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NEVEU-SCHWARZ EXCITED STRING SCATTERING;

A SUPERCONFORMAL GROUP COMPUTATION

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A B S T R A C T

The approach to string theory of the authors is generalized to the superstring. It provides an elegant derivation of the tree scattering of arbitrary Neveu-Schwarz strings. The computation reduces to an exercise in superconformal group theory.

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In previous papers, we have advocated a new approach to string scattering, based on group theory [1,2]. Besides providing a possible route to a deeper understanding of strings, it greatly increases our technical efficiency for the computation of S-matrix elements between excited string states, and their gauge properties. This efficiency comes from the virtually complete elimination of all the cumbersome oscillator algebra which appears when sewing together string propagators and vertices. In the bosonic string, we remarked that this could be replaced by much simpler multiplications of conformal transformations together with overlap conditions between the external string degrees of freedom. The existence of the overlap conditions follows from the form of the light-cone or covariant string interaction vertices, which are conformally equivalent to the old Caneschi-Schwimmer-Veneziano vertex. In this new approach, however, the existence of overlaps is one of the axioms, which are spelled out in Refs. [1,2]. In these papers, we observed that string vertices are group theoretic properties which determine the vertices and the integration measure.

In this paper, we only need the tree overlaps, which for bosonic trees we took to be

$$V \{ \varphi^{\mu(j)}(z_j) - \varphi^{\mu(i)}(z_i) \} \stackrel{(1)}{=} 0$$

between the string position operators

$$\varphi^{\mu}(z) = - \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} \alpha_n^{\mu} z^{-n} + \alpha_0^{\mu} \ln z + \frac{\overleftarrow{\partial}}{\partial \alpha_0^{\mu}} \quad (2)$$

of lines i and j, where

$$Z_j(z) = \frac{(z_j - z)(z_{j-1} - z_{j+1})}{(z_{j-1} - z)(z_j - z_{j+1})} \equiv V_j^{-1}(z) \quad (3)$$

this choice reproduces for V the Lovelace-Olive vertex [3]. The connection between Z_j and Z_{j+1} , via z and Eq. (3) is an $SU(1,1)$ transformation which cycles the lines. The z_i 's are the Koba-Nielsen variables. We then showed that the integration measure over the Koba-Nielsen variables, and, in the case of loops, the moduli can be obtained by solving a simple differential equation which follows from the group properties of V. We refer the reader to Refs. [1,2] for more details.

In this short paper, we apply these ideas to the tree scattering of N-arbitrary excited Neveu-Schwarz strings. The case $N=3$ can be found in Ref. [4].

It has already been noticed that for N ground state scattering, this process can be written as an integration over N-3 commuting and N-2 anticommuting and two anticommuting Koba-Nielsen variables [5]. Since three commuting and two anticommuting variables disappear when super SU(1,1) invariance is taken into account. In order to apply the ideas of Refs. [1,2] to the present case we work in superspace and the corresponding superconformal transformations [6]. We start from the superconformal field $Q^\mu(z) + \theta/\sqrt{z} H^\mu(z)$ which has superconformal weight zero. As usual

$$H^\mu(z) = \sum_{\nu \in \mathbb{Z} + \frac{1}{2}} b_\nu z^{-\nu} \quad (4)$$

and we use the notations and transformation laws of Ref. [4].

We then look for a super SU(1,1) transformation, using the variables (z_i, θ_i) and (z, θ) , which generalizes Eq. (3). It is rather natural to consider the generalized cross-ratio

$$\bar{Z}_j(z, \theta) = \frac{(z_j - z - \theta_j \theta) (z_{j-1} - z_{j+1} - \theta_{j-1} \theta_{j+1})}{(z_{j-1} - z - \theta_{j-1} \theta) (z_j - z_{j+1} - \theta_j \theta_{j+1})} \quad (5)$$

