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COMMENTS ON THE ALGEBRA OF STRAIGHT, TWISTED AND INTERTWINING VERTEX OPERATORS

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ABSTRACT

The extent to which the 'intertwining' operator, which converts the untwisted or straight bosonic string to a twisted string and vice-versa, also enhances the algebra constructed in the two sectors separately is explored. The relationship between the $E_8 + E_8$ and the $so(32)$ algebras is discussed in the context of the reflection twist, as is the special role of E_8 . The possibility of the other twists (particularly of third order) playing a fundamental role in string theory is briefly explored.

1. Introduction

Twisted bosonic strings were invented and used in several contexts many years ago [1-3]. At the time, the discussion was limited to the reflection, $x \rightarrow -x$, (an example of a Z_2 twist), for natural reasons. Firstly, the reflection twist generalised to bosonic strings the most obvious features of the Neveu–Schwarz–Ramond (NSR) spinning string [4-6]; that is, mixed boundary conditions and a pair of distinct sectors. Secondly, their use in ref[3] was not anticipated but derived by factorising, as an expectation value of a product of vertex operators, a specific amplitude, in turn derived from what was essentially the calculation of a Green's function in the Mandelstam picture [7,8]. The resulting off-shell amplitude was in general agreement with an earlier proposal of Schwarz, who had adopted quite a different starting point to define an off-shell state based on the dual model gauge conditions [9,10].

More recently, some of the properties of strings associated with general twists have been explored [11-13] and provide a framework within which to describe a variety of unified string models. These are based on the heterotic principle [14,15], but display a greatly reduced gauge symmetry and potentially interesting phenomenology.

One of the main results of ref[3] was the construction of a vertex operator with the property that it (or its conjugate) converts the untwisted sector to the twisted sector (or vice-versa), and was therefore to be regarded as a twisted state emission vertex, (or TSEV for short). This operator was a bosonic analogue of the ancient fermion emission vertex [16-18], which was necessary to complete the description of the NSR spinning string, and shares many of its properties. For example, each behaves the same way with respect to the conformal group generators and has conformal weight $\Delta = d + D/16$. Here, D is the dimension of the relevant space in which the string and its twisted counterpart take their values, and d is the occupation number of the emitted state. More precisely, D was the dimension of flat space-time for the original applications and is the dimension of a more exotic space, (Z_2 orbifold), in the contemporary arena [11-13].

Originally, since dual model vertices are supposed to have unit conformal weight, the value of Δ was deemed to be peculiar (for D set equal to 10 or 26). However, it is now seen to be naturally complementary to the BRST ghostly contributions needed in either of the two cases [19-21]. On the other hand, if D is eight or sixteen then the conformal weight comes out correctly provided d is chosen to be one half or zero, respectively.

The TSEVs corresponding to some other more general twists have been constructed recently [22,23], but for the most part we shall not discuss them here.

The problem we do wish to discuss is the following. It is clear, as we mentioned above, that for the reflection twist, in an appropriate number of dimensions and for suitably chosen states, the conformal weight of the associated TSEV can be unity. If we suppose the piece of the string we are interested in is constrained to move on a torus, (for example, it is a closed left-mover), then we know, for suitably chosen tori, that the untwisted sector allows the construction of a Kac-Moody algebra via the Frenkel-Kac-Segal (KFS) mechanism [24-27]. Moreover, we also know that the twisted sector allows the construction of an infinite algebra [28-30] whose finite zero-graded part, (i.e. commuting with L_0), is a representation of a subalgebra of the zero-graded component in the untwisted sector. The question is, do the TSEVs mapping from one sector to the other enlarge the algebra in these special and rare cases? The NSR version of this same question in ten-dimensional spacetime led eventually to an understanding of the fermion emission vertex as being closely related to the spacetime supersymmetry generator acting in the combined Fock space of suitably projected Neveu-Schwarz and Ramond sectors [19,31-35]. It seems to us that the bosonic version of such a relation is interesting to work out in detail and may, as we mention in the final section, lead to interesting conjectures when generalised to other sorts of twist. At any rate, we shall see that the results illuminate some of the statements in the literature on the relationship between the $E_8 \times E_8$ and $spin(32)/Z_2$ heterotic strings [13,36-38], and provide a further example of the rather unique behaviour of E_8 [39].

In section (2) we shall define the vertex and review some of its properties, including the way in which the KFS ‘cocycle factors’ relate to their twisted counterparts. Of course, this aspect was not considered at all in olden times. In subsequent sections, we shall do what we can to compute the various operator product expansions which are necessary steps towards establishing the algebra we are seeking. These calculations are not straightforward and, for the moment, remain incomplete as we shall explain. Our most detailed example will be worked out for the case when D is sixteen, but we shall give an indication of similar results for other choices as well. The final section is devoted to some comments and conjectures. In particular, we are intrigued by a six dimensional example, for which there is some amusing numerology but as yet no detailed computations.

2. Preliminaries

A standard or ‘straight’ string field will be denoted by $X(z)$ and is defined by

$$X(z) = q - ip \ln z + i \sum_{\substack{n \neq 0 \\ n \in \mathbb{Z}}} \frac{\alpha_n}{n} z^{-n}, \quad (2.1)$$

where

$$[\alpha_n, \alpha_m] = n \delta_{n+m,0}, \quad \alpha_n^\dagger = \alpha_{-n}, \quad p = \alpha_0 \quad (2.2)$$

for each component, (the space labels are suppressed) and

$$[q, p] = i. \quad (2.3)$$

We shall be interested mainly in a totally left (or right) moving string on a torus defined by a simply-laced root lattice, in which case $z = e^{i(\tau+\sigma)}$. The standard vertex [24,25,40] is

$$V(\alpha, z) = : e^{i\alpha \cdot X(z)} : C(\alpha), \quad (2.4)$$

where α is a lattice vector and $C(\alpha)$ is the usual ‘cocycle’ operator function of momentum. The moments of the vertex operator define a Kac–Moody algebra via the Frenkel–Kac–Segal mechanism [24–27], in the by now familiar way.

An analogous reflection twisted string field is defined similarly [1-3,12,13]

$$X_T(z) = q + i \sum_{r \in \mathbb{Z} + \frac{1}{2}} \frac{c_r}{r} z^{-r}, \quad (2.5)$$

where

$$[c_r, c_s] = r \delta_{r+s,0}, \quad c_r^\dagger = c_{-r} \quad (2.6)$$

for each component, and $2q$ is a lattice vector. As the origin is encircled once by the variable z , $X_T(z)$ returns to minus itself up to a lattice translation. For our purposes, in considering the algebras defined by such fields, just which fixed point q we pick is immaterial, the associated degeneracy of the vacuum being understood in terms of a

matrix valued operator ‘cocycle’. Thus, we take $q = 0$ and the twisted vertex operator analogous to (2.4) to be [28-30,41-44],

$$V_T(\alpha, z) = 2^{-\alpha^2} z^{-\frac{1}{2}\alpha^2} : e^{i\alpha \cdot X_T(z)} : C_T(\alpha), \quad (2.7)$$

where the power of two is conventional, and $C_T(\alpha)$ is the twisted ‘cocycle’ matrix. For each simply laced root lattice $C_T(\alpha)$ has a well defined dimension and $\frac{1}{4}C_T(\alpha)$, with $\alpha^2 = 2$, itself represents the twist invariant subalgebra of the Lie algebra associated to the root lattice. We shall take it to satisfy

$$C_T(\alpha)C_T(\beta) = (-)^{\alpha \cdot \beta} C_T(\beta)C_T(\alpha) = \varepsilon(\alpha, \beta) C_T(\alpha + \beta) \quad (2.8)$$

$$C_T(-\alpha) = (-)^{\frac{1}{2}\alpha^2} C_T(\alpha) = C_T^\dagger(\alpha), \quad (2.9)$$

where $\varepsilon(\alpha, \beta)$ are real, but not necessarily trivial, cocycles ensuring the associativity of the product (2.8). If α and β are root vectors then $\varepsilon(\alpha, \beta)$ are actually the structure constants of the original untwisted algebra:

$$[E_\alpha, E_\beta] = \varepsilon(\alpha, \beta) E_{\alpha+\beta}, \quad E_\alpha^\dagger = E_{-\alpha}. \quad (2.10)$$

The reflection twist can be taken to act consistently on the Lie algebra, sending

$$E_\alpha \rightarrow -E_{-\alpha}, \quad \mathbf{H} \rightarrow -\mathbf{H}, \quad (2.11)$$

in which case the invariant subalgebra is the set of combinations of step operators $\frac{1}{2}\{E_\alpha - E_{-\alpha}\}$. The moments of (2.7) satisfy

$$\begin{aligned} [V_T(\alpha, z)]_r &= (-)^{\frac{1}{2}\alpha^2 + 2r} [V_T(-\alpha, z)]_r \\ &= (-)^{\frac{1}{2}\alpha^2 + 2r} [V_T(\alpha, z)]_{-r}^\dagger, \end{aligned} \quad (2.12)$$

and hence match nicely with the particular representation on the algebra picked out by the action of the twist, equation (2.11).

