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INVARIANTS AND THE PATTERN OF QUARK MASSES AND MIXINGS

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A B S T R A C T

We suggest that the study of weak-basis invariants may play an important rôle in finding the essential features of the pattern of quark masses and mixings. In particular, we show that the most popular ansätze for the quark mass matrices can be translated into invariant conditions on those matrices. One of these invariant conditions is common to all the ansätze that we have analyzed.

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One of the most intriguing puzzles in particle physics is the pattern of quark masses and mixings. In the standard electroweak model [1], the Yukawa couplings which generate the up and down quark mass matrices M_u , M_d , are arbitrary and, as a result, the quark masses and mixings cannot be predicted. This very unsatisfactory feature of the standard model is clearly one of the major unsolved problems in particle physics.

In the past, various attempts have been made to obtain relations between quark masses and mixings [2] through the imposition of special ansätze for M_u , M_d . These ansätze are presented in a special weak basis where typically some of the matrix elements of M_u , M_d are either vanishing or related to each other in a simple way. As a result, one obtains [3] relations between quark masses and mixings. However, since the ansätze are presented in a special weak basis, it is difficult to infer what properties of the Yukawa couplings are implied by them.

In this letter, we suggest that the study of weak-basis invariants may provide some important clues for the search of the essential features of the observed pattern of quark masses and mixings. The use of weak-basis invariants has been particularly successful in the study of CP violation in the standard model [4,5] and beyond [6].

We shall restrict ourselves to the standard model with three generations and assume that M_u , M_d are Hermitian matrices. In the standard model, due to the non-existence of right-handed charged currents, this is not a restriction, since M_u , M_d can always be made Hermitian through redefinitions of the right-handed quark fields, which leave the charged current invariant. For our analysis, it is convenient to define a matrix A by:

$$M_d = p M_u + A \quad (1)$$

where

$$p = \frac{\text{tr } M_d}{\text{tr } M_u} \quad (2)$$

From Eqs. (1) and (2), it follows that

$$\text{tr } A = 0 \quad (3)$$

Although Eqs. (1) and (2) resemble Stech's formula [7], it should be noted that at this stage no constraints have been imposed on the mass matrices, Eq. (1) being just the definition of A . By construction, A is a traceless Hermitian matrix and therefore it has eight parameters. However, only six of them are physically meaningful. This can best be seen in the weak basis where M_u is diagonal and real.

Indeed, by making a change of weak basis, one can keep M_u diagonal, real, and change the phases of the non-diagonal elements of A . As a result, there is only one physically meaningful phase in A , namely $\arg(A_{12}A_{23}A_{31})$ which is, of course, invariant under the above weak-basis transformations. The six physical parameters of A correspond to the four parameters of the Kobayashi-Maskawa (KM) [8] mixing matrix, and to two down-quark masses, the third one being fixed by Eq. (2), once p is given.

Although Eqs. (1) and (2) do not imply a priori any constraint on M_u , M_d , the physical motivation for introducing [7] the matrix A was the idea that in some sense A is small. If one attempts to give a more precise meaning to the smallness of A , one is naturally led to consider the basic invariants of A . Since by construction $\text{tr} A = 0$, one is left with the other two coefficients of the characteristic equation, which we will designate by $\text{tr}^2 A$, $\det A$. One may easily express these two invariants in terms of quark masses and mixing angles. Let us first note that in the weak basis where M_u is diagonal, real, Eq. (1) can be written:

$$V d V^\dagger = p u + A' \quad (4)$$

where V is the KM mixing matrix, u, d stand for the diagonal up and down quark mass matrices, and A' denotes the matrix A in the new basis. Since A is a Hermitian traceless matrix, one has:

$$\text{tr}^2 A = -\frac{1}{2} \text{tr} (A^2) \quad (5)$$

Using Eqs. (4), (5) and the unitarity of V , one obtains:

$$\text{tr}^2 A' = \text{tr}^2 A = -\frac{1}{2} \text{tr} [(d - p u)^2] + p Q \quad (6)$$

with

$$Q = \sum_{i \neq j} u_i (d_i - d_j) |V_{ij}|^2 \quad (7)$$

where $|u_i|$, $|d_i|$ are the up and down quark masses. In the limit of zero mixing angles (i.e., $V=1$), Q vanishes. Therefore $\text{tr}^2 A$ has two contributions, one arising from the fact that the observed up and down quark mass eigenvalues are not proportional and another one resulting from having non-zero mixing angles. Similarly, one obtains from Eq. (4):

$$\det A = \frac{1}{3} \text{tr} A^3 = \frac{1}{3} \text{tr} d^3 - p \text{tr} (V d^2 V^\dagger u) + p^2 \text{tr} (V d V^\dagger u^2) \quad (8)$$

Using the unitarity of V one can write Eq. (8) as:

$$\det A = \frac{1}{3} \prod_{i=1}^3 (d_i - p u_i) + Q' \quad (9)$$

where

$$Q' = p \sum_{i \neq j} u_i (d_i^2 - d_j^2) |V_{ij}|^2 - p^2 \sum_{i \neq j} u_i^2 (d_i - d_j) |V_{ij}|^2 \quad (10)$$

We have again written $\det A$ as a sum of two terms, one which is independent of the mixing angles and another one which vanishes in the limit of zero mixing angles.