One immediately realizes that this is indeed invariant under a simultaneous super SU(1,1) transformation of (z, θ) and the Koba-Nielsen variables (z_i, θ_i) . The corresponding anticommuting part of (5) is then determined by demanding that the whole transformation $(z, \theta) \rightarrow (\bar{Z}_j, \bar{\theta}_j)$ be superconformal. One finds (up to an irrelevant overall sign):

$$\begin{aligned} \bar{\theta}_j(z, \theta) &= \frac{\theta(z_{j-1} - z_j) + \theta_{j-1}(z_j - z) + \theta_j(z - z_{j-1}) - 2\theta\theta_{j-1}\theta_j}{(z_{j-1} - z - \theta_{j-1}\theta)} \\ &\quad \cdot \sqrt{\frac{(z_{j-1} - z_{j+1} - \theta_{j-1}\theta_{j+1})}{(z_{j-1} - z_j - \theta_{j-1}\theta_j)(z_j - z_{j+1} - \theta_j\theta_{j+1})}} \\ &\equiv \Omega(z, \theta; z_{j-1}, \theta_{j-1}; z_j, \theta_j) \bar{Z}_j^{\frac{1}{2}} \end{aligned} \quad (6)$$

where

$$\Omega(z, \theta; z_{j-1}, \theta_{j-1}; z_j, \theta_j) = \frac{\theta(z_{j-1} - z_j) + \theta_{j-1}(z_j - z) + \theta_j(z - z_{j-1}) - 2\theta\theta_{j-1}\theta_j}{(z_{j-1} - z_j - \theta_{j-1}\theta_j)(z - z_j - \theta\theta_j)(z_{j-1} - z - \theta_{j-1}\theta)} \quad (7)$$

is itself a super $SU(1,1)$ invariant.

It is instructive to eliminate (z, θ) and write down the cycling transformation $(T_{j+1})^{-1}$ from (Z_{j+1}, Θ_{j+1}) to (Z_j, Θ_j) . One obtains the simple result

$$\begin{aligned} Z_j &= \frac{1}{1 - Z_{j+1} a_{j,j+1} + \Theta_{j+1} \Omega_j (a_{j,j+1})^{\frac{1}{2}}} \\ \Theta_j &= \frac{(a_{j,j+1})^{\frac{1}{2}} \Theta_{j+1} - \Omega_j}{1 - Z_{j+1} a_{j,j+1}} \end{aligned} \quad (9)$$

In these equations ε_j is the sign of $(z_{j+1} - z_{j-1})(z_{j+2} - z_j)(z_{j+1} - z_j)$, and

$$a_{j,j+1} = \frac{(z_{j+2} - z_{j+1} - \theta_{j+2}\theta_{j+1})(z_{j-1} - z_j - \theta_{j-1}\theta)}{(z_{j+2} - z_j - \theta_{j+2}\theta_j)(z_{j-1} - z_{j+1} - \theta_{j-1}\theta_{j+1})} \quad (10)$$

and [see Eq. (7)]

$$\Omega_j = \Omega(z_{j-1}, \theta_{j-1}; z_j, \theta_j; z_{j+1}, \theta_{j+1}) \quad (11)$$

When all anticommuting parameters θ_i are set equal to zero, one recovers trivially the cycling transformation of Eq. (2.17) in Refs. [1,2]. It is also interesting to consider the case of the three-point function, where one sets $j+2 = j-1$, hence $a_{j,j+1} = 1$ and the transformation of Eqs. (8) and (9) reduces to the one used in Ref [4].

Just as in the bosonic case, the cycling transformations T_j can be written as a product of two factors

$$T_j = V_j^{-1} V_{j-1} \quad (12)$$

where

$$V_j^{-1} : (z, \theta) \rightarrow (z_j, \theta_j) \quad (13)$$

as in Eqs. (5) and (6). Hence, in accord with our general strategy,

$$T_N T_{N-1} \dots T_1 = 1 \quad (14)$$

We expect that the factorization of T_j into the product $V_j^{-1} V_{j-1}$ is a consequence of demanding that the resulting tree and loop graphs factorize as required by unitarity. One can see this in the sense that the vertex is an exponential of a bilinear form in the oscillators, and the matrix of this form will factorize if the T 's do. The integrand V of the multi-string tree S -matrix thus satisfies the overlap

$$V \left[\left\{ \varphi^{\mu(s)}(z_j) + \frac{\theta_j}{\sqrt{z_j}} H^{\mu(s)}(z_j) \right\} - \left\{ \varphi^{\mu(s+1)}(z_{j+1}) - \frac{\theta_{j+1}}{\sqrt{z_{j+1}}} H^{\mu(s+1)}(z_{j+1}) \right\} \right] \quad (15)$$

for all values of (z, θ) . V is a function of the Koba-Nielsen variables (z_i, θ_i) only; it is the N -point generalization of the same object considered in the case $N=3$ in Ref. [4]. When expanding V in powers of θ_i , the order zero term V_0 contains a bosonic part, built with the commuting oscillators α_n^μ , which is the Lovelace-Olive vertex, and fermionic part, built with b_r^μ , which is the vertex studied by Corrigan and Montonen in Ref. [7]. These vertices use the infinite dimensional representations of $SU(1,1)$ which the α_n^μ and b_r^μ oscillators provide. As noticed in Ref. [4], the integration over the anticommuting θ_i variables does not select V_0 , but rather some other term in the θ_i expansion of V .

