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LATTICE QUANTUM CHROMODYNAMICS^{*)}

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A B S T R A C T

I give an introduction to the main aspects of lattice calculations in quantum chromodynamics. I review some of the most recent results in the pure gauge sector ($q\bar{q}$ potential, glueballs, finite temperature deconfining transition) and in the one including fermions (hadron spectroscopy, hadronic matrix elements of composite operators).

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INTRODUCTION

When the energies involved in hadronic processes are of the order of the hadron masses, quantum chromodynamics - the theory which is supposed to describe them - enters into a strong coupling regime where a perturbative expansion becomes inadequate. This can be seen from the behaviour of the running coupling constant (Fig. 1) which exceeds one below some energy scale E_0 of the order of one GeV.

Fig. 1 The behaviour of the running coupling constant in QCD.

Perturbation theory can be seen as a tool to perform difficult integrals, functional integrals, by which one can express all vacuum expectation values (vev) of quantum fields. For example, the vev of two scalar fields at positions x and 0 can be written as:

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$$\langle \phi(x) \phi(0) \rangle = \frac{\int d[\phi] \exp\{i/\hbar S(\phi)\} \cdot \phi(x) \phi(0)}{\int d[\phi] \exp\{i/\hbar S(\phi)\}} \quad (1)$$

This is the generalisation to quantum field theory of the Feynman path integral for ordinary quantum mechanics which provides the amplitude $K(b,a)$ for a particle to move from a point "a" to a point "b":

$$K(b,a) = \int_a^b d[x(t)] \exp\left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} d\tau L(\dot{x}(\tau), x(\tau)) \right\} \quad (2)$$

The fluctuations over the trajectories around the classical one in Eq. (2) correspond to fluctuations of the fields around the classical solution in Eq. (1) and in both cases are governed by the value of the action.

If the action is quadratic in the fields ϕ , i.e. $S(\phi)=S_2(\phi)$, everything is simple. One only gets two-point functions: in terms of Feynman diagrams, propagators and mass insertions. The theory is non-interacting. Instead, if $S(\phi)=S_2(\phi)+S_{int}(\phi)$, where the interacting part of the action $S_{int}(\phi)$ is at least trilinear in the fields, one is not able to put the result in a closed analytic form and is forced to use approximate methods to evaluate the functional integrals.

The perturbative method consists of developing the exponential of the interacting part of the action. It amounts to a development in the number of interactions and leads to Feynman diagrams whose complexity grows with the order of the expansion. However, it is justified only when the strength of the interaction is sufficiently small. When this is not the case one has a "non-perturbative" problem.

THE LATTICE FORMULATION

The lattice approach consists of a numerical evaluation of the functional integral¹. It needs a discretisation of the four-dimensional space-time by the introduction of a basic length scale, the lattice spacing "a". Points on the lattice, which will be imagined for simplicity at the vertices of hypercubes, are at a distance "a" apart.

If we define by N the number of points in each direction the lattice will be also characterised by its total volume $(Na)^4$. On the lattice we have therefore two cut-offs: a finite space resolution and a finite volume.

For standard integrals the sampling is done on a finite set of points. For functional integrals, the sampling is over a set of functions, i.e., - in the simple case of a scalar field - of specific values of the field at each lattice point. These are also called configurations. The sum over points of ordinary integrals is replaced by a sum over configurations.

The vev in Eq. (1) can be seen as a kind of statistical average of the product of fields $\phi(x)\phi(0)$ taken with the field distribution:

$$d[\phi] \exp\left\{ \frac{i}{\hbar} S(\phi) \right\} \quad (3)$$

However, the presence of the imaginary unit in the exponent forbids the identification of the expression in Eq. (3) with a probability distribution. This can be obviated by making an "Euclidean rotation", i.e. by going to an imaginary time formulation with the replacements:

$$t \rightarrow -it \equiv \tau_E$$

$$\rightarrow iS(\phi) = i \int dt d^3x \dots \rightarrow - \int d\tau_E d^3x \dots \equiv -S_E(\phi) \quad (4)$$

The field distribution can now be interpreted as a probability distribution:

$$d[\phi] \exp \{ -S_E(\phi) \} \quad (5)$$

and vev's can be obtained as averages over configurations distributed according to Eq. (5).

The "equilibrium" configurations are obtained as follows. One typically starts from an arbitrary configuration and then updates the system with an algorithm which, after some "thermalisation sweeps", leads to equilibrium. The application of the same algorithm from then on will produce equilibrium configurations. It is a process similar to the one of putting a system with an initial random temperature in contact with a thermal bath at some fixed temperature. After some thermalisation time - corresponding in our case to the "computer time, i.e. to the number of sweeps through the lattice - the system will reach the equilibrium. Starting from then, thermodynamic averages will have their values determined by a Boltzmann distribution with the temperature of the bath. I will briefly describe two types of algorithms. The first is called "Metropolis" after its author² and consists of the following steps.

For each lattice site, one has to update the value of the field by choosing between the original value ϕ (I am making the case of a scalar theory for simplicity) and a new candidate value, ϕ' , chosen so that the measure remains invariant under the change $\phi \rightarrow \phi'$. This is the case if ϕ' differs from ϕ by a random constant (for ordinary integrals dx is invariant for a translation of the variable). In order to choose between the two values (the "old" and the "new" one), one considers the corresponding change of the action :

$$\Delta S = S(\phi) - S(\phi') \quad (6)$$

If ΔS is positive, one replaces the old value with the new one.

If ΔS is negative, one compares the quantity $\exp(\Delta S)$ with a random number R between 0 and 1. If $\exp(\Delta S)$ is greater than R , one replaces the old value with the new, if not, one keeps the old value. Therefore, if ΔS is negative, the new value is accepted with a probability $\exp(\Delta S)$.

The above procedure is repeated for each lattice site and for the whole lattice several times (one performs several "sweeps"). It leads eventually to the equilibrium distribution because it respects the detailed balance: if $W(\phi \rightarrow \phi')$ represents the probability for passing from the value ϕ to the value ϕ' , then:

$$\frac{W(\phi \rightarrow \phi')}{W(\phi' \rightarrow \phi)} = \exp[-S(\phi') + S(\phi)] \quad (7)$$

The ratio of transition probabilities is the ratio of the corresponding equilibrium distributions. This is enough to reach equilibrium. A rough

argument can be given as follows. The change in the probability distribution at each update of the field is given by:

$$\delta P(\phi) = \sum_{\phi' \neq \phi} W(\phi' \rightarrow \phi) P(\phi') - \sum_{\phi' \neq \phi} W(\phi \rightarrow \phi') P(\phi) \quad (8)$$

At the equilibrium the distribution is stationary, i.e., $\delta P_{eq}(\phi) = 0$ which implies:

$$\frac{W(\phi' \rightarrow \phi)}{W(\phi \rightarrow \phi')} = \frac{P_{eq}(\phi)}{P_{eq}(\phi')} = e^{-S(\phi) + S(\phi')} \quad (9)$$

Notice that accepting the new value only for ΔS positive leads to the configuration which realizes the minimum of the action, i.e., to the solution of the classical equations of motion. Quantum mechanics instead sometimes also tolerates increases of the action.