Notice, too, that another different representation of the invariant Lie algebra may be obtained from the straight string taking the combination

$$\frac{1}{2} \{ [V(\alpha, z)]_0 - [V(-\alpha, z)]_0 \}, \quad (2.13)$$

which mimics the invariant combination for the abstract Lie algebra. The simply laced Lie algebras and their reflection twisted subalgebras are: $(su(n) \supset so(n))$, $(so(2n) \supset so(n) + so(n))$, $(E_8 \supset sp(4))$, $(E_7 \supset so(16))$, $(E_8 \supset so(16))$. In each case there is a specified ‘cocycle’, for which we list a mnemonic in appendix A. For more details on how the algebras work and why the cocycles in either sector are as they are see refs[26,27,30].

In either sector there is a representation of the Virasoro algebra L_n and L_n^T . Thus,

$$\begin{aligned} L_n &= : \frac{1}{2} \sum_{m \in \mathbb{Z}} \alpha_m \alpha_{n-m} : \\ L_n^T &= : \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} c_r c_{n-r} : + \frac{D}{16} \delta_{n,0} \end{aligned} \quad (2.14)$$

and an extra constant piece is added to L_0 in the twisted case so that the standard commutation relations are obeyed, namely

$$[L_n, L_m] = (n - m) L_{n+m} + \frac{D}{12} n(n^2 - 1) \delta_{n+m,0}. \quad (2.15)$$

Here, D is the dimension of the root lattice. In particular the vertex in either sector transforms with conformal weight $\frac{1}{2} \alpha^2$. They each have conformal weight unity when α is a root with $\alpha^2 = 2$. Notice, the vacua in the two sectors never have the same conformal weight. However, it may happen, when D is a multiple of eight, that certain states in the two sectors do match up at least as far as their conformal weights are concerned. For example, the first excited level in the ‘straight’ sector has conformal weight one and matches the vacuum in the twisted sector if $D = 16$, and the first excited state in the twisted sector when $D = 8$. On the other hand, if D is 24 the first excited state in the twisted sector matches the *second* level in the straight sector. It is this fact which underlies the construction of the ‘moonshine module’ in ref[39]. We shall explore some of the consequences of this matching in section (4).

The next ingredient we need is the operator which converts one sector to the other. Apart from the ‘cocycle’ pieces it was constructed many years ago in a different context [3], and is the bosonic analogue of the fermion emission vertex [16–18].

The basic object we shall denote by $W(\phi, z)$, where ϕ refers to the ‘emitted’ twisted state. In fact, ϕ will always either be the degenerate vacuum or the first excited state in the twisted sector; notice these are the only states annihilated by L_{+1}^T . (None of them are annihilated by L_{-1}^T or L_0^T remember, but that doesn’t matter).

It is convenient to write [3,18,23],

$$W(\phi, z) = e^{zL_{-1}^T} I(\phi, z), \quad (2.16)$$

where $I(\phi, z)$ has the form:

$$I(\phi, z) = \langle v | : \exp\{A + B^+ + B^-\} : | \phi \rangle \quad (2.17)$$

with

$$\begin{aligned} A(z) &= \sum_{n,m \geq 0} \frac{1}{2} \alpha_n A_{nm}(z) \alpha_m \\ B^\pm(z) &= \sum_{\substack{n \geq 0 \\ r \geq \frac{1}{2}}} \alpha_n B_{nr}^\pm(z) c_{\pm r}. \end{aligned} \quad (2.18)$$

The matrices A , $B^\pm$ are defined by the generating functions

$$\sum_{n,m \geq 0} y^n A_{nm}(z) y'^m = -2 \ln \left(\sqrt{y-z} + \sqrt{y'-z} \right) \quad (2.19)$$

$$\sum_{n \geq 0} y^n B_{nr}^\pm(z) = \mp \frac{(y-z)^{\mp r}}{r}, \quad (2.20)$$

and thus, in particular,

$$A_{nm}(z) = \frac{(-z)^{-n-m}}{m+n} \binom{-\frac{1}{2}}{m} \binom{-\frac{1}{2}}{n} \quad (2.21)$$

with m and n not both zero, and,

$$A_{00}(z) = -\ln(-4z). \quad (2.22)$$

The left hand state in (2.17) is rather special. It is annihilated by all the creation

operators in the straight Fock space and is a special momentum state. Specifically,

$$\langle v | = \sum_{\alpha \in \Lambda} \langle \alpha | C_T(\alpha), \quad (2.23)$$

a sum over the whole simply-laced root lattice, Λ . The right hand state in (2.17) is the ‘emitted’ twisted state $|\phi\rangle$, it is a vector on which the twisted ‘cocycle’ acts as well as (perhaps) being excited in the twisted Fock space. It was found in ref[23] that the intertwining of the matrix factor and the straight momentum factor, $C_T(\alpha)$ and $C(\alpha)$ respectively, was essential to the intertwining of the Fock spaces as a whole. Ref[22] does not discuss this point.

The expression (2.16) has a number of nice properties. First of all, it is relatively easy to check that

$$V_T(\alpha, y - z) I(\phi, z) = I(\phi, z) V(\alpha, y) \quad (2.24)$$

at least in some region of the y - z complex plane. The result is obviously not true everywhere because of the different singularities in the vertex operators pertaining to each of the two sectors. The manipulations are in fact valid in the region $|z| > |y|$ and $|y - z| > |z|$ and there is little point in reviewing them again here [3]. We shall, however, return to a subtlety related to this later, in section (3). The only new piece relative to the old calculations is the ‘cocycle’ contribution. This part can be seen to work easily and explicitly. The relevant slice of the calculation is:

$$\begin{aligned} \sum_{\beta} \langle \beta | C_T(\beta) e^{i\alpha \cdot q} C(\alpha) &= \sum_{\beta} \langle \beta - \alpha | C_T(\beta) \epsilon(\alpha, \beta - \alpha) \\ &= \sum_{\beta} \langle \beta | C_T(\beta + \alpha) \epsilon(\alpha, \beta) \\ &= C_T(\alpha) \sum_{\beta} \langle \beta | C_T(\beta), \end{aligned} \quad (2.25)$$

where we have made use of a neat representation of the straight ‘cocycle’,

$$C(\alpha) = \sum_{\beta} \epsilon(\alpha, \beta) |\beta\rangle \langle \beta|, \quad (2.26)$$

to be found in refs[24-27].

For the generalisations of equations (2.16)—(2.23) involving general twists there is a more complicated inter-dependence between the Fock space part and the ‘cocycles’ [23].

The second important property relates to the conformal transformations. The vertex (2.16) has conformal weight Δ_ϕ , the same as that of the emitted state (as one expects since this is a general feature of all string vertices). In other words, for example, for a Mobius transformation it transforms according to,

$$O_T(\gamma)W(\phi, z)O_S^{-1}(\gamma) = (cz + d)^{-2\Delta_\phi}W(\phi, \gamma(z)), \quad (2.27)$$

where the subscripts T, S refer to the twisted and straight representations of $SU(1, 1)$, respectively, and γ is the transformation

$$z \rightarrow \gamma(z) = \frac{az + b}{cz + d}.$$

The conjugate vertex to $W(\phi, z)$, that is the vertex $\overline{W}(\phi, z)$ mapping the sectors in the opposite sense to equation (2.24), is defined by

$$\overline{W}(\phi, z) = z^{-2\Delta_\phi} (W(\phi_c, 1/z^*))^\dagger, \quad \phi_c = C^{-1}\phi^*, \quad (2.28)$$

where C is the charge conjugation matrix appropriate to the Clifford algebra in which the ‘cocycles’ lie. It can be chosen to satisfy [31]

$$CC^\dagger = C^\dagger C = 1, \quad CC^* = \epsilon, \quad (2.29)$$

where $\epsilon = 1$ if the number of elementary gamma matrices is $8k$ or $8k + 2$ and $\epsilon = -1$ if their number is $8k + 4$ or $8k + 6$. Thus, we shall have

$$\begin{aligned} C^*(\alpha) &= CC(\alpha)C^{-1} \\ C^T C(\alpha) &= C^*(\alpha)C^T, \end{aligned} \quad (2.30)$$

as useful properties to be used later. Notice, the existence of C is rather special to these cocycles, constructed from gamma matrices, which are in turn special to the $\mathbf{Z}_2$ nature of the twist. It is not true in general, that a set of cocycles is equivalent to its conjugate set. For example, there is a cubic twist in $so(8)$ [45], analogous to the reflection in the sense that it leaves nothing in the lattice but the origin fixed, for which the twisted vacua are a three dimensional representation (and hence complex) of the invariant $su(3)$ algebra.

The definition (2.28) may seem odd at first but this is almost an invariance for the vertices (2.4) and (2.7); only the momentum α changes sign in those cases. To check this is straightforward. It follows from the definitions of the vertices and the properties of the ‘cocycles’. Specifically, equation (2.9) in the twisted case, and the fact

$$\left(C(\alpha)e^{i\alpha.q}\right)^\dagger = e^{-i\alpha.q}C(-\alpha)$$

in the untwisted case. The definition (2.28) meshes nicely with (2.27), and the conjugate vertex transforms similarly with the labels S and T interchanged.