We now consider the computation of the full vertex including its dependence on the anticommuting Koba-Nielsen variables θ_j . This can be done directly from the overlap identities of Eq. (15) which in fact contains several component equations. However, it is easier to use the integrated identity which is given by

$$V \left\{ \sum_{j=1}^N \oint_{\xi_j=0} \frac{d\xi_j}{\xi_j} d\psi_j N^\mu(\xi_j, \psi_j) \right\} = 0 \quad (16)$$

where

$$N^\mu(z, 0) = \sqrt{z} \left(H^\mu(z) + \frac{Q}{\sqrt{z}} P^\mu(z) \right) \quad (17)$$

$$(\xi_j, \psi_j) = (T_j \dots T_2)(\xi_1, \psi_1) \quad (18)$$

and ϕ is an arbitrary analytic function of ξ_i, ψ_i with the exception of possible poles at $\xi_j = 0$.

The above method was used in Ref. [4] for the case of $N=3$, but in this paper we use a new method which works directly with the group theoretic properties of the vertex. Recalling the general discussion of Refs. [1,2] and in particular Eq. (2.23) we can find a conformal transformation $\mathcal{U}$ that introduces the θ_j into the vertex:

$$V(z_j, \theta_j) = V(z_j, 0) \prod_{j=1}^N \mathcal{U}^{(j)} \quad (19)$$

The two vertices in the above equation must satisfy the corresponding cycling conditions and equation (2.24) of Ref. [2] now reads

$$\mathcal{U}^{j+1} = T^{j+1}(z, 0) \mathcal{U}^j [T^{j+1}(z, 0)]^{-1} \quad (20)$$

where

$$T^j = (V^j)^{-1} V^{j-1} \quad (21)$$

The unique solution to Eq. (20), up to transformations which leave the vertex inert and are therefore irrelevant for this purpose, is

$$\mathcal{V}^j = [V^j(z, 0)]^{-1} V^j(z, \theta) \quad (22)$$

Using the definition of v of Eqs. (13), (5) and (6) it is straightforward to compute $[\mathcal{V}^j]^{-1}$ which on Z has the action

$$[\mathcal{V}^j]^{-1} : z \rightarrow z' = \left[1 - \frac{\theta_{j-1} \theta_{j+1}}{z_{j-1} - z_{j+1}} + \frac{\theta_j \theta_{j+1}}{z_j - z_{j+1}} - \frac{\theta_{j-1} \theta_j}{(z_j - z_{j-1})} \right] \quad (23)$$

$$\cdot \left[\frac{z + \theta_j \theta \sqrt{z_{j-1} - z_{j+1}}}{\sqrt{(z_j - z_{j-1})(z_j - z_{j+1})}} - \frac{z \theta_{j-1} \theta \sqrt{(z_j - z_{j+1})}}{\sqrt{(z_j - z_{j-1})(z_{j-1} - z_{j+1})}} - \frac{z \theta_{j-1} \theta}{(z_j - z_{j-1})} \right]$$

From this equation we identify

$$\mathcal{V}^j = \exp \left[\frac{\theta_j \sqrt{z_{j-1} - z_{j+1}}}{\sqrt{(z_j - z_{j-1})(z_j - z_{j+1})}} G_{-\frac{1}{2}} \right] \exp \left\{ \frac{-\theta_{j-1} \sqrt{z_j - z_{j+1}}}{\sqrt{(z_j - z_{j-1})(z_{j-1} - z_{j+1})}} G_{\frac{1}{2}} \right\}$$

$$\exp \left\{ \frac{\theta_{j-1} \theta_{j+1}}{(z_{j-1} - z_{j+1})} - \frac{\theta_j \theta_{j+1}}{(z_j - z_{j+1})} + \frac{\theta_{j-1} \theta_j}{(z_{j-1} - z_j)} \right\} L_0 \quad (24)$$

and as a result we have found the full vertex and not just the bosonic piece $V(z_j, 0)$.