The second updating procedure is based on a Langevin equation of the type³:

$$\frac{d\phi}{d\tau} = - \frac{\partial S}{\partial \phi} + \eta \quad (10)$$

where η is a Gaussian random variable. The above equation is the same regulating the Brownian motion of a particle subject to a force in a medium with a given viscosity. However, in this case the "time" τ acts like a fictitious fifth dimension. If ϕ satisfies the Langevin equation, it can be shown that the associated probability distribution $P(\phi, \tau)$ satisfies the Fokker-Planck equation:

$$\frac{\partial}{\partial \tau} P(\phi, \tau) = \frac{\partial}{\partial \phi} \left(\frac{\partial}{\partial \phi} P(\phi, \tau) + P(\phi, \tau) \frac{\partial S}{\partial \phi} \right) \quad (11)$$

The stationary solution is again obtained when $\delta P(\phi, \tau) = 0$ which implies the solution:

$$P = e^{-S(\phi)} \quad (12)$$

The correct equilibrium distribution is only obtained for a continuous "time" τ . However, this cannot be realized in practice since one has to use a discretised version of the Langevin equation. The time τ is then proportional to the number of iterations that one performs on the equation, the "computer time", and it is a discrete variable. This introduces an error in the final distribution that one has to correct for by suitable methods.

In general there are several ways to lead the fields to their equilibrium distributions; each of them has advantages and disadvantages. The Metropolis has typically a reasonably fast convergence but needs a site-by-site updating. The Langevin is rather slow - the "computer time" steps must produce a very slow evolution of the system to imitate a continuous time evolution - but it has the advantage that the algorithm can be run in parallel on all lattice sites, since it is not based on the detailed balance.

THE CONTINUUM LIMIT

The theory on the lattice becomes the continuum theory only in the limit where the lattice spacing goes to zero. How can one take such a

limit? This is not straightforward, given that the computer code does not contain an instruction of the type:

$$A = \text{ONE_FERMI} \quad (13)$$

It is obvious: on the computer, we only have pure numbers and everything comes in lattice spacing units.

The essential point is to realize that "a" acts as an ultra-violet cut-off in momentum space: one cannot define wavelengths smaller than the lattice spacing. The lattice can be seen as a particular scheme for regularizing the ultra-violet divergences arising in renormalisable theories. The standard procedure of perturbative calculations consists of eliminating the values of the original couplings (the bare couplings) and of the cut-off in favour of the renormalised couplings and of a reference energy scale where their values are related to experimental quantities.

On the lattice one keeps bare couplings and their explicit cut-off dependence. These two quantities are in fact related by the renormalisation group equation. For small values of the coupling, such a dependence can be obtained from a perturbative expansion. The result for QCD is given in Fig. 2: it shows that $g_{\text{BARE}}(a)$ and "a" are redundant parameters, the only physically relevant quantity being the dimensionful parameter Λ_{QCD} which labels the different curves. According to asymptotic freedom the coupling goes to zero at short distance ("a" $\rightarrow$ 0).

The relation between g_{BARE} and "a" provides a way to define the limit $a \rightarrow 0$: for QCD it means taking $g_{\text{BARE}} \rightarrow 0$.

The continuum limit can be described in an alternative language by considering the lattice as a four-dimensional statistical system. If we call ξ_L the correlation length in lattice units, i.e., the typical distance at which the fields on the lattice are correlated, and ξ_{phys} the same quantity in physical units, the following trivial relation holds:

$$\xi_L = \xi_{\text{phys}}/a \quad (14a)$$

Therefore, if we want to keep a finite correlation length in physical units when the lattice spacing goes to zero, we must have correspondingly $\xi_L \rightarrow \infty$. The continuum limit is obtained when the bare coupling approaches a critical value (0 for QCD) where a phase transition characterised by an infinite correlation length occurs.

The reliability of the lattice approximation can be judged by performing the scaling tests. They aim to test the relation between the bare coupling and the cut-off expressed by the curves in Fig. 2 which, for small values of the coupling, takes the asymptotic form:

$$a \cdot \Lambda = \exp \left\{ - \text{const.} / g_{\text{BARE}}^2(a) \right\} \quad (14b)$$

There are two types of scaling tests: the first consists of looking at the behaviour of a dimensionful quantity (dimensionless on the lattice) as a function of the bare coupling. From Eqs. (14) follows:

$$M_{\text{physical}} \cdot a \equiv M_{\text{lattice}} = \left[M_{\text{physical}} / \Lambda \right] e^{-\text{const.} / g_{\text{B}}^2(a)} =$$

$$= \left[M_{\text{physical}} / \Lambda \right] \cdot e^{-\text{const}' \cdot \beta} \quad (15)$$

M_{lattice} is a well-defined function of $\beta \equiv 6/g_{\text{BARE}}^2(a)$.

The second method relies on the real space renormalisation group technique, widely used in statistical mechanics⁴. A complete description is beyond the scope of these lectures: I will only introduce some basic concepts. The general idea is to study the behaviour of the system after a dilatation of the lattice spacing by some factor "b". This is realised by defining block variables obtained as some "averages" of the original site variables over a cluster of neighbouring lattice sites. The site variables are then integrated out by keeping the block ones fixed. If the clusters are at a distance "b*a" apart, the block variables are defined on a lattice with a correspondingly increased lattice spacing.

The interaction of the block variables will be different in general from the one for the site variables. This can be represented by a change in the coupling constants including the possibility that some couplings which were zero in the site action acquire a non-zero value for the blocked action. The change of the couplings under the blocking operation defines a flow diagram in the coupling constant space. Figure 3 represents a typical flow diagram for two coupling constants and it contains the main ingredients. The black point is a fixed point, the continuous line is called a critical line and the arrows on it indicate the flow of the couplings which are attracted toward the fixed point; the dashed line is the renormalised trajectory where the arrows flow away from the fixed point.

The fixed point defines the values of the parameters where a continuum limit exists for the lattice theory, for, if the lattice correlation length is very large, the integration of field fluctuations between the scales "a" and "b*a" will not affect the system, i.e., the coupling will not move under the renormalisation. The combination of couplings defining the critical line is called irrelevant: under repeated applications of the blocking transformation the couplings move following the arrows and end up at the fixed point independently of their initial values. Instead, the combination defining the renormalised trajectory is relevant: the continuum limit can only be reached by its careful tuning. In the case of QCD with only one coupling (β), one has the flow of Fig. 4 where β behaves as a relevant coupling with a fixed point at $\beta = \infty$.