3. Operator products of Twisted State Emission Vertices

In this section, we shall conjecture and to a certain extent verify formulae for the operator product expansion of a pair of twisted state emission vertices. There are two possibilities and we need to compute both of them. They are:

$$(a) \quad \overline{W}(\psi, z)W(\phi, z') \tag{3.1}$$

and

$$(b) \quad W(\psi, z)\overline{W}(\phi, z'). \tag{3.2}$$

Clearly, the operator product in (a) maps the straight sector to another straight sector and so provided the conformal weights of the states $|\phi\rangle$ and $|\psi\rangle$ are unity, we will expect a double pole factor $(z - z')^{-2}$, a single pole term $(z - z')^{-1}$ whose coefficient is a combination of standard straight vertices, together with pieces entirely regular at $z = z'$. The straight vertices will, however, appear in an unusual form involving a mapping between two identical Fock spaces—in rather the way the three ‘Reggeon’ vertex [46,47] would do if a specific state were appended to one of the legs. This calculation is relatively straightforward and suggests the conjecture. The product (b) is expected to come out to be a similar expression to that in (a) but with twisted vertices replacing the straight ones. This calculation is less straightforward, but many of the ingredients have been computed before [9,10,48] and we can make use of some of the old results. We shall also see that it is sometimes necessary to modify the lattice Λ by the addition of special weights.

Let us suppose $|\phi\rangle$ and $|\psi\rangle$ are vacuum states with equal conformal weight $\Delta_\phi = \Delta_\psi = \Delta_0$. We can easily compute the expectation value of the product (a) between momentum states $|\alpha\rangle$ and $|\beta\rangle$ in the two straight Fock spaces to obtain,

$$\begin{aligned} \langle \alpha | \bar{W}(\psi, z) W(\phi, z') | \beta \rangle &= \left\{ \psi^T C C_T^\dagger(\alpha) C_T(\beta) \phi \right\} (-4)^{-\frac{1}{2}\alpha^2 - \frac{1}{2}\beta^2} \\ &\times (zz')^{-\frac{1}{2}\beta^2} (z - z')^{\frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2 - 2\Delta_0} \left(\frac{\sqrt{-z} - \sqrt{-z'}}{\sqrt{-z} + \sqrt{-z'}} \right)^{\alpha\beta}. \end{aligned} \quad (3.3)$$

In deriving (3.3), it is useful to use the identity

$$e^{\frac{1}{z}L_{+1}} e^{z'L_{-1}} = e^{\frac{zz'}{z-z'}L_{-1}} \left(1 - \frac{z'}{z} \right)^{-2L_0} e^{\frac{1}{z-z'}L_{+1}} \quad (3.4)$$

and to spot the nice facts that the Mobius transformations represented by

$$\left(1 - \frac{z'}{z} \right)^{-L_0} e^{\frac{1}{z-z'}L_{+1}}$$

and

$$\left(1 - \frac{z}{z'} \right)^{L_0} e^{-\frac{zz'}{z-z'}L_{-1}}$$

have $-z'$ and $-z$, respectively, as fixed points. If in (3.3) we put $\varepsilon = z - z'$ and expand near $\varepsilon = 0$ we find:

$$\begin{aligned} \langle \alpha | \bar{W}(\psi, z) W(\phi, z') | \beta \rangle &= \left\{ \psi^T C C_T^\dagger(\alpha) C_T(\beta) \phi \right\} \times \varepsilon^{\frac{1}{2}(\alpha+\beta)^2 - 2\Delta_0} \\ &\times (-4)^{-\frac{1}{2}(\alpha+\beta)^2} (z')^{-\alpha\beta - \beta^2} \left(1 - \frac{1}{2}(\alpha + \beta) \cdot \beta \frac{\varepsilon}{z'} + O(\varepsilon^2) \right), \end{aligned} \quad (3.5)$$

a curiously asymmetrical result in α and β . However, we notice that if $\Delta_0 = 1$, as it must be when $D = 16$, there is a double pole (and then no single pole) only when $\alpha + \beta$ vanishes, with coefficient $\psi^T C \phi$. Moreover, if α and β are roots of squared length two there will be just a simple pole when $\alpha + \beta$ is also a root. It has a coefficient equal to

$$\frac{1}{4}\varepsilon(\alpha, \beta) \left\{ \psi^T C C_T(\alpha + \beta) \phi \right\} (z')^{-\alpha\beta - \beta^2}, \quad (3.6)$$

which also looks strange, but is actually the sort of term we ought to expect. Indeed, both terms follow straightforwardly from the conjecture

$$\bar{W}(\psi, z) W(\phi, z') \sim \frac{1}{\varepsilon^2} \psi^T C \phi \mathbf{1} + \frac{1}{4\varepsilon} \sum_{\alpha^2=2} \psi^T C C_T(\alpha) \phi V(\alpha, z') + \text{regular}, \quad (3.7)$$

where the tilde indicates that we should be considering the straight vertex in a different but equivalent representation mapping between two straight Fock spaces. Noting

the reflection property (2.8) of the twisted cocycle, we observe in the second term of (3.7) that we may replace the sum over roots with a sum over positive roots, and the combination of straight vertices then involved is precisely the one invariant under the reflection twist, equation (2.13), with the conventions we adopted in (2.11). In other words we may conjecture instead,

$$\begin{aligned} \overline{W}(\psi, z)W(\phi, z') &\sim \frac{1}{\epsilon^2}\psi^T C\phi \mathbf{1} + \frac{1}{\epsilon} \sum_{\substack{\alpha > 0 \\ \alpha^2 = 2}} \frac{1}{2}\psi^T C C_T(\alpha)\phi \\ &\times \frac{1}{2}\{V(\alpha, z') - V(-\alpha, z')\}. \end{aligned} \quad (3.8)$$

It is worth remarking that despite the fact we have no direct proof of (3.8), the equation in operator language does have the correct conformal properties [49], and the expectation values we evaluated strongly support the validity of the expressions occurring as coefficients of the Fock space pieces. A more general argument based on coherent states is likely to make use of the technology developed in refs[50,51].

The next step in our program is to evaluate the product (b) in sufficient detail to convince us that it comes out to be similar to (3.8), but with twisted vertices on the right hand side. This computation is less straightforward and again we shall proceed by appending vacua $\langle \mu |$ and $| \lambda \rangle$ in the twisted sector and computing the vacuum expectation value. Here, μ and λ refer to vacuum vectors in the twisted sector.

With these assumptions the quantities $B^\pm$ in each of the vertices disappear and we are left with

$$\begin{aligned} \langle \mu | W(\psi, z) \overline{W}(\phi, z') | \lambda \rangle \\ = \langle \mu | \langle v | e^{A(z)} | \psi \rangle \langle \phi_c | e^{A^\dagger(1/(z')^*)} | v \rangle | \lambda \rangle (z')^{-2\Delta_0}, \end{aligned} \quad (3.9)$$

remembering that L_{+1}^T and L_{-1}^T annihilate to the right and left, respectively, (3.9) can be rendered more symmetrical-looking by using the facts

$$\begin{aligned} z^{L_0} e^{A(1)} z^{-L_0} &= e^{A(z)} \\ (z')^{L_0} e^{A^\dagger(1)} (z')^{-L_0} &= e^{A^\dagger(1/(z')^*)}, \end{aligned} \quad (3.10)$$

to obtain,

$$\begin{aligned} (z')^{-2\Delta_0} \sum_{\alpha} \left\{ \mu^\dagger C_T(\alpha) \psi \right\} \left\{ \phi^T C C_T^\dagger(\alpha) \lambda \right\} \\ \times \langle \alpha | e^{\frac{1}{2}\alpha_n A_{nm}(1/\sqrt{u})\alpha_m} e^{\frac{1}{2}\alpha_{-n} A_{nm}(1/\sqrt{u})\alpha_{-m}} | \alpha \rangle, \end{aligned} \quad (3.11)$$

where the sum on α extends over the whole root lattice, u is the ratio z'/z , and the

matrices $A_{nm}(1/\sqrt{u})$ are defined in equations (2.21) and (2.22). Techniques for computing an expectation value such as that in (3.11) were developed long ago, and indeed the combination of factors occurring in (3.11) is precisely the same as a computation of Schwarz [9,10]. We simply quote the old results, although they may be derived rather easily and group theoretically following and expanding the techniques of ref[48]. Thus,

$$\begin{aligned} \langle \alpha | e^{\frac{1}{2}\alpha_n A_{nm}(1/\sqrt{u})\alpha_m} e^{\frac{1}{2}\alpha_{-n} A_{nm}(1/\sqrt{u})\alpha_{-m}} | \alpha \rangle \\ = \left[\frac{2}{\pi} K(\sqrt{u})(1-u)^{\frac{1}{4}} \right]^{-\frac{D}{2}} \exp \left(-\alpha^2 \frac{\pi K'(\sqrt{u})}{2K(\sqrt{u})} \right), \end{aligned} \quad (3.12)$$

where $K(\sqrt{u})$ is the complete elliptic integral of the first kind [52], and $K'(\sqrt{u})$ is the same function evaluated at $\sqrt{1-u}$.