The deduction of the measure is now rather trivial. We can deduce the conformal transformation which induces the change $\theta_j \rightarrow \theta_j + \varepsilon$ in the vertex from the general theory or directly by making this change in Eq. (24). The result is the relation

$$\begin{aligned} \frac{\partial V}{\partial \theta_j} = V & \left[+ \left\{ \left(- \frac{\theta_{j-2}}{z_{j-2} - z_j} + \frac{\theta_{j-1}}{z_{j-1} - z_j} \right) L_0^{(j-1)} \right\} \right. \\ & + \left\{ - \left(\frac{\theta_{j+1} + \theta_{j-1}}{z_j - z_{j+1}} \frac{1}{z_{j-1} - z_j} \right) L_0^{(j)} + h^j G_{-\frac{1}{2}}^{(j)} \right\} \\ & + \left\{ + \left(\frac{\theta_{j+2}}{z_j - z_{j+2}} + \frac{\theta_{j+1}}{z_j - z_{j+1}} \right) L_0^{(j+1)} \right\} \end{aligned}$$

$$\left[\frac{-\sqrt{(z_{j+1} - z_{j+2} - \theta_{j+1} \theta_{j+2})}}{\sqrt{(z_{j+1} - z_j - \theta_{j+1} \theta_j)(z_j - z_{j+2} - \theta_j \theta_{j+2})}} G_{\frac{1}{2}}^{(j+1)} \right] \quad (25)$$

where

$$h^j = \sqrt{\frac{z_{j-1} - z_{j+1} - \theta_{j-1} \theta_{j+1}}{(z_j - z_{j-1} - \theta_j \theta_{j-1})(z_j - z_{j+1} - \theta_j \theta_{j+1})}}$$

We now implement the final axiom of Refs. [1,2], namely the decoupling of zero norm physical states from a vertex on which all legs are physical states. The full vertex is given by

$$W = \int d z_j d \theta_j \bar{f}(z_j, \theta_j) V(z_j, \theta_j) \quad (26)$$

where the usual modification in the measure due to super SU(1,1) invariance is to be understood. We find that

$$\begin{aligned} W |x\rangle_1 \dots G_{-\frac{1}{2}}^{(1)} |\psi\rangle_{(1)} \dots |x\rangle_N &= 0 \\ &= \int d z_j d \theta_j [h^j]^{-1} \bar{f} \left(\frac{\partial V}{\partial \theta_j} + \frac{1}{2} \left(\frac{\theta_{j-2}}{z_{j-2} - z_j} - \frac{\theta_{j-1}}{z_{j-1} - z_j} \right) \right. \\ &\quad \left. - \frac{1}{2} \left(\frac{\theta_{j+2}}{z_j - z_{j+2}} + \frac{\theta_{j+1}}{z_j - z_{j+1}} \right) \right) \end{aligned} \quad (27)$$

and as a result

$$\begin{aligned} \frac{\partial}{\partial \theta_j} \ln([h^j]^{-1} \bar{f}) &= \frac{1}{2} \left(\frac{\theta_{j-2}}{z_{j-2} - z_j} - \frac{\theta_{j-1}}{z_{j-1} - z_j} \right) \\ &\quad - \frac{1}{2} \left(\frac{\theta_{j+2}}{z_j - z_{j+2}} + \frac{\theta_{j+1}}{z_j - z_{j+1}} \right) \end{aligned} \quad (28)$$

Consequently, the solution for the measure is

$$\overline{f} = \frac{2}{\pi} \sqrt{\frac{(z_{j-2} - z_j - \theta_{j-2} \theta_j)}{(z_{j-1} - z_j - \theta_{j-1} \theta_j)^2}}. \quad (29)$$

One realizes that f is a straightforward generalization of the corresponding bosonic result with a square root thrown in.

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After this work was completed, we received a paper by A. Clarizia and F. Pezzella, entitled "N-Reggeon Vertex for Neveu-Schwarz String", which addresses the same problem. Our starting point for the cycling transformation, Eq. (5), is different from Eq. (17) of their paper. The latter corresponds to replacing θ_{j+1} by zero in our Eq. (5), thus spoiling super $SU(1,1)$ invariance of Z_j . Consequently, their final results are different from ours, and it is not clear to us that their vertex is physical in the sense that it decouples zero norm states. We thank these authors for sending us their paper before publication, which prompted us to write up our results.

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