The second way of performing a scaling test is to apply a blocking transformation and to determine the corresponding shift in β that one has to make in order to keep a reference physical quantity unchanged. This can be seen from Fig. 5 which shows again the dependence of $g_{\text{BARE}}(a)$ on "a" for a fixed value of Λ_{QCD} : by changing "a" to "2*a" the physics (i.e., Λ_{QCD}) stays constant if correspondingly the coupling is shifted by the quantity Δg . Notice that the flow agrees with the behaviour shown in Fig.4.

The two scaling tests are just different ways of testing the same law.

OUTLINE AND THE VARIABLES

The discussion can be divided into two main sections: the case of pure gauge fields (i.e., a world with only gluons) and the one including fermions.

The first will cover string tension estimates, glueballs and the finite temperature deconfinement transition. The second will treat hadron

Fig. 2 The dependence of the bare coupling constant squared upon the lattice spacing "a".

Fig. 3 A typical flow diagram in a two-coupling constant space with a fixed point (black), a critical line (continuous) and a renormalized trajectory (dashed).

Fig. 4 The flow with one coupling in QCD.

Fig. 5 The change of g_{BARE}^2 for a change of the cut-off from "a₀" to "2.a₀".

spectroscopy and hadronic matrix elements within the "quenched approximation", i.e., by neglecting fermion loops, and will present some results for the unquenched case.

Gauge variables are represented as 3×3 unitary matrices acting on the colour space and are defined on the links of the lattice: they "transmit" the colour information between adjacent sites and make possible a local gauge invariance. The relation with the familiar vector fields of continuum QCD is given by:

$$U_{\mu}^{AB}(x) \equiv \left\{ \exp \left[i g \hat{A}_{\mu}(x) \cdot a \right] \right\}^{AB} \quad (16)$$

i.e. the matrices represent the exponential of the line integral of the vector field on the link connecting two sites. Fermions are defined on the sites and are represented by complex spinors with colour and spin indices.

The action in terms of "links" (the gauge variables) is not unique: it must only satisfy the requirement of gauge invariance. It can be written in general as a superposition of the trace of the product of links around different closed loops. If $\{G(x)\}$ are a set of arbitrary 3×3 unitary matrices defining independent gauge rotations at each lattice site, the tranformation law for the links reads:

$$U_{\mu}^{AB}(x) \rightarrow \left\{ G(x) U_{\mu}(x) G^{\dagger}(x+\hat{\mu}) \right\}^{AB} \quad (17)$$

and it leaves the trace of the product of links on a closed loop invariant. There have been several proposals in the literature for the choice of the action; I will only discuss results obtained with the action originally proposed by Wilson¹, which consists of the simplest closed loop that one can form on a hypercubic lattice, i.e., the "plaquette", a square made out of four adjacent sites. Its explicit form is:

$$S_{\text{gauge}} = \sum_{\mu, \nu} \left\{ \text{const} + \beta \text{Real} \left\{ \text{Tr} [U_P] \right\}_{\mu\nu} \right\} \quad (18)$$

where the plaquette in the plane μ - ν is given by:

$$[U_P]_{\mu\nu} \equiv U_{\mu}(x) U_{\nu}(x+\hat{\mu}) U_{\mu}^{\dagger}(x+\hat{\nu}) U_{\nu}^{\dagger}(x) \quad (19)$$

To the leading order in "a" the above action reduces to the usual continuum action. (Here, on paper, we can take the "naïve" limit.)

The fermion action has to deal with the so-called "doubling" problem. A theorem states that for a local action it is impossible to have a chiral invariant theory⁵ on the lattice without having at the same time a doubling of fermion species. For example, the propagator of a chiral invariant theory⁶ is of the form:

$$\frac{1}{\sum_{\mu} \sin [p_{\mu} \cdot a] \gamma^{\mu}} \quad (20)$$

and it has poles at $p_{\mu} = 0$ and $p_{\mu} = \pi/a$ for each value of μ . It therefore describes a theory with sixteen massless fermions (the doubling phenomenon). The alternative is to deal with a lattice action which breaks explicitly chirality with terms which vanish with "a". Unwanted fermion species acquire a mass of order $1/a$ and decouple in the continuum

limit. This is the solution proposed by Wilson¹; the corresponding action is:

$$S_{\text{fermions}}^{\text{Wilson}} = \sum_x \left\{ -\bar{\psi}(x) \psi(x) + K \left(\sum_{\mu} \bar{\psi}(x) (1 - \gamma_{\mu}) U_{\mu}(x) \psi(x + \hat{\mu}) \right) + \text{h.c.} \right\}$$

K is the "hopping" parameter governing the value of the quark mass and $\{\psi(x)\}$ are the fermion fields defined on the sites. The presence of links connecting quark fields at different points preserves the gauge invariance which, for fermions, consists of the following transformation law:

$$\psi(x)_{\alpha}^A \rightarrow \sum_B G(x)^{AB} \psi(x)_{\alpha}^B \quad (22)$$

where A and α are colour and spin indices respectively.

THE STRING TENSION

The most intriguing property of QCD is the confinement of colour sources. In a pure gauge world this implies a potential between two static charges which increases without bound with their separation. A suitable parametrisation is one where it rises linearly with the distance according to:

$$V(R) \sim \sigma R \quad (23)$$

The constant σ is called the string tension. From the spacing of resonances with increasing angular momentum one can estimate its value: $\sqrt{\sigma} \approx 420$ MeV.

It can be calculated as follows. Imagine creating an infinitely heavy quark-antiquark pair at some point on the lattice, separating them up to a distance R, keeping them there for a time T and then annihilating them again at the point where they were originally created. The only trace of their passage is a colour "trail" along their world line (i.e., their path on the lattice) given by the exponential of the line integral along the path of the colour vector field. This is just the trace of the product of link matrices around the "Wilson loop" consisting of the rectangle of sites of length R and T.

The expectation value of Wilson loops is then a measure of the time evolution of a $q\bar{q}$ pair at distance R. This is proportional to:

$$e^{-\frac{i}{\hbar} E t} \Rightarrow e^{+\frac{i}{\hbar} \sigma R t} \quad (24)$$

where the last equality follows because the kinetic energy does not contribute for infinitely heavy quarks. The imaginary unit in the exponential disappears after the Euclidean rotation and the expression in Eq. (24) becomes:

$$e^{-\sigma R t} \quad (25)$$

i.e., the expectation value of Wilson loops is proportional to a term decreasing exponentially with the area of the loop. The slope of the exponential is the string tension in lattice units. From the self-energy interactions of quarks, one also expects a dependence in the exponential

upon the perimeter of the loop which can be cancelled by taking suitable ratios of loops of different sizes.