Expression (3.12) is convenient enough if one is interested in the behaviour near $u = 0$. However, on the contrary, we are interested in the ‘dual’ region $u = 1$, or in terms of the original variables $z' = z$. To examine that region it is convenient to rewrite (3.12) in terms of theta functions. Thus,

$$\begin{aligned} \ln q &\equiv -\frac{\pi K'(\sqrt{u})}{K(\sqrt{u})}, \quad \sqrt{u} = \left(\frac{\theta_2}{\theta_3} \right)^2, \quad \sqrt{1-u} = \left(\frac{\theta_4}{\theta_3} \right)^2 \\ K(\sqrt{u}) &= \frac{\pi}{2} \theta_3^2, \quad K(\sqrt{1-u}) = -\frac{\ln q}{2} \theta_3^2, \end{aligned} \quad (3.13)$$

where the θ_i are, as usual [52], given by

$$\begin{aligned} \theta_1 &= 2q_0 q^{\frac{1}{4}} \prod_n (1 - q^{2n})^2 \\ \theta_2 &= 2q_0 q^{\frac{1}{4}} \prod_n (1 + q^{2n})^2 \\ \theta_3 &= q_0 \prod_n (1 + q^{2n-1})^2 \\ \theta_4 &= q_0 \prod_n (1 - q^{2n-1})^2 \end{aligned} \quad (3.14)$$

and

$$q_0 = \prod_n (1 - q^{2n}). \quad (3.15)$$

Inserting (3.13) and (3.12) into (3.11) we obtain finally,

$$\begin{aligned} & \langle \mu | W(\psi, z) \overline{W}(\phi, z') | \lambda \rangle \\ &= (z')^{-2\Delta_0} \left(\sum_{\alpha} \left\{ \mu^\dagger C_T(\alpha) \psi \right\} \left\{ \phi^T C C_T^\dagger(\alpha) \lambda \right\} q^{\frac{1}{2}\alpha^2} \right) \frac{[\theta_3(q)]^{-D}}{(1-u)^{D/8}}, \end{aligned} \quad (3.16)$$

which is a suitable starting point for the ‘dual’ discussion.

In order to obtain an expression to match with (3.7) there are two problems. One we have mentioned already; (3.16) is written in a form manifestly good for small q (or $\sqrt{u}$). The other involves a rearrangement of the inner products inside the sum, since we actually want a sum of terms of the form $\psi^T C C_T(\alpha) \phi$. A hint as to how this might work is contained already in the computation of the four fermion scattering amplitude [53,54], where the Fierz transformation meshes mavelously with the ‘duality’ transformation. There, the sums involved were finite, but here they run over the whole lattice. Nevertheless, the completeness relation for the ‘cocycles’ is the important ingredient, fitting nicely with a Jacobi transformation of the variable q !

As we have summarised in appendix A, it is easy to select a representation of the cocycles within a Clifford algebra of suitable dimension, starting with an assignment to the simple roots of the Dynkin diagram. The whole Clifford algebra is required, however, if we wish to make use of a completeness relation. (Although, it may sometimes happen by accident that the ‘cocycles’ on the roots are complete in a restricted sense. For example, in the E_8 case, they are a complete set of antisymmetric matrices within the Clifford algebra). In some cases, such as the root lattices for $su(2k+1)$, E_6 , E_7 and E_8 , the full Clifford algebra is generated at higher levels of the lattice. That is easy to see from our summary in appendix A. In other cases, such as $so(2k)$ or $su(2k)$ this does not work. For $so(4k)$, which is the only case of this type we will worry about in detail, the ‘cocycle’ matrices generated by the root lattice are the subset commuting with γ^{2k+1} . In other words, the set of products of elementary gamma matrices of even length. In order to obtain a complete set we ought to enhance the lattice by adding a single set of spinor weights, (i.e. S or $\overline{S}$, not S and $\overline{S}$), and attaching to them the products of gamma matrices with odd length. This is obviously a consistent procedure, and the lattice so obtained is actually self-dual provided k is also even. In fact, the lattice so obtained is both self-dual and even if k is a multiple of four (since the shortest spinor weights have length-squared $\frac{k}{2}$).

We shall suppose this procedure has been carried out and there will be a set Λ_0 of (positive) vectors on the root (or roots plus certain weights) lattice for which

$$\sum_{\tau \in \Lambda_0} C^\dagger(\tau)_{ij} C(\tau)_{kl} = 2^{D/2} \delta_{il} \delta_{jk}. \quad (3.17)$$

We emphasise; the τ 's will not, in general, have the same length. Using (3.17) and (2.8) we may rearrange the inner products on the right hand side of eq (3.16). Thus,

$$\begin{aligned} & \left\{ \mu^\dagger C_T(\alpha) \psi \right\} \left\{ \phi^T C C_T^\dagger(\alpha) \lambda \right\} \\ &= 2^{-D/2} \sum_{\tau} \left\{ \mu^\dagger C_T^\dagger(\tau) \phi \right\} \left\{ \lambda^T C_T^*(\alpha) C^T C_T(\tau) C_T(\alpha) \psi \right\} \\ &= 2^{-D/2} \sum_{\tau} \left\{ \mu^\dagger C_T^\dagger(\tau) \phi \right\} \left\{ \lambda^T C^T C_T(\tau) \psi \right\} (-)^{\alpha \cdot \tau} (-)^{\frac{1}{2} \alpha^2}, \end{aligned} \quad (3.18)$$

where we have made use of the properties of the conjugation matrix C , equation (2.30), and the matrix 'cocycles' equations (2.8) and (2.9). Thus, we have removed the α dependence of the 'cocycle' in equation (3.16) and the remaining sum over α defines a kind of partition function,

$$\Gamma(q, \tau) = \sum_{\alpha} (-q)^{\frac{1}{2} \alpha^2} (-)^{\alpha \cdot \tau}. \quad (3.19)$$

for which we shall need a variant of the Jacobi imaginary transformation [52].

If we suppose q and q' to be related by

$$\ln q \ln q' = \pi^2, \quad (3.20)$$

then we demonstrate in appendix B that

$$\Gamma(q, \tau) = (-)^{\frac{1}{2} \tau^2} \left(\frac{\ln q'}{i\pi} \right)^{D/2} \Gamma(q', \tau), \quad (3.21)$$

for any lattice vector τ , provided the dual of the (enhanced) lattice is 'integral', by which we mean every vector in Λ^* has integer length-squared. This fact is guaranteed if the (enhanced) root lattice is self-dual. In any case, we see D must be a multiple of eight which fits nicely with the other observation that the operator product expansion only

works as we expect provided D is so constrained. We recall that the theta function part of (3.16) also transforms nicely under (3.20),

$$(\theta_3(q))^2 = \left(\frac{\ln q'}{\pi} \right) (\theta_3(q'))^2, \quad (3.22)$$

and thus, together, (3.21) and (3.22) transform equation (3.16) conveniently in the sense that there are no embarrassing factors of $\ln q'$ or i left over. At this stage we have,

$$\begin{aligned} \langle \mu | W(\psi, z) \overline{W}(\phi, z') | \lambda \rangle &= 2^{D/2} (z')^{-2\Delta_0} \sum_{\tau, \alpha} \left\{ \mu^\dagger C_T^\dagger(\tau) \phi \right\} \left\{ \lambda^T C^T C_T(\tau) \psi \right\} \\ &\times (-)^{\frac{1}{2}\tau^2} (-q')^{\frac{1}{2}\alpha^2} (-)^{\tau \cdot \alpha} (i)^{D/2} \frac{[\theta_3(q')]^{-D}}{(1-u)^{D/8}}, \end{aligned} \quad (3.23)$$

which is still not quite in the form we want. Note however,

$$\begin{aligned} \lambda^T C^T C_T(\tau) \psi &= \psi^T C_T^T(\tau) C \lambda \\ &= \psi^T C C_T(\alpha) C_T(\tau) C_T(\alpha) \lambda (-)^{\frac{1}{2}(\tau+\alpha)^2}, \end{aligned} \quad (3.24)$$

using the ‘cocycle’ properties. Hence, we can use the completeness properties of the ‘cocycles’ again to get,

$$\begin{aligned} \langle \mu | W(\psi, z) \overline{W}(\phi, z') | \lambda \rangle &= (z')^{-2\Delta_0} \sum_{\alpha} \left\{ \phi^T C^T C_T^\dagger(\alpha) \psi \right\} \left\{ \mu^\dagger C_T(\alpha) \lambda \right\} (q')^{\frac{1}{2}\alpha^2} \frac{(\theta_3(q'))^{-D}}{(1-u)^{D/8}}. \end{aligned} \quad (3.25)$$

The ϕ - ψ inner product in equation (3.25) can be reversed, again using equations (2.29) and (2.30) to show

$$\phi^T C^T C_T^\dagger(\alpha) \psi = \psi^T C C_T(\alpha) \phi \quad (3.26)$$

a useful identity.