At this point, one may wonder whether any of the two invariants of A can vanish, without running into obvious conflict with the observed pattern of quark masses and mixings. We will show next that while $\text{tr}^2 A$ cannot be zero for realistic mixing matrices, $\det A$ may vanish. In fact, it will be seen in the sequel, that in the most popular ansätze for the quark mass matrices, $\det A$ vanishes. Since $\det A$ is expressed in terms of directly measured quantities through Eq. (9), one has a powerful test of the most popular ansätze for quark mass matrices. In order to see why $\text{tr}^2 A$ cannot vanish in realistic schemes, note that since A is a traceless Hermitian matrix, its eigenvalues are in general given by $(a_1, a_2, -(a_1 + a_2))$, with a_i real, but not necessarily positive. One then obtains:

$$\text{tr}^2 A = - \left[\frac{3}{4} (a_1 + a_2)^2 + \frac{1}{4} (a_1 - a_2)^2 \right] \quad (11)$$

From Eq. (11), it follows that in order to have $\text{tr}^2 A = 0$, the matrix A itself would have to vanish. This would be in obvious conflict with experiment since non-zero mixing angles require the commutator $[M_u, A]$ not to vanish. Note, however, that $\text{tr}^2 A$ may be, after proper normalization, a good invariant measure of the smallness of A .

Next we give a systematic and basis independent description of the various ansätze for the quark mass matrices, translating them into invariant restrictions on the quark mass matrices. These ansätze do not provide an explanation for the hierarchical structure of the quark masses, which can be expressed in terms of basic invariants as:

$$[\text{tr} S_\alpha]^{1/2} \gg [\text{tr}^2 S_\alpha]^{1/4} \gg [\det S_\alpha \cdot \text{tr} S_\alpha]^{1/8} \quad (12)$$

where $S_\alpha = M_\alpha M_\alpha^\dagger$ ($\alpha = u, d$), and $\text{tr} S_\alpha$, $\text{tr}^2 S_\alpha$, $\det S_\alpha$ denote the basic invariants of S_α

(i.e., the coefficients of its characteristic equation). Instead, the measured values of the quark masses are used as input, together with the ansätze, in order to constrain the parameters of the KM matrix.

STECH ANSATZ

The scheme [7] proposed by Stech requires that in the weak basis where M_u is diagonal, A be purely imaginary. Therefore, in this basis, the diagonal elements of A vanish and the product $A_{12}A_{23}A_{31}$ is purely imaginary. The matrix A which, as we have seen, has in general six parameters, is left with only three free parameters, namely $|A_{12}|$, $|A_{23}|$, $|A_{31}|$. This shows that Stech's ansatz implies three independent constraints. We have found that the three constraints can be translated into the following three invariant conditions:

$$\text{tr}(M_u A) = 0 \quad (13a)$$

$$\text{tr}(M_u^2 A) = 0 \quad (13b)$$

$$\det A = 0 \quad (13c)$$

It is obvious that the conditions of Eqs. (13) are necessary in order for A to satisfy Stech's ansatz. It can be easily verified that they are also sufficient. Indeed, starting from an arbitrary traceless Hermitian matrix A , and assuming non-degenerate up-quark masses, Eqs. (13a,b) constrain the diagonal elements of A to vanish in the weak basis where M_u is diagonal and real. Then Eq. (13c) constrains $(A_{12}A_{23}A_{31})$ to be pure imaginary in the same basis. As a result, the matrix A is constrained by Eqs. (13) to assume, in the basis where M_u is diagonal, real, the form suggested by Stech:

$$A_S = \begin{pmatrix} 0 & A_{12} & A_{13} \\ A_{12}^* & 0 & A_{23} \\ A_{13}^* & A_{23}^* & 0 \end{pmatrix} \quad (14)$$

with $\text{Re}(A_{12}A_{23}A_{31}) = 0$. We have thus shown that Eqs. (13) are a set of necessary and sufficient conditions for the quark mass matrices to satisfy Stech's ansatz. Each one of these invariant conditions can be written in terms of directly measurable quantities. Using Eq. (4), together with unitarity of the KM matrix, one can write Eqs. (13a,b,c) respectively as:

$$\sum_{i \neq j} (d_i - d_j) u_i |V_{ij}|^2 = \sum_{i \neq j} u_i (d_i - p u_i) \quad (15a)$$

$$\sum_{i \neq j} (d_i - d_j) u_i^2 |V_{ij}|^2 = \sum_{i \neq j} u_i^2 (d_i - p u_i) \quad (15b)$$

$$Q' = \frac{3}{\pi} \sum_{i=1}^3 (d_i - p u_i) \quad (15c)$$

where Q' is given by Eq. (10). We will not make a detailed phenomenological analysis of the various ansätze. However, we would like to point out that the condition $\det A = 0$, which is common to the most popular ansätze for quark mass matrices, predicts, in the limit of very large top quark-mass, a simple relation for $|V_{31}|^2$, $|V_{32}|^2$, the matrix elements which control $B_d^0 - \bar{B}_d^0$ and $B_s^0 - \bar{B}_s^0$ mixing. From Eq. (15c), one obtains in the limit of very large top quark mass:

$$d_2 (d_3 - d_1) |V_{31}|^2 + d_1 (d_3 - d_2) |V_{32}|^2 = - \frac{d_1 d_2 (d_1 + d_2)}{3(d_1 + d_2 + d_3)} \quad (16)$$