By examining Wilson loops for various values of R and for large T , one can extract the whole shape of the potential. Figure 6 reports the results obtained at $\beta = 6.2$ on a $16^3 \times 32$ lattice for the force as a function of the distance in lattice units⁷. The curve through the points can be parametrised as:

$$F = \frac{0.234}{r^2} + \frac{0.036}{a^2} \quad (26)$$

i.e., as a superposition of a confining and of a Coulomb part.

Are these results a good approximation of what happens in the continuum theory? You should have learned by now that in order to answer this question, one has to perform scaling tests.

Figure 7a gives $\sigma \cdot a^2$ as a function of β . The straight line represents the expected asymptotic scaling law of the type in Eq. (15). At $\beta \sim 5.6 \div 5.7$ and beyond $\beta \sim 6.4$ the scaling is badly violated. This becomes more evident from the plot in Fig. 7b which shows $\sigma/\Lambda_{\text{lattice}}^2$ ^{*} as a function of β .

The discrepancy at low β is easy to understand in terms of a too coarse lattice spacing: the holes of our space time are too big. In order to understand why the results are bad at high β it is useful to look in Fig. 8 at the force divided by σ as a function of the distance multiplied by $\sqrt{\sigma}$ (the string tension has the dimensions of a mass squared). The units are therefore physical units: if one replaces $\sqrt{\sigma}$ with its physical value one can obtain the distance in Fermi. The figure shows that the points at the largest value of β explore the potential only up to a distance of about 0.4 Fermi, where the Coulomb contribution is still relevant. This leads to an overestimate of the string tension and explains why the highest β point violates the scaling. It is worth going to high values of β where the lattice spacing is small only if the total number of lattice points can be correspondingly increased to keep the total size big enough in physical units.

An alternative way of extracting the string tension is by measuring the correlation of "Polyakov loops"⁸. They are defined by:

$$\Omega(\vec{x}) \equiv \frac{1}{3} \text{Tr} \left\{ \prod_{t_i} U_4(\vec{x}, t_i) \right\} \quad (27)$$

i.e., as the trace of the product of links in the time direction along a loop wrapping around the lattice. This quantity is gauge invariant if periodic boundary conditions are used - as is normally the case - where opposite boundary points are identified, giving to the lattice the topology of a torus. The static potential can be obtained from the correlation of two such loops:

$$\langle \Omega(\vec{x}) \Omega(\vec{y}) \rangle = e^{-N_\tau V_{q\bar{q}}(|\vec{x} - \vec{y}|)} \quad (28)$$

The comparison given in Fig. 9 of the values for the string tension obtained from Wilson and from Polyakov loops allows us to estimate the

^{*}) Λ depends in general upon the regularisation scheme used; the one obtained on the lattice is called Λ_{lattice} .

Fig. 6 The force between static quarks as a function of the distance versus a linear plus Coulombic fit.

Fig. 7 a) The string tension in lattice units as a function of β .
b) The string tension in physical (Λ_{LATTICE}) units as a function of β .

Fig. 8 The force as a function of the distance in physical units.

Fig. 9 A comparison of string tension determinations with Polyakov loops (solid curves) and with Wilson loops (open circles).

amount of "systematic" error involved. Beyond $\beta \sim 6.0$ the results start to coincide.

The expectation value of a single Polyakov loop is used as a monitor of finite size effects. In the infinite volume limit its value is zero, and a non-zero result indicates the presence of spurious correlations produced by the finite total lattice size. In physical units this happens when the lattice becomes smaller than about one Fermi in each direction. The shadowed region in Fig.10 defines the values of β and N (the number of lattice points per direction) where finite size effects are important⁹. Since the lattice spacing decreases exponentially with β , N has to increase exponentially to keep the physical dimensions constant.

Renormalisation group tests confirm that beyond $\beta \sim 6.0 \div 6.2$ scaling holds. Fig.11 shows the shift in β after a $\sqrt{3}$ renormalisation factor¹⁰: the asymptotic scaling prediction is represented by the dashed line. For a comparison with the scaling tests of the string tension what matters is the value of the renormalised constant, i.e. $\beta - \Delta\beta$. Scaling starts to show up at $\beta \sim 6.5$, i.e. $\beta - \Delta\beta \sim 6.0$, in agreement with the previous tests. The value of the lattice spacing in physical units can be determined from the value of the string tension measured at $\beta = 6.2$ - where scaling is visible. It is

$$a_{\beta=6.2} \approx (2.1 \text{ GeV})^{-1} \approx 0.1 \text{ Fermi} \quad (29)$$

GLUEBALLS

Gluons have a colour charge and can tie together to form bound states. The method used to estimate their masses is the following. One looks at the correlation of two operators at different positions in space time with the quantum numbers of the bound state to be analysed. The mass is extracted from its behaviour as a function of the distance between the operators. The corresponding equations are:

$$\begin{aligned} \int d^3x \langle 0 | O(x) O^\dagger(0) | 0 \rangle &= \\ &= \sum_n \int d^3x \langle 0 | O(x) | n \rangle \langle n | O^\dagger(0) | 0 \rangle = \\ &= \sum_n \int d^3x e^{i p_n x} C_n \rightarrow \sum_n e^{-E_n t} C_n \end{aligned} \quad (30)$$

where

$$C_n \equiv |\langle 0 | O(0) | n \rangle|^2 \quad (31)$$

The states $|n\rangle$ are a complete set with the operator's quantum numbers. The periodicity of the lattice changes the exponentials in Eq. (30) into hyperbolic cosines (for bosons):

$$\sum_n \left\{ \cosh \left[\left(t - \frac{T}{2} \right) E_n \right] \right\} \cdot C_n \quad (32)$$

For large times the lowest energy eigenstate dominates and from a fit of the correlation one can extract the value of the slope of the exponential, i.e., of the mass.

For glueballs, the correlations die very fast: at $\beta \approx 6.0$, after a few lattice spacings the statistical fluctuations are bigger than the signal. A fit from only a few lattice spacings leads in general to an overestimate of the lowest mass because of the non-negligible

Fig. 10 Finite size effects in the β - N plane. For reference, at $\beta = 6$ is reported the approximate value of the lattice size in physical units.

Fig. 11 Renormalization group scaling test for a change from "a" to " $\sqrt{3}a$ ".