Equation (3.25) is finally a suitable expression for the discussion of the operator product. It is clear, from the definition of u in terms of θ functions, that to order $(1-u)$

$$q' = \frac{1}{16}(1-u) + O(1-u)^2$$

and

$$\begin{aligned}
(\theta_3(q'))^{-D} &= 1 - 2Dq' + O(q')^2 \\
&= 1 - \frac{D}{8}(1-u) + O(1-u)^2, \\
z' &= z(1 - (1-u)), \\
\frac{(z')^{-D/8}}{(1-u)^{D/8}} &= \frac{z^{-D/8}}{(1-u)^{D/8}} \left(1 + \frac{D}{8}(1-u) + O(1-u)^2 \right),
\end{aligned} \tag{3.27}$$

and so, when $\Delta_0 = \frac{D}{16}$,

$$\frac{(z')^{-D/8}(\theta_3(q'))^{-D}}{(1-u)^{D/8}} = \frac{1}{(z-z')^{D/8}} (1 + O(1-u)^2). \tag{3.28}$$

Putting these together in sixteen dimensions we find

$$\begin{aligned}
&\langle \mu | W(\psi, z) \bar{W}(\phi, z') | \lambda \rangle \\
&= \frac{\{\phi^T C \psi\} \{\mu^\dagger \lambda\}}{\epsilon^2} + \frac{(z')^{-1}}{16\epsilon} \sum_{\alpha^2=2} \left\{ \psi^T C C_T(\alpha) \phi \right\} \left\{ \mu^\dagger C_T(\alpha) \lambda \right\} + \text{regular}
\end{aligned} \tag{3.29}$$

and recognise the result as the expectation value of our expression, similar to (3.8) but with the right hand side replaced by twisted vertices, (2.7). As before, we are unable to prove the operator expression for the product, only to check its expectation value.

It is perhaps worth emphasising that we have proved and made use of an interesting property of certain lattices (self-dual is an example) which can be summarised as,

$$\begin{aligned}
&\frac{1}{(\theta_3(q'))^D} \sum_{\alpha \in \Lambda} \left\{ \phi^T C^T C_T^\dagger(\alpha) \psi \right\} \left\{ \mu^\dagger C_T(\alpha) \lambda \right\} (q')^{\frac{1}{2}\alpha^2} \\
&= \frac{1}{(\theta_3(q))^D} \sum_{\alpha \in \Lambda} \left\{ \phi^T C C_T^\dagger(\alpha) \lambda \right\} \left\{ \mu^\dagger C_T(\alpha) \psi \right\} (q)^{\frac{1}{2}\alpha^2},
\end{aligned} \tag{3.30}$$

where q and q' are related by (3.20). The modular transformation is intimately linked with the Fierz rearrangement, as we expected it should be.

The next problem we need to tackle is how to elaborate the discussion following equation (2.24) in order to be able to take moments of the various vertex operators. As it stands, this is not possible since the equation is invalid on at least part of any contour surrounding the origin in the y, z planes. This problem occurred before for the fermion vertex, and was solved in ref[55]. Here, we shall borrow heavily from that discussion. Again, we shall restrict our analysis to $|\phi\rangle$ a vacuum state in order to simplify it.

To analyse equation (2.24) carefully we need to separate out those parts of the computation that are valid in the region I, $|y| < |z|$, from those valid in the region II, $|y - z| > |z|$. In fact, when the 'emitted' state $|\phi\rangle$ is a vacuum, the piece $B^+(z)$ plays no role and the left hand side of (2.24) can be simplified in region II, the right hand side in region I.

Thus, we obtain

$$V_T(\alpha, y - z)I(\phi, z) = \frac{S(y, z)}{(y - z)^{\frac{1}{2}\alpha^2}}, \quad \text{for } |y - z| > |z| \quad (3.31)$$

$$I(\phi, z)V(\alpha, y) = \frac{S(y, z)}{(y - z)^{\frac{1}{2}\alpha^2}}, \quad \text{for } |y| < |z|, \quad (3.32)$$

where we have used the same symbol $S(y, z)$ for the quantity on the right hand sides because they can be shown to be equal when the regions I and II overlap. The function $S(y, z)$ is not quite good enough for us, however, because although we do not need the full expression for it and it is not singular at $y = z$, it does have square root branch points there. These are expected, of course, since the twisted vertex would have integer and half integer moments if we expanded it out in powers of $(y - z)$. The solution to this difficulty is already suggested in the conjecture (3.8). We ought to antisymmetrise the straight and twisted vertices with respect to the momentum. When we do that, defining

$$\begin{aligned} \tilde{V}(\alpha, z) &= \frac{1}{2} \{V(\alpha, z) - V(-\alpha, z)\} \\ \tilde{V}_T(\alpha, z) &= \frac{1}{2} \{V_T(\alpha, z) - V_T(-\alpha, z)\}, \end{aligned} \quad (3.33)$$

the problem disappears and we find instead

$$\tilde{V}_T(\alpha, y - z)I(\phi, z) = \frac{\tilde{S}(y, z)}{(y - z)^{\frac{1}{2}\alpha^2}} \quad (3.34)$$

$$I(\phi, z)\tilde{V}(\alpha, y) = \frac{\tilde{S}(y, z)}{(y - z)^{\frac{1}{2}\alpha^2}}, \quad (3.35)$$

with $\tilde{S}(y, z)$ regular at $y = z$.

Now, we may repeat the rest of the argument of ref[55]. Considering

$$\oint_C \frac{dy}{2\pi i} \frac{\tilde{S}(y, z)}{(y - z)^{\frac{1}{2}\alpha^2}} f = \oint_{C_0} \frac{dy}{2\pi i} \frac{\tilde{S}(y, z)}{(y - z)^{\frac{1}{2}\alpha^2}} f + \oint_{C_z} \frac{dy}{2\pi i} \frac{\tilde{S}(y, z)}{(y - z)^{\frac{1}{2}\alpha^2}} f, \quad (3.36)$$

where C is a contour encircling both $y = 0$ and $y = z$, (hence deformable to region II), C_0 encircles the origin $y = 0$ only avoiding $y = z$, (hence deformable to region I) and C_z encloses the point $y = z$. The function f may have poles at $y = 0, z$ but no branch points there. Thus, equation (3.36) allows us to smoothly extrapolate from region I to region II replacing $\tilde{S}$ by its value in the appropriate region, given either by (3.34) or (3.35). We also remark at the same time that we have in mind α should be a root, $\frac{1}{2}\alpha^2 = 2$, and that we choose $f(y) = y^n$ where n is any integer. Thus, the third term is evaluated to be

$$\tilde{S}(z, z) z^n. \quad (3.37)$$

Summarising, we have

$$\oint_C \frac{dy}{2\pi i} \tilde{V}_T(\alpha, y - z) I(\phi, z) y^n = \oint_{C_0} \frac{dy}{2\pi i} I(\phi, z) \tilde{V}(\alpha, y) y^n + \tilde{S}(z, z) z^n. \quad (3.38)$$

Finally, on the given contours we may multiply the factor e^{zL-1} back to restore the full TSEV and using the explicit expressions for the Fock space functions $A(z)$, $B^-(z)$, evaluate $\tilde{S}(z, z)$:

$$\begin{aligned} \tilde{S}(z, z) &= C_T(\alpha) \langle v | e^{A+B^-} | \phi \rangle (-)^{\alpha.p} \\ &= \langle v | e^{A+B^-} | C_T(\alpha) \phi \rangle. \end{aligned} \quad (3.39)$$

Then (3.38) becomes,

$$\left[\tilde{V}_T(\alpha, y) \right]_n W(\phi, z) - W(\phi, z) \left[\tilde{V}(\alpha, y) \right]_n = W(C_T(\alpha) \phi, z) z^n, \quad (3.40)$$

where we have chosen to evaluate the contour integrals in terms of the moments of the two types of vertex operator. We find equation (3.40) an attractive summary of how the TSEV, corresponding to a vacuum state, transforms with respect to the Kac–Moody generators in each of the sectors. Notice that this calculation is also rather general, no mention being made of any particular dimension, or indeed style of lattice, (provided only that it be a simply-laced root lattice). The ‘intertwining’ nature of the TSEV is also clarified.

At this stage, with equation (3.40) and the two conjectures for the operator products (a) and (b), equations (3.1) and (3.2), we may proceed to make some comments on the algebras relating the three different types of vertex operator.

4. Comments on the algebra containing both twisted and untwisted vertex operators

We have remarked already that the zeroth moments of the straight or twisted vertices, antisymmetrised with respect to their momenta, generate the twist invariant subalgebra, g_0 , of the original algebra g corresponding to the simply-laced root lattice Λ . Equation (3.40) tells directly that the TSEV, or ‘intertwining’ operator, between the two sectors, $W(\phi, z)$, has zeroth moments transforming under g_0 in the same representation as the vacuum states $|\phi\rangle$. Thus, for example if g is taken to be $so(32)$ with a twist invariant algebra subalgebra $so(16) + so(16)$, the intertwining operator transforms as a $(1, 128) + (128, 1)$. On the other hand, if g is $E_8 + E_8$ with the same twist invariant subalgebra then the appropriate intertwining operator transforms as a $(16, 16)$ under g_0 . If needed, the appropriate facts for any of the other algebras of rank sixteen can be determined from the summary in appendix A.

