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REPHASING-INVARIANT PARAMETRIZATION OF THE QUARK MIXING MATRIX

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A B S T R A C T

We suggest that within the context of the standard model with three generations, the most convenient way of parametrizing the Kobayashi-Maskawa matrix is through four independent moduli of its matrix elements. We extend this parametrization to the case of four generations and analyze in detail the unitarity constraints on the independent moduli. The strength of CP violation is completely determined from the knowledge of a set of independent moduli