Fig. 12 Glueball correlation with the source method.

contamination of the higher mass states contributing to the sum in Eq. (30). The "source" method allows us to measure a signal up to seven or eight lattice spacings at the same value of β^{11} . It is based on the idea of measuring the response to a perturbation in the field configuration in a given region of the space-time as a function of the separation from the perturbed region. The probe used to measure the perturbation determines the quantum numbers of the states propagating the response. In practice, one "freezes" the value of the links in a given region, for example on a time slice of the lattice, by setting all of them to one (the "source") and studies the quantity:

$$C_J \equiv \int d^3x \langle 0 | O(x) | J \rangle \quad \text{J = source at } t=0 \quad (33)$$

which measures the response to the perturbation through the operator $O(x)$. The signal survives to larger distances because the source does not fluctuate. However, in this case the coefficients C_n are not positively defined:

$$C_n(J) \equiv \langle 0 | O(0) | n \rangle \langle n | J | 0 \rangle \quad (34)$$

and a determination of the glueball masses from a finite number of lattice spacings does not represent an upper bound to the true value anymore.

Figure 12 shows the results obtained with the source method¹². The full line is the fit made with a single hyperbolic cosine and it appears to reproduce the "data" for $t > 4$.

The source method has recently been further improved. The general idea can be understood from a simple example of ordinary quantum mechanics. The Feynman amplitude $K(x_b, T; x_a, 0)$ for the propagation of a particle from x_a to x_b in the time T can be written in terms of the eigenfunctions of the Hamiltonian $\Phi_n(x)$ as:

$$K(x_b, T; x_a, 0) = \sum_{n=0}^{\infty} e^{-E_n T} \Phi_n(x_b) \Phi_n^*(x_a) \quad (35)$$

The similarity between Eq. (35) and Eq. (30) is apparent. One can project out the lowest energy state by multiplying by the appropriate eigenfunction and integrating over the x 's:

$$\int dx_1 dx_2 \psi_2^*(x_2) K(x_2, T; x_1, 0) \psi_1(x_1) \quad (36)$$

with $\psi_{1,2} \equiv \Phi_0$

i.e., by smearing the original Feynman amplitude.

The case in Eq. (35) can be recovered by using a point-like wave function:

$$\begin{aligned} \psi_2(x) &= \delta(x - x_b) \\ \psi_1(x) &= \delta(x - x_a) \end{aligned} \quad (37)$$

The example shows that point-like wave functions couple to all states while a more efficient projection can be realised by using "smeared" functions, the best being the appropriate eigenfunctions. One can therefore expect that glueball estimates can improve by using "smeared" operators.

The simplest operator for the 0^{++} state is a sum of plaquettes with a common corner and different orientations and is a "point-like" operator, i.e., formed out of fields at neighbouring points. The "APE

collaboration" has instead used a class of operators extending over an increasing number of lattice points¹³. The dependence of the results upon the operator's size allows one to minimize the influence of the higher excited and leads to the results in Table 1. Finite lattice size effects are negligible for lattices with more than 12+14 points per direction. In physical units, the results can be summarised as follows:

$$\begin{aligned} m(0^{++}) &\sim (1.4 \pm .15) \text{ GeV} \\ m(2^{++}) &\sim m(0^{++}) \end{aligned} \quad (38)$$

FINITE TEMPERATURE DECONFINEMENT TRANSITION

It was already suggested a long time ago that at sufficiently high temperature, hadrons melt into a plasma of quarks and gluons. The corresponding phase transition is called the deconfinement transition. The quantum averages of a system at finite temperature can be calculated by a simple modification of the averages at zero temperature: the time direction has periodic boundary conditions and a finite extent equal to the inverse of the temperature. On the lattice, the smaller extent of the time direction is realised by using asymmetric lattices with a direction much shorter than the others. Another possibility is to take a symmetric lattice but asymmetric couplings, i.e., a value of β for the plaquettes containing a link in the time direction different from that for the plaquettes containing only links in the space directions. Since β determines the value of the lattice spacing, this leads to asymmetric lattices in physical units.

Most of the groups use the first method where the temperature in lattice units is simply obtained as the inverse of the number of lattice points in the time direction. The occurrence of the phase transition can be monitored by an "order parameter" which is non-zero in the high temperature phase and zero otherwise. This is the case for the expectation value of a single Polyakov loop. We already know that the expectation value of two loops can be used to extract the static quark-antiquark potential: imagine separating the pair more and more until they are at an infinite distance. The correlation of the two loops will be then non-zero only if the quarks are not confined, otherwise the energy required for this separation will be infinite and, according to Eq. (28), the correlation zero. The single loop expectation value can be considered as the limit of the two loops one when the separation goes to infinity. Figures 13 and 14 show the behaviour of L as a function of β for different lattice sizes. By varying β , one varies the lattice units until $1/N_\tau$ in physical units crosses the value of the deconfining temperature. The τ transition looks discontinuous, i.e., of first order. This is confirmed by the co-existence at the transition temperature of the two phases. The series of plots in Fig. 15 shows the value of L averaged over contiguous clusters of iterations for two independent lattices with different starting configurations: a "cold" one where all the links are the unit matrix and a "hot" one where they take random values in the group space. Here hot is synonymous with disorder and cold with order. An ordered configuration has always $L \neq 0$ and a disordered one $L \approx 0$ (to be exactly zero one would need an infinite volume to average on). The evolution of the two systems depends upon the value of the temperature. Away from the critical temperature, the system is in the wrong phase and soon or later collapses in the right phase while at the critical temperature each phase can survive indefinitely. In fact the coexistence

Fig. 13 The dependence of $\langle |L| \rangle$ upon β on a $8^3.2$ lattice.

Fig. 14 The dependence of $\langle |L| \rangle$ upon β on a $8^3.4$ lattice.

Fig. 15 The change of $\langle |L| \rangle$ with the iteration number; the crosses are points obtained from a cold start, the solid circles from a hot start. The physical temperature decreases going from the results at the top-left to those at the bottom-right.

Fig. 16 The scaling test for $T_c / \Lambda_{\text{lattice}}$ compared to the one for $\frac{1}{2} \sqrt{\sigma} / \Lambda_{\text{lattice}}$.

of the two phases is used as a precise way of determining the critical temperature¹⁴.

The temperature is a physical quantity and can be used for scaling tests. Figure 16 gives $T_c/\Lambda_{\text{lattice}}$ as a function of β . The crosses represent the value of $0.5\sqrt{\sigma}/\Lambda_{\text{lattice}}$. The two quantities approach the scaling in a similar way around $\beta \approx 6.0$.

From the value of the string tension one could determine the lattice spacing in physical units [Eq. (29)]. We can now also obtain the physical value of the transition temperature:

$$T_c \sim 250 \text{ MeV} \quad (39)$$

The value of the latent heat of the phase transition can be estimated of the order of 3 GeV/fm^3 .

QUENCHED FERMIONS

The quenched approximation consists of neglecting internal fermion loops. The physics justifications of the approximation are based on the "Zweig rule" and on the results of a large N_c (N_c = number of colours) expansion. According to the former, pair creation by gluons is inhibited. This is supported by the small width of resonances - like the Φ - for which kinematically favourable channels are accessible only if a quark-antiquark pair is produced.

