bar{b}$ are complex $SU(4)$ indices and S is some scalar function constructed from the Riemann tensor and its covariant derivatives on K_6 of the form $S \sim \alpha'^3 (Riem)^3 + \alpha'^4 (Riem)^4 + \dots$. Although it is not feasible to solve (4.3) explicitly, one can nevertheless deduce general properties of the solutions; in particular, it follows from the fact that S is a globally-defined function on K_6 that the Ricci-form is exact. This implies that K_6 must have vanishing first Chern class, and hence (3.39) or (3.43) is satisfied.

In the examples discussed so far of spaces that admit nowhere-vanishing Majorana-Weyl spinors, there have always been two such spinors, of the same helicity. This is because in these examples the structure group is $SU(4) \subset SO(8)$, and

the two spinors correspond to the singlets in the decomposition $8_c \rightarrow 6 \oplus 1$ under the reduction of $SO(8)$ to $SU(4)$. There also exists the possibility of M_6 admitting just one nowhere-vanishing Majorana-Weyl spinor. This arises if the structure group can be reduced from $Spin 8$ to $Spin 7$ but not down to $Spin 6 \approx SU(4)$ and hence corresponds to the decomposition $8_c \rightarrow 7 \oplus 1$. Examples exist of eight-manifolds with a metric whose holonomy group is $Spin 7$ [19]. Such a space will admit a reduction from $Spin 8$ to $Spin 7$ and will be Ricci-flat with the nowhere-vanishing spinor being covariantly constant.

Appendix

The generators A_{ij} ($ij=1...7$) of the $Spin7_s$ subalgebra of $Spin8$ are related to the generators F_{ij} ($ij=1...8$) as follows:

$$\begin{aligned}
 2A_{12} &= F_{12} - F_{34} - F_{56} + F_{78} \\
 2A_{34} &= -F_{12} + F_{34} - F_{56} + F_{78} \\
 2A_{56} &= -F_{12} - F_{34} + F_{56} + F_{78} \\
 2A_{13} &= F_{13} + F_{24} - F_{57} - F_{68} \\
 2A_{14} &= F_{14} - F_{23} - F_{58} + F_{67} \\
 2A_{15} &= F_{15} + F_{26} + F_{37} + F_{48} \\
 2A_{16} &= F_{16} - F_{25} + F_{38} - F_{47} \\
 2A_{17} &= F_{17} - F_{28} - F_{35} + F_{46} \\
 2A_{23} &= F_{23} - F_{14} - F_{58} + F_{67} \\
 2A_{24} &= F_{24} + F_{13} + F_{57} + F_{68} \\
 2A_{25} &= F_{25} - F_{16} + F_{38} - F_{47} \\
 2A_{26} &= F_{26} + F_{15} - F_{37} - F_{48} \\
 2A_{27} &= F_{27} + F_{18} + F_{36} + F_{45} \\
 2A_{45} &= F_{45} - F_{36} - F_{18} + F_{27} \\
 2A_{46} &= F_{46} + F_{35} + F_{17} + F_{28} \\
 2A_{47} &= F_{47} - F_{16} - F_{25} + F_{38} \\
 2A_{67} &= F_{67} + F_{14} + F_{23} + F_{58}
 \end{aligned}$$

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