Equation (3.40) is valid in any number of dimensions and is naturally part of a Kac-Moody algebra. Defining the moments of the various operators in a shorthand notation:

$$\begin{aligned} V_n(\alpha) &= \oint \frac{dz}{2\pi i} \tilde{V}(\alpha, z) \\ V_n^T(\alpha) &= \oint \frac{dz}{2\pi i} \tilde{V}_T(\alpha, z) \\ W_n(\phi) &= \oint \frac{dz}{2\pi i} W(\phi, z) \\ \overline{W}_n(\phi) &= \oint \frac{dz}{2\pi i} \overline{W}(\phi, z), \quad n \in \mathbb{Z} \end{aligned} \tag{4.1}$$

equation (3.40) becomes

$$V_n^T(\alpha) W_m(\phi) - W_m(\phi) V_n(\alpha) = W_{n+m}(C_T(\alpha)\phi). \tag{4.2}$$

Similarly, the conjugate vertex satisfies

$$\overline{W}_m(\phi) V_n^T(\alpha) - V_n(\alpha) \overline{W}_m(\phi) = \overline{W}_{n+m}(C_T(\alpha)\phi), \tag{4.3}$$

as a consequence of (2.30).

The interesting feature about dimension sixteen is that then and only then do the intertwining operators $W(\phi, z)$, with $|\phi\rangle$ a vacuum state in the twisted sector, have the same conformal weight as the conventional vertex operators. This has two consequences. Firstly, equations (4.2) and (4.3) are actually conjugates of each other. Secondly, the conjectured operator product expansions, (equation (3.8) and its twisted counterpart), are dimension dependent but in sixteen dimensions the pole structure in $z - z'$ is precisely appropriate to a Kac–Moody algebra. However, to obtain a commutator, (rather than an anticommutator), for the moments of W and $\bar{W}$ we have to assume that C , the charge conjugation matrix, is symmetric, (rather than antisymmetric). That is, ϵ in equation (2.29) must be chosen to be unity. This in itself provides a further constraint on the type of algebra g we are allowed to consider since it implies that although the rank of g is already required to be sixteen, the twisted vacuum representation of g_0 must also involve gamma matrices pertaining to $8k$ or $8k + 2$ dimensions. Assuming this to be the case, the moments of equation (3.8) inform us:

$$\begin{aligned} \bar{W}_n(\psi)W_m(\phi) - \bar{W}_m(\phi)W_n(\psi) \\ = \psi^T C \phi n \delta_{n+m,0} + \sum_{\substack{\alpha > 0 \\ \alpha^2=2}} \frac{1}{2} \psi^T C C_T(\alpha) \phi V_{n+m}(\alpha), \end{aligned} \quad (4.4)$$

while the twisted counterpart gives

$$\begin{aligned} W_n(\psi)\bar{W}_m(\phi) - W_m(\phi)\bar{W}_n(\psi) \\ = \psi^T C \phi n \delta_{n+m,0} + \sum_{\substack{\alpha > 0 \\ \alpha^2=2}} \frac{1}{2} \psi^T C C_T(\alpha) \phi V_{n+m}^T(\alpha), \end{aligned} \quad (4.5)$$

(We also recall that the Fierz rearrangement necessary to obtain (4.5) requires the dual lattice to Λ to be ‘integral’. That is, each vector in Λ^* has integer length squared). We conclude, from (4.4) and (4.5) that the subalgebra g_0 , which we have represented within each sector, is actually enlarged to an algebra g_1 (not necessarily the same as g) which mixes the two sectors in a remarkable way. Actually, because (4.4) and (4.5) are essentially identical we may combine the vertex operators attractively into a 2×2 matrix structure:

$$U_n(\alpha, \phi) = \begin{pmatrix} V_n(\alpha) & \bar{W}_n(\phi) \\ W_n(\phi) & V_n^T(\alpha) \end{pmatrix}, \quad (4.6)$$

where the ‘diagonal’ elements generate g_0 within each sector and the off diagonal elements enlarge the algebra g_0 to g_1 .

What is the structure of the algebra? One might have thought, as a first guess, that it would contain all the generators of g odd under the reflection twist, in addition to the invariant ones which generate g_0 . However, that turns out not to be the case. For example, in the instances we have already mentioned the odd generators of $E_8 + E_8$ would transform as a $(128, 1) + (1, 128)$ of the $so(16) + so(16)$ subalgebra, whereas the zeroth moments of the intertwining operator are a $(16, 16)$ representation. On the other hand, the odd generators of $so(32)$ would transform as a $(16, 16)$, but the moments of the intertwining operator are a $(128, 1) + (1, 128)$. So, in neither case can g_1 be the same as g . Rather, we are drawn to the conclusion,

$$\begin{aligned} (E_8 + E_8)_1 &= so(32) \\ (so(32))_1 &= E_8 + E_8, \end{aligned} \tag{4.7}$$

which is an algebraic statement of previous observations [13,36-38].

Notice that the root lattice of $E_8 + E_8$ is by itself even and self-dual and satisfies all the requirements for our arguments. The same is not true of $so(32)$ unless we enlarge the root lattice by the addition of a set of spinor weights in order to obtain a complete set of ‘cocycle’ matrices on the enhanced lattice. The enhanced lattice is even and self-dual. It is interesting that this requirement was forced by a kind of ‘tree-level duality’, (i.e. re-expressing the operator product (3.2) in terms of $z - z'$), and not by any considerations of modular invariance of partition functions relevant to the duality of one-loop amplitudes.

As pieces to contribute to rank sixteen, the unitary group algebras $su(2k + 1)$ fail to have ‘integral’ weight lattices and cannot meet our requirements. The only other orthogonal group we might consider is $so(16)$ and this is interesting. On the one hand, if we enhance its root lattice by adding a set of spinor weights we change g . This is because the shortest spinor weights of $so(16)$ are the same length as the roots, and indeed the enhanced lattice is precisely the E_8 lattice. On the other hand, the ‘cocycle’ matrices defined on the roots *alone* generate the representation $(8, 1) + (1, 8)$ of the twist invariant algebra $so(8) + so(8)$. Thus, they may be regarded as a couple of sets of eight by eight antisymmetric matrices which are complete in a restricted, antisymmetric sense. In other words, in this case, our arguments will work without enhancing the lattice in the way we have described.

So far we have discussed only the case where the TSEV involves a vacuum state. Another possibility is to take the next state in the twisted sector in which case the TSEV has unit conformal weight in eight dimensions. For the case of $g = E_8$, the twist invariant subalgebra is $so(16)$ and the first excited state in the twisted Fock space is a 128 of $so(16)$. Thus, the TSEV also transforms as a 128 of $so(16)$ in the sense of equation (4.2), (with a suitably modified right hand side). We have checked, to the same precision as the details in section (3), that the enhanced algebra, (in this case containing the $so(16)$ generators in both sectors and the 128 zeroth moments of the TSEV, or intertwining operator), generate E_8 again [39]. In other words,

$$(E_8)_1 = E_8. \quad (4.8)$$

However, the details are more complicated and we shall omit them here.

Our results, partial though they are, suggest it may be worthwhile to look elsewhere, at other twists, to see if any similar phenomena can occur. The reflection twist is rather special since it relates to the spacetime supersymmetry of the spinning string in ten dimensions, to a way of extending off-shell for a purely bosonic string in twenty-six dimensions, and provides intriguing representations of certain Lie algebras in sixteen dimensions. It is also the simplest twist one could imagine since it satisfies,

$$\sigma + 1 = 0, \quad (4.9)$$

if we denote the lattice mapping by σ . Higher order twists involve a huge variety of possibilities. However, if we decide to look at a third order generalisation of (4.9), i.e. those twists satisfying

$$\sigma^2 + \sigma + 1 = 0, \quad \sigma^3 = 1, \quad (4.10)$$

there are remarkably few [45]. Actually, if we suppose g to be simple then the only root lattices allowing a third order twist satisfying (4.10) are those for $su(3)$, G_2 , $so(8)$, F_4 , E_6 and E_8 , and *no* others. The twist invariant subgroups g_0 are $u(1) + u(1)$, $u(2)$, $su(3)$, $su(3) + su(3)$, $su(3) + su(3)$, $su(3) + su(3) + su(3)$ and $su(9)$, respectively, which may (at least for the simply-laced cases) be represented in each of the three stringy sectors—a straight (FKS) sector and two twisted ones. In each of the two twisted sectors the vacuum will be degenerate, (except for the $su(3)$ example), and represent g_0 via the

matrix ‘cocycles’. Moreover, the twisted vacua have conformal weight $D/18$, where D is the rank of g and the dimension of the root lattice. None of the simple algebras in the above list has rank eighteen. Thus, there is no choice of simple algebra for which the vacuum has unit conformal weight. Indeed, it is worth remarking that only for E_6 do any states at all in the twisted sectors have unit conformal weight, and then they are at the second excited level; that is, these states are of the form

$$c_{-\frac{2}{3}}^a |\phi\rangle, \quad c_{-\frac{1}{3}}^a c_{-\frac{1}{3}}^b |\phi\rangle, \quad a, b = 1, 2, 3. \quad (4.11)$$

Notice, there is a set of bose operators moded $\mathbb{Z} \pm \frac{1}{3}$ for each sector, and a, b take the values 1, 2 or 3. This is because every sector carries a representation of the Virasoro algebra, and the six real dimensions are best regarded as three complex ones for discussing the twist whose eigenvalues are ω or ω^2 , with $\omega^3 = 1$, ($\omega \neq 1$). We expect there to be a TSEV with unit conformal weight corresponding to the states in (4.11). (The states are annihilated by L_{+1}). We can also say what representation of $su(3) + su(3) + su(3)$ it must be, in an intertwining sense, generalising (3.40). In this case, the matrix ‘cocycles’ yield a three dimensional representation of g_0 (in terms of $su(3)$ λ or λ^* matrices, for some further details see appendix C), and thus the states (4.11) are a $\underline{27}$, suitably distributed among the simple pieces of g_0 , (i.e. a ‘three’ with respect to each). This is a remarkable numerical ‘accident’; each sector contributes a twenty-seven dimensional TSEV, fifty-four operators in all, whose zeroth moments will combine, (although we have not checked it), with g_0 to give an enhanced algebra g_1 of dimension seventy-eight. We conjecture this algebra to be E_6 again. If this is a correct conjecture, E_6 plays a role for a triple twist similar to that of E_8 for a double twist. A quick check through the list of examples generalising (4.10) yields no other opportunities for this phenomenon.

The reflection twist also has a meaning when formulated in space-time. One question we might ask is whether this could ever be true for a triple twist. At first sight not, since the conformal weights of the intertwining operators would not be unity unless we append a ghostly piece. However, there are a couple more numerical accidents in this context. First of all, in twenty-six dimensions, we note a ghost contribution, (in the sense of refs[20,21]), of ghost number $\frac{1}{3}$ would give a sensible vertex since

$$\frac{D_B}{18} + \frac{1}{2}q_B(q_B - 3) = \frac{26}{18} - \frac{1}{6} \cdot \frac{8}{3} = 1. \quad (4.12)$$

Secondly, in ten dimensions and fermi fields a ghost contribution, (in the sense of ref[19]),

with ghost number $-\frac{2}{3}$ gives,

$$\frac{D_F}{18} - \frac{1}{2}q_F(q_F + 2) = \frac{10}{18} + \frac{8}{18} = 1, \quad (4.13)$$

also numerically sensible. If we were to consider a general twist there would be several sectors usually, with vacua of different conformal weight. If we took a pair of conjugate sectors and considered the analogues of equations (4.12) and (4.13) we should have to require

$$\frac{D_B}{4}\lambda(1 - \lambda) + \frac{1}{2}q_B(q_B - 3) = 1 \quad (4.14)$$

and

$$\frac{D_F}{4}\lambda(1 - \lambda) - \frac{1}{2}q_F(q_F + 2) = 1 \quad (4.15)$$

where λ is a fraction representing the moding of the Fock space operators. We expect D_B and D_F to be twenty-six and ten, respectively, so a single relation between q_B and q_F determines everything. Remarkably, the assumption,

$$q_B - q_F = 1, \quad (4.16)$$

relating ghost numbers in the different sectors leads to just two solutions: $q_B = \frac{1}{2}$ or $\frac{1}{3}$. We do not have a strong basis for the assumption (4.16). However, it seems to us interesting that using this assumption the only possible solution, other than the one appropriate to the reflection twist, is the one we were led to consider earlier, on purely algebraic grounds.

5. Comments

We have been intrigued by the fact that twisted strings and associated vertex operators seem to play a role in string theory and in mathematics. We set out to explore the algebra composed of straight string vertex operators, twisted string operators and TSEVs, not necessarily in order to compute scattering amplitudes, (since this has already been done to some extent elsewhere [56]), but rather to try and gain a better understanding of how apparently different heterotic string theories are related to each other. To the extent that we are able to check the algebra we can see directly how

the $E_8 + E_8$ and $so(32)$ algebras are related and why it is that the latter root lattice needs to be enhanced to be self-dual. In this context, the argument is really a tree level argument nothing to do with a loop calculation at all. We are also intrigued by the fact that what is essentially the same vertex can play so many different roles depending on the context, the statistics and the dimension, both in the purely Lie algebraic sense and in space-time. It seems to us that this is not properly understood even for the case of the reflection twist; despite the fact there are many more pieces of the puzzle known to be relevant they still remain to be assembled to a coherent whole.

Beyond that we are intrigued by the possibility that there may be other structures yet to be discovered, dependent upon the property of higher order twists, which also have their analogues in space-time. As we have remarked it does not seem to us inconceivable that there are structures other than supersymmetry that may be defined on the string world sheet and promoted to space-time.

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APPENDIX A

In this appendix we shall give a mnemonic for all the ‘cocycle’ matrices appropriate to a reflection twist for any of the simply-laced root lattices. They have been computed in detail before [27], but we shall attach them in a systematic way to each Dynkin diagram. Knowing the matrices for the simple roots then defines them and the cocycle, up to a ‘gauge’ transformation, for all the other roots.

We shall denote a product of distinct elementary hermitian gamma matrices by a single label. Thus,

$$\gamma^1 \gamma^2 \gamma^3 \dots \equiv \gamma^{123\dots} \equiv (123\dots).$$

If the number of labels is k then the hermitian matrices in the Clifford algebra are those of ‘length’ $k = 4N$ or $4N + 1$, where N is any integer; the rest are antihermitian.

1. $su(N)$:

Clearly, the set (ab) generate an $so(N)$ representation with dimension $2^{\frac{N-1}{2}}$ if N is odd, $2^{\frac{N}{2}}$ if N is even and reducible in the latter case.

2. $so(2N)$:

These neatly generate the set (ab) and $(N+1 a b)$. If N is even $(N+1) = (123\dots N)$ and the representation of $so(N) + so(N)$ has dimension $2^{\frac{N}{2}}$; otherwise, it has dimension $2^{\frac{N+1}{2}}$. As far as the vertex operators are concerned we may identify the two $so(N)$ algebras as follows. Notice, the integer moments of the twisted vertex operator fail to distinguish a positive root from its negative. Hence, we may split the roots into two sets $\{\alpha_{ij}\} = \{e_i - e_j\}$, $\{\tilde{\alpha}_{ij}\} = \{e_i + e_j\}$, where $e_i, i = 1, \dots, N$

are an orthogonal basis for the $so(2N)$ root lattice. Then the two commuting $so(2N)$ algebras are defined to be the zeroth moments of

$$\hat{V}_T^\pm(\alpha_{ij}, z) = \frac{1}{2} \gamma^{ij} (V_T(\alpha_{ij}, a) \pm \gamma^{N+1} V_T(\tilde{\alpha}_{ij}, z)),$$

as is easy to check.

3. E_6 :

These generate the set (ab) , (abc) , $(7) = (123456)$ on the positive roots, and give an eight dimension representation of $sp(4)$.

4. E_7 :

These generate, on the positive roots, the whole Clifford algebra in six dimensions, except for 1, and give an eight dimensional representation of $su(8)$.

5. E_8 :

These generate, on the positive roots, the set (ab) , $(9ab)$, (abc) and $(9a)$, in fact all the antihermitian matrices in the Clifford algebra, yielding a sixteen dimensional representation of $so(16)$. The γ matrices can also be chosen to be real in this case, (as always in dimension $8k$, or $8k + 2$), in which case the 'cocycles' are antisymmetric on the positive roots of E_8 .

APPENDIX B

Here we prove the relation (3.21).

Recall the fundamental relation relating a lattice sum to its dual:

$$\sum_{\omega \in \Lambda^*} e^{2\pi i \beta \cdot \omega} = \sum_{\alpha \in \Lambda} \delta(\beta - \alpha). \quad (\text{B1})$$

Multiplying each side by $e^{-2\pi i \beta \cdot \beta' - z \beta^2}$ and integrating over all β yields:

$$\left(\frac{\pi}{z}\right)^{\frac{\dim \Lambda}{2}} \sum_{\omega \in \Lambda^*} e^{-\pi^2 (\omega - \beta')^2 / z} = \sum_{\alpha \in \Lambda} e^{-2\pi i \alpha \cdot \beta'} e^{-\alpha^2 z}. \quad (\text{B2})$$

Now, writing $z = -i\pi\sigma$, $\beta' = \frac{1}{2}\tau$ and $q = e^{i\pi(2\sigma-1)}$, we recognise the right hand side of (B2) to be

$$\sum_{\alpha \in \Lambda} (-q)^{\frac{1}{2}\alpha^2} (-)^{\alpha \cdot \tau} = \Gamma(q, \tau). \quad (\text{B3})$$

Next, define σ' , q' via,

$$\sigma' = \frac{\sigma}{1-2\sigma}, \quad q' = e^{i\pi(2\sigma'+1)},$$

then

$$\ln q \ln q' = -\pi^2 (2\sigma - 1)(2\sigma' + 1) = \pi^2,$$

which represents the transformation we are trying to achieve.

Consider

$$\begin{aligned} \Gamma(q', \tau) &= \sum_{\alpha \in \Lambda} (-)^{\alpha \cdot \tau} \exp \left\{ i\pi \alpha^2 \frac{\sigma}{1-2\sigma} \right\} \\ &= \left(i \frac{1-2\sigma}{\sigma} \right)^{\frac{\dim \Lambda}{2}} \sum_{\omega \in \Lambda^*} \exp \left\{ -i\pi \left(\omega - \frac{1}{2}\tau \right)^2 \frac{1-2\sigma}{\sigma} \right\}, \end{aligned}$$

using (B2). Provided Λ^* is an ‘integer’ lattice, (in the sense that each vector in Λ^* has

an integer squared length), we notice

$$e^{2i\pi(\omega - \frac{1}{2}\tau)^2} = e^{2i\pi(\omega^2 - \omega\tau + \frac{1}{4}\tau^2)} = (-)^{\frac{1}{2}\tau^2},$$

and we can use equation (B2) again to get

$$\begin{aligned} \Gamma(q', \tau) &= \left(i \frac{1-2\sigma}{\sigma}\right)^{\frac{\dim \Lambda}{2}} (-)^{\frac{1}{2}\tau^2} (-i\sigma)^{\frac{\dim \Lambda}{2}} \sum_{\alpha \in \Lambda} (-)^{\alpha \cdot \tau} \exp \{i\pi \alpha^2 \sigma\} \\ &= (1-2\sigma)^{\frac{\dim \Lambda}{2}} (-)^{\frac{1}{2}\tau^2} \Gamma(q, \tau) \\ &= \left(\frac{-\ln q}{i\pi}\right)^{\frac{\dim \Lambda}{2}} (-)^{\frac{1}{2}\tau^2} \Gamma(q, \tau). \end{aligned} \tag{B4}$$

This is precisely equivalent to (3.21). It is interesting that the argument seems to be perfectly good even if Λ is not self-dual. Both the Fierz rearrangement and the modular transformation require two stages, the dual lattice only occurring at an intermediate step.

APPENDIX C

Here, we should like to gather some information about the twisted algebras associated with the triple twist, σ , satisfying $\sigma^2 + \sigma + 1 = 0$. We pay particular attention to $so(8)$ and E_6 .

For $so(8)$ we can represent the twist as an action on the weights taking $V \rightarrow S \rightarrow \bar{S}$. Thus, setting

$$\sigma = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \end{pmatrix} \tag{C1}$$

we can check easily that $\sigma^3 = 1$, and that σ has no real eigenvectors. Notice, this is not the automorphism considered in ref[57]. However, up to a choice of basis in $\mathbf{R}^4$ all cubic automorphisms are either of the above type, or that considered in ref[57], (with eigenvalues $1, 1, \omega$ and ω^2). Note, however, there are three cubic automorphisms, up to conjugation, within the group of lattice transformations fixing the origin, one of which is (C1) (and outer), the other two like that of ref[57] with one inner and one outer [45].

In order to set up the algebra corresponding to the vertex operators built from string fields twisted by M , we shall need to identify with each root a matrix factor whose main task is to ensure the algebra is that of commutators by removing certain phase factors. Thus, associated to each root there is a matrix factor $C_T(\alpha)$ satisfying

$$C_T(\alpha)C_T(\beta) = \omega^{-\alpha \cdot (1-\sigma)\beta} C_T(\beta)C_T(\alpha), \quad (C2)$$

where ω is a complex cube root of unity, $e^{\frac{2\pi i}{3}}$. It is useful to examine the phase factors for the simple roots and to note,

$$\alpha_i \cdot (1-\sigma)\alpha_j = \begin{pmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \text{ mod } 3, \quad (C3)$$

where it is the fourth column that refers to the centre point on the Dynkin diagram in our conventions. Notice, for a double twist the analogous matrix to (C3) is the Cartan matrix mod 2. That is why, in that case, we can attach the matrix factors so neatly to the Dynkin diagram, as outlined in appendix A. Here, the Dynkin diagram is not so useful. However, it is easy to convince oneself that the eight root orbits under M may be labelled by the three outer points of the Dynkin diagram and the extended point ψ , together with their negatives. In other words, we can summarize (C3) in a Dynkin-like way as follows:

where the arrows denote the signs in (C3). With our convention, the centre point is on the same orbit as α_2 . Hence it is omitted. The phase factors (C2) may now be mimicked by assigning special matrices to each vertex of (C4):

where

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and

$$gh = \omega hg. \quad (C6)$$

Naturally, this elementary argument is in complete agreement with a proposition of Lepowsky [30]. The dimension of the factor matrices is given in this case by the formula,

$$\dim C_T = \sqrt{\dim(\Lambda/R)}, \quad (C7)$$

where the lattice R is a sublattice of the root lattice Λ and defined by taking the set of vectors α in Λ for which the phase factors, (C2), are unity for any choice of β . More succinctly, R is defined by

$$R.(1 - \sigma)\Lambda = 0 \pmod{3}. \quad (C8)$$

A quick computation reveals the coset Λ/R to have dimension nine. A more detailed argument on the twisted string vertices confirms the vacuum to be a $\underline{3}$ (or $\underline{3}^*$) of $su(3)$ (the subalgebra invariant under σ), the next excited state to be a $\underline{6}^*$ (or $\underline{6}$), and so on. Actually, comparing the weights of the first two excited states with respect to a commuting set of vertex operators is enough to pin down the dimension of the vacuum. This latter argument generalises the remarks in ref[58].

We can discuss E_6 in terms of $so(8)$. The roots of E_6 are conveniently summarised as,

$$\begin{aligned} &(\alpha, 0) \quad (S, \pm\alpha_2) \\ &(V, \pm\alpha_1) \quad (\bar{S}, \pm\alpha_3), \quad \alpha_3 = -(\alpha_1 + \alpha_2). \end{aligned} \quad (C9)$$

Here, α is any root of $so(8)$; V , S and $\bar{S}$ are any of the vector or two kinds of spinor weights, respectively (eight of each), and α_1 , α_2 , α_3 are the roots of $su(3)$ scaled to have unit length. The automorphism we discuss is σ (C1) on the four dimensional subspace in which the roots of $so(8)$ lie and a rotation through 120° in the two dimensional subspace in which α_i , $i = 1, 2, 3$, lie. The vertex operator algebra is quite complicated and will not be reproduced here. Nevertheless, it is easy to see that to each $so(8)$ vector weight

V we may assign two orbits labelled by the sign of the accompanying scaled $su(3)$ root, and we have already labelled the orbits within the $so(8)$ root space, (C4). We need to know how to put these orbits together to manufacture the three $su(3)$ s invariant under the automorphism. It is important that the vertex operators corresponding from within each type of orbit to a given $su(3)$ generator actually *commute*. This means that if we pick a vector weight V , and let $V^\pm$ denote the corresponding E_6 roots, we need to pick a root of $so(8)$ orthogonal to V *and* its images under σ . Up to signs there is just one and we call it V^0 . The correspondance is

$$V : e_1, e_2, e_3, e_4 \simeq V^0 : \psi, 2, 1, 3,$$

for our choice of basis and conventions. In terms of these, the $su(3)$ s are just

$$\frac{1}{3} \left(V_0(V^0) + \omega^{\lambda_1} V_0(V^+) + \omega^{\lambda_2} V_0(V^-) \right) C_T(V^0), \quad (C10)$$

where the zero subscript indicates the zeroth moment of the vertex operator, and $C_T(V^0)$ is the matrix factors assigned in (C5). Moreover,

$$\lambda_1 + \lambda_2 = 0 \pmod{3}. \quad (C11)$$

Thus, (C10) defines three mutually commuting sets of eight generators. We note immediately that the vacuum is a singlet under two of the $su(3)$ s (since the zeroth moments of a twisted vertex operator give one on the Fock space vacuum), and a triplet under the other.

The matrix corresponding to (C3) for E_6 is

$$\alpha_i \cdot (1 - \sigma) \alpha_j = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad (C12)$$

where the third column represents the central node of the Dynkin diagram. Using (C12) we may check the dimension of Λ/R and find it to be nine once again. So, once more the elementary argument agrees, as expected of course, with the mathematical literature

[30]. A detailed comparison of the weights of the first two levels in the Fock space tells us that if the vacuum is a $(3, 1, 1)$, say, then the next excited state is a $(1, 3^*, 3^*)$, and so on. However, the manipulations with the vertex operators on the states are rather tedious and will be omitted.

It is interesting to notice that although we are discussing an apparently different situation to that of ref[57] we, nevertheless, need to consider eight special triplets of orbits, each assigned (in our case) to one of the $su(3)$ generators within the E_6 . The relationship between $V^\pm$ and V^0 needs to be elucidated and may turn out to be related to the octonion multiplication table.

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