Diagrams with fermion loops are suppressed by a factor $1/N_c$ per each loop and disappear in the large N_c limit. Technically, the inclusion of fermion loops is rather complicated and (computer) time-consuming.

As we have seen from Eqs. (21) and (22), the fermion action is quadratic and can be written in general as:

$$S_{\text{fermions}} = \sum \bar{\psi}(x) \Delta(U)_{xx'} \psi(x') \quad (40)$$

Fermion fields can be integrated out and the two-point function has the form:

$$\begin{aligned} \langle 0 | \bar{\psi}(x) \psi(0) | 0 \rangle &= \frac{\int dU d\bar{\psi} d\psi e^{-[S_g + S_f]} \bar{\psi}(x) \psi(0)}{\int dU e^{-[S_g + S_f^{\text{eff}}]} [\Delta(U)]^{-1}} = 1 \end{aligned} \quad (41)$$

The matrix $\Delta(U)$ acts on the standard space-time co-ordinates and in spin and colour space. The presence of fermion loops produces an extra term in the gauge action:

$$S_g^{\text{eff}} = \frac{1}{2} \text{Tr} \{ \ln [\Delta(U)] \} \quad (42)$$

The quenched approximation amounts to setting¹⁵:

$$S_g^{\text{eff}} = 0 \quad (43)$$

The calculation of the hadron spectrum employs the same technique described for the estimate of glueball masses. This time the correlation is made with operators containing fermion fields. For example, for the

pion, one can use $O(x) = \bar{\Psi}(x)\gamma_5\Psi(x)$ and for the rho with polarisation α $O_\alpha(x) = \bar{\Psi}(x)\gamma_\alpha\Psi(x)$.
The "hadron" correlation between the points 0 and x in the quenched approximation is given by:

$$D_h(x,0) = \frac{1}{Z} \int dV d\bar{\psi} d\psi O_h(x) O_h^\dagger(0) \exp[-S_g - S_{\text{fermions}}]$$

(quenched) $\longrightarrow \frac{1}{Z} \int dV \exp[-S_g] \cdot \langle O_h(x) O_h^\dagger(0) \rangle_{\text{at } V \text{ fixed}} \quad (44)$

A corresponding diagram is given in Fig. 17. The full lines represent the propagation of the fermion-antifermion pair correlating the operators between the points x and 0.

The hardest part of the calculation is the inversion of the matrix Δ needed to evaluate the fermion propagator. This is done by relaxation methods which become more and more unstable as the quark mass gets small. The fermion propagator is calculated for a set of uncorrelated gauge configurations: for each of them fermion and anti-fermion propagators are tied together with the operator quantum numbers at the position 0 and x. The result is then summed over the configurations to provide the correlation in Eq. (44).

Figures 18a and b report the correlations of operators having the quantum numbers of the pion, rho, proton and delta as a function of the time in lattice units¹⁶. The full lines are the fit with one hyperbolic cosine (see the equations for the glueball case). The two figures differ by the quark mass value which is smaller in Fig. 18b. As a consequence, in the latter the correlation decreases less rapidly, especially for the pion, whose mass squared is expected to go to zero linearly with the quark mass. The hadron masses extracted for different quark masses are given in Fig. 19¹⁷: the quark mass value in the Wilson formulation is defined to be zero when the pion mass is zero. This is a consequence of the explicit breaking of the chirality with the Wilson fermion action which makes the renormalised mass not proportional to the bare mass.

The hadron and the quark masses (up and down quarks are taken to be degenerate) are both in lattice spacing units: in order to fix the scales in physical units, we can use as input the values of, say, the rho and of the pion mass. This leads to Fig. 20 and to an inverse lattice spacing at $\beta = 6.3$ equal to ~ 2.9 GeV. The full lines are rough fits to the dependence of the hadron mass from the quark mass which are needed to extrapolate the values obtained to the interesting region of very small quark masses of the order of 5 MeV. Instabilities in the inversion of the quark propagator make that region inaccessible at present.

The proton mass comes out typically too high. This is a general disease of quenched calculations of the hadron spectrum as can be seen from Fig. 21⁹ which reports the ratio M_p/M_π as a function of M_π/M_ρ . The results are still very close to the naive quark-parton model value based on quark counting. The reasons for this lack of success are to be found partly in the systematic error still present in the proton mass determination due to the finite lattice size and to the difficulty in isolating the lowest mass state contribution to the correlation, partly in the error introduced by the extrapolation to low quark masses and partly in the quenched approximation.

The lattice spacing in physical units determined from a fit to the pion and the rho in the quenched approximation is confronted with its expected scaling law in Fig. 22. The points indicate:

Fig. 17 A diagram contributing to the correlation of two q-q operators at positions "0" and "X" in the quenched approximation.

Fig. 18 a) Propagators for the π , ρ , p and Δ channels. For the baryons, the points on the right-hand side show the $p(\Delta)$ propagator and on the left-hand side those for the opposite parity partner, both averaged over upper and lower Dirac components. The curves are the fit results.

Fig. 18 b) The same as Fig. 18a for a lower value of the quark mass.

Fig. 19 Hadron masses versus the quark mass ($m_{\text{up}} = m_{\text{down}}$) in lattice units.

Fig. 20 The same as Fig. 19 in physical units.

Fig. 21 The ratios m_N/m_ρ and m_π/m_ρ are confronted with the naïve quark parton model result (NPM)^o and the experimental value (EXP).

Fig. 22 Scaling law of the lattice spacing extracted from π - ρ masses in the quenched approximation as a function of β .

$$\Lambda_{\text{lattice}} = 4.7 \text{ MeV} \quad (45)$$

This value is in good agreement with the one derived from the string tension. By using the relation between Λ_{lattice} and $\Lambda_{\overline{\text{MS}}}^{18}$:

$$\Lambda_{\overline{\text{MS}}} = 28.9 \Lambda_{\text{lattice}} \quad (46)$$

one obtains a value for $\Lambda_{\overline{\text{MS}}}$ also consistent with the one extracted from hadron processes at large momentum transfer where perturbative QCD is applicable.

This is a success.

The explicit breaking of chirality in the Wilson fermion formulation still allows the definition of a partially conserved axial - flavour non-singlet - current provided one introduces suitable counterterms for the mass and the current renormalisation constants. Their determination is a non-perturbative problem. The values can be obtained by enforcing standard current algebra relations among matrix elements of product of currents. This approach has been originally advocated in Ref. 19). For example, if one defines the following non-local axial current:

$$A_\mu \equiv \frac{1}{2} \left\{ \bar{u}(x) \gamma_\mu \gamma_5 U_\mu(x) d(x+\mu) + \bar{u}(x+\mu) \gamma_\mu \gamma_5 U_\mu^\dagger(x) d(x) \right\} \quad (47)$$

and the local pseudoscalar density:

$$P_5 \equiv \bar{u}(x) \gamma_5 d(x) \quad (48)$$

the matrix element of the equation of partial axial current conservation has the form:

$$\langle \alpha | Z_A \nabla^\mu A_\mu - 2m P_5 | \beta \rangle = 0 \quad (49)$$

where Z_A and M are two unknown non-perturbative quantities. Their ratio can be obtained by estimating the following ratio of matrix elements:

$$R_A(t) \equiv \frac{\sum_x \langle \nabla_t A_0(x,t) P_5^\dagger(0) \rangle}{\sum_x \langle P_5(x,t) P_5^\dagger(0) \rangle} = \frac{2m}{Z_A} + \mathcal{O}(a) \quad (50)$$

The result of a calculation on a $10^3 \cdot 20$ lattice at $\beta = 6.0$ is shown in Fig. 23²⁰: for large values of T the result becomes stable and allows an estimate of R_A . In order to get Z_A and M separately, one has to use current algebra relations involving three-point functions and other currents besides the ones of Eqs. (47) and (48).

The knowledge of the renormalisation constants provides the correct values for the rho and phi electromagnetic decay constants:

$$\frac{1}{f_\rho} = 0.38(7) [0.29 \text{ exp}]; \quad \frac{1}{f_\phi} = 0.30(4) [0.23 \text{ exp}] \quad (51)$$

and for quark masses in the lattice regularisation scheme:

$$\frac{1}{2} (m_u + m_d) = 3.3(6) \text{ MeV}$$

$$m_s = 80(15) \text{ MeV} \quad (52)$$

A solid understanding of the chiral properties of the Wilson action provides the tools to calculate the hadronic matrix elements of the weak Hamiltonian^{21,22}. According to the Wilson expansion, the isospin 3/2 contribution can be written as:

$$H_{3/2} = G_F \frac{\sin \theta_c \cos \theta_c}{2\sqrt{2}} C_4(\mu) O_4(\mu) \quad (53)$$

where $C_4(\mu)$ and $O_4(\mu)$ are a coefficient function and a four-fermion operator respectively, both renormalised at some scale μ . $SU(3) \times SU(3)$ chiral perturbation theory relates the matrix element of the $\Delta I = 3/2$, $\Delta S = -1$ operator (I =isospin, S =strangeness) to a $\Delta S = 2$ matrix element:

$$\langle \pi^+ \pi^0 | H_{3/2} | K^+ \rangle = G_F \frac{\sin \theta_c \cos \theta_c}{2\sqrt{2}} C_4(\mu) \frac{3}{4} \frac{m_K^2 - m_\pi^2}{F_\pi} \cdot \frac{\lambda_{LL}}{m_K^2} \quad (54)$$

where

$$\lambda_{LL} \equiv \langle K^0 | O_{LL}(\mu) | \bar{K}^0 \rangle$$

$$O_{LL} \equiv \bar{d} \gamma_\mu (1 - \gamma_5) s \bar{d} \gamma^\mu (1 - \gamma_5) s \quad (55)$$

The quantity λ_{LL}/m_K^2 can be obtained in the quenched approximation by evaluating diagrams of the type in Fig. 24 corresponding to the correlation:

$$\sum_{\vec{x}, \vec{y}, y_0} \langle P_5(\vec{x}, t) O_{LL}(\vec{y}, y_0) P_5(0) \rangle \quad (56)$$

The matrix element in Eq. (55) can be deduced from its time dependence.

The result for different values of the meson mass in lattice units is given in Fig. 25. The two marks on the abscissa indicate the values corresponding to the pion and to the kaon mass. The extrapolation of the curve to its chiral limit gives:

$$\frac{\lambda_{LL}}{m_K^2} \xrightarrow{\text{chiral limit}} \frac{\langle \pi^+ \pi^0 | H_{3/2} | K^+ \rangle}{m_K} = (8 \pm 3) \cdot 10^{-8} [3.7 \cdot 10^{-8} \text{ exp}] \quad (57)$$

which compares with the experimental value.

Similar techniques lead to the estimate of the "D/F" ratio, the Ξ -P mass splitting and of the " σ term"²³:

$$D/F = -0.4 \pm 0.2 [-0.3 \text{ exp}] \quad \sigma_{\text{term}} = (15 \pm 4) \text{ MeV}$$

$$M_\Xi - M_P = (210 \pm 60) \text{ MeV} [380 \text{ MeV exp}] \quad [26 \text{ MeV exp. quenched}]^{(58)}$$

The last two results are in clear disagreement with the experimental values: as in the case of the proton-rho mass ratio, one can invoke the uncertainty in the quark mass extrapolations used, but also a systematic error of the quenched approximation.

It is then time to remove it.

Fig. 23 $R_A(t)$ (see text).

Fig. 24 Insertion of a four-fermion operator in the "meson-meson" correlation function.

Fig. 25 The mesonic matrix element of the product of two local left-handed currents λ_{LL} as a function of the meson mass in lattice units.

INCLUDING FERMION LOOPS

The extra term in the gauge action due to the presence of fermion loops has a basic property which makes its inclusion difficult. While the standard plaquette term is "local", i.e., it involves only neighbouring links, the new term introduces in general the interaction among links which are far apart on the lattice. This is because fermion pairs, especially if very light, can propagate over large distances and introduce long-range link interactions. Updating algorithms like Metropolis are based on the variation of the action for a local change of a link variable. If the action is local, this implies a calculation involving neighbouring links only; this is not the case for the full effective action.

One uses approximation methods.

The first one is called the "pseudofermion method" and consists of considering the variation of the effective action for small changes of the links:

$$\frac{\delta S}{\delta U} \delta U \sim \frac{1}{\Delta} \frac{\delta \Delta}{\delta U} \delta U + O(\delta U^2) \quad (59)$$

The inversion of the fermion matrix is a hard task and is done by a trick which gives the name to the method. The following identity is used:

$$\left[\frac{1}{\Delta} \right]_{ij} = \langle \phi_i \phi_j \rangle = \frac{\int d[\phi] e^{-\phi \Delta \phi} \phi_i \phi_j}{\int \dots \cdot 1} \quad (60)$$

which relates the inverse to a two-point function of an auxiliary system of bosonic variables having the same quadratic action of the fermions (the "pseudofermions"). This method has two sources of error: higher-order terms in the variation of the action are neglected and the inverse of the matrix Δ is calculated only once for a lattice sweep. The last operation violates the detailed balance and leads to an approximate equilibrium distribution.