At this point, we should recall that the eigenvalues of M_u , M_d are real, due to their hermiticity, but not necessarily positive. In fact, consistency of Stech's ansatz requires at least one of the eigenvalues in each quark sector to be negative, the usual choice being $d_2 = -m_s$, $u_2 = -m_c$. With this choice, Eq. (16) gives:

$$|V_{31}|^2 \approx \frac{m_d}{m_s} |V_{32}|^2 + \frac{m_s m_d}{3 m_b^2} \quad (17)$$

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In this scheme [9], the matrix A is postulated to be of the form:

$$A_F = \begin{pmatrix} 0 & \alpha & 0 \\ \alpha^* & 0 & \beta \\ 0 & \beta^* & 0 \end{pmatrix} \quad (18)$$

in the weak basis where M_u is of the form:

$$M_u = \begin{pmatrix} 0 & a & 0 \\ a & 0 & b \\ 0 & b & c \end{pmatrix} \quad (19)$$

with a, b real. It can easily be verified that both the phases of α and β are physically meaningful and in fact they can be expressed in terms of invariants:

$$\begin{aligned} c|\beta|^2 &= \text{tr } M_u A^2 \\ 2bc \text{Re } \beta &= \text{tr } M_u^2 A \\ |\alpha|^2 + |\beta|^2 &= -\text{tr }^2 A \\ 2a \text{Re } \alpha + 2b \text{Re } \beta &= \text{tr } M_u A \end{aligned} \quad (20)$$

where $-a^2c = \det M_u$; $-(a^2+b^2) = \text{tr}^2 M_u$ and $c = \text{tr } M_u$. Therefore in the Fritzsch ansatz there are eight parameters ($a, b, c, \alpha, \beta, \gamma$) to specify ten physical quantities: the six quark masses and the four parameters of the KM mixing matrix. One thus expects to have two independent invariant conditions, and indeed one of the conditions is $\det A = 0$. The other condition can be obtained by noting that

$$\text{tr}^2 (M_u A) = a^2 |\alpha|^2 + b^2 |\beta|^2 + 2ab \text{Re } (\alpha \beta) \quad (21)$$

Using Eqs. (20) to express $a, b, c, \text{Re } \alpha, \text{Re } \beta, |\alpha|, |\beta|$ in terms of invariants and substituting in Eq. (21), one obtains a second invariant condition. However, this condition is rather complicated and probably not of much practical use.

We turn now to the scheme proposed by Gronau, Johnson and Schechter [10] where the matrix A is postulated to be of the form:

$$A = \begin{pmatrix} 0 & i\alpha' & 0 \\ -i\alpha' & 0 & i\beta' \\ 0 & -i\beta' & 0 \end{pmatrix} \quad \alpha', \beta' \text{ real} \quad (22)$$

in the weak basis where M_u is of the form of Eq. (19). Since this scheme combines the two previous ones, its invariant content consists of the two invariant conditions of Fritzsch's scheme, together with the invariant conditions of Eqs. (13a,b).

Recently, there has been increasingly more accurate data [11] on the quark mixing matrix and this has motivated a detailed analysis [3] of the phenomenological implications of the various ansätze and their comparison with experiment. With the appearance of better experimental constraints on the elements of the KM matrix, it is likely that some of the ansätze will be in conflict with experiment. Note that since the various quark schemes can be translated into a set

of invariant conditions, even if a particular ansatz is ruled out by experiment, a subset of its invariant conditions may still be verified.

At this point, it is worth recalling that the various ansätze are not natural in the technical sense. For example, if one considers the invariant condition $\det A = 0$, even if the Yukawa couplings of the standard model are fine-tuned in such a way that $\det A$ vanishes in tree approximation, this quantity will in general receive infinite radiative corrections. This would in turn require a non-vanishing $\det A$ in tree approximation, in order to guarantee renormalizability. Even if one succeeds in constructing models where a particular ansatz is naturally realized in tree approximation, there will still be finite radiative corrections, which would have to be taken into account in a detailed comparison with experiment.

In conclusion, we have shown that weak-basis invariants can be useful in the study of the pattern of quark masses and mixings. In particular, we have indicated how the most common ansätze for quark mass matrices can be translated into weak-basis invariant conditions on the quark mass matrices. Each one of the invariant conditions can be written in terms of directly measurable quantities and therefore can be experimentally tested. It is remarkable that one of these invariant conditions is common to all the ansätze which we have analyzed. We find particularly attractive the possibility of having the regularities of the quark masses and mixings expressed as invariant conditions of the quark mass matrices.

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