The second method is based on the Langevin equation²⁵. The necessary discretisation of the equation again leads only to an approximate equilibrium distribution. The error can be partially eliminated by repeating the calculation for different discretisation steps and by extrapolating the answer to the continuum case.

In any case, the updating time expands considerably with respect to the pure gauge case.

The effect of dynamical fermions on the string tension can be mostly reabsorbed by a shift in the value of β : the values obtained in the pure gauge case are similar to the ones including fermion loops obtained at a lower β value. Fermion loops propagating over large distances correlate the gluon fields and produce an effect similar to the one of increasing β . This reflects the change in the β function due to the vacuum polarisation of fermion pairs.

One also expects a screening of the colour force and a decrease of the value of the string tension at large distances when the energy accumulated in the string is equal to that required for the formation of a quark-antiquark pair. The potential becomes flatter, as can be seen in Fig. 26, obtained with dynamical fermions of about a hundred MeV²⁶.

The shift in the β value is confirmed by renormalisation group studies and by some preliminary hadron mass spectrum calculations. In the latter,

Fig. 26 Comparison between the quenched and the unquenched case of the potential (up to a constant) as a function of the distance. The horizontal axis reports the values of the energy accumulated in the string as a function of the distance in the unquenched case.

the quark mass values used are generally rather high and do not allow us to draw any conclusion about the influence of dynamical fermions.

The biggest impact of including fermion loops is on the finite temperature deconfining transition. The calculations done so far indicate that for quark masses of the order of or larger than the deconfining temperature T_c there are no effects. When $m_q \approx 0.4 \cdot T_c$ the phase transition weakens and becomes probably of second order. When the mass is very small, $m_q \approx 0.1 \cdot T_c$ or less, the transition appears to be of the first order again. The influence of the quark mass values and of the number of quark flavours on the order of the transition has not been clearly established.

Given the constraint on the number of lattice points used for dynamical fermion calculations, the existing results are typically obtained at a β value still away from the scaling region. Scaling tests of the deconfinement temperature have shown sizeable scaling violations with a pattern - reported in Table 2 - similar to the one obtained in the quenched case with the same lattice sizes²⁷.

A SHORT SUMMARY

For the pure gauge case, the results obtained for the string tension and for the deconfinement temperature approximately follow, at the highest β values, the scaling law expected from asymptotic freedom. Finite size effects are also under control. Glueball mass determinations are improving and the present results have been checked against finite volume problems, while convincing scaling tests are still lacking. Quenched fermions with the quark mass values achieved so far lead to a hadron spectroscopy with a wrong rho over proton mass ratio. The lattice spacing obtained from pion and rho mass determinations fits the expected scaling curve in β and agrees with the value extracted from the string tension and from the fits to high energy experiments. The technology for the evaluation of hadronic matrix elements is in progress and has already provided interesting numbers for the weak effective Hamiltonian.

Dynamical fermions are at a preliminary stage: screening effects in the quark-antiquark potential are visible and one finds an interesting dynamics of the deconfinement transition as a function of the quark mass. Most of the results simply amount to a redefinition of β of the analogous quenched calculations. They are obtained on lattice sizes which correspond to those used for quenched calculations about five years ago.

PERSPECTIVES

The goal is to reach a lattice having $2\div 3$ Fermi of extension in each direction and a space resolution (i.e., a lattice spacing) of say 0.03 Fermi. This implies a cut-off of about 6 GeV which seems reasonable for the study of a dynamics characterised by bound states with one GeV masses. This implies:

$$N = 60 \div 80 \quad (61)$$

or larger.

When will it be possible to achieve these sizes? Lacking a theory of computer performance evolution, we can look at a scaling law of past performances.

The logarithm of the typical lattice sizes used at a given time for quenched calculations as a function of the year in natural units more or less follows a straight line as shown in Fig. 27. The second horizontal scale represents the corresponding type of computer used and my personal estimate of its average speed in Megaflops (millions of floating operations per second). The third scale is the logarithm of the speed and shows that $\ln(N)$ is approximately linear in $\ln(\text{speed})$. This last relation is understandable: in most of the calculations the computer time required is proportional to the total volume of the lattice. Therefore, at a given time, the maximum lattice size will follow roughly the law:

$$\frac{N^4}{\text{CPU power}} \sim \text{constant} \quad (62)$$

This implies a linear relation among the logarithms with a slope which is close to the one of Fig. 27. Some corrections are needed to get the numbers of the figure, since the total time dedicated in a year to a single calculation also increased. Today, long calculations easily take of the order of a thousand cpu hours and the "unitarity limit" of a full year of CPU time in a year is not so far. This explains also why some collaborations moved to the construction of homemade "dedicated" computers, which, as in the APE case, are now starting to produce the first results.

The goal size for quenched calculations will be presumably reached around 90'÷91'.

The prediction for unquenched calculations is much harder, given the preliminary status of the art. I believe that by using the fastest computers one could reach today a size of about 12 lattice sizes. This amounts to a shift downwards of the straight line of the quenched case by a constant. An optimist would interpret the extra factor to be volume independent: the unquenched line would be the run parallel to the other. This is the lower full line in the figure. However, if the extra factor is interpreted as a different volume dependence the line will be flatter. A size dependence like N^5 instead of the N^4 of the quenched case agrees with the possibility today of running quenched lattices of 24 points and unquenched of 12 and produces the dashed curve in Fig. 27.

The corresponding date is around 96'.

How powerful must be the computers to sustain such calculations? A speed of about 30 Gigaflops for the quenched case and of 3 Teraflops for the full case. The memory must be adequate.

A three Teraflops computer means an impressive parallel architecture: three thousand parallel processors with a clock time of about one nanosecond each. This technology is not available but is conceivable.

The middle of the 1990's will be the era of tera-machines: TeV accelerators (SSC,LHC) and Teraflops computers.

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Fig. 27 A personal view of the logarithm of the lattice size (N = number of points/direction) as a function of:
 i) the year,
 ii) the type of computer and (in parentheses) its average speed,
 iii) the logarithm of the speed.
 For the unquenched case two possible extrapolations are presented.

N	m(0 ⁺⁺)	m(2 ⁺⁺)	m(0 ⁺⁺)/√σ
10	0.65(3)	0.6 (3)	3.0
12	0.76(4)	0.8 (2)	3.2
16	0.82(5)	1.00(15)	3.4

Table 1 - The values of 0⁺⁺ and 2⁺⁺ glueball masses for various lattice sizes. σ is the string tension. The numbers in parentheses are the statistical errors on the last significant digit.

	N _t	N _f = 4	N _f = 0
T _c /Λ _{lattice}	4	2.77	2.61
T _c /Λ _{lattice}	6	2.14	2.12

Table 2 - The ratio of the deconfining temperature over Λ_{lattice} as a function of the numbers of light flavours and of the lattice size in the time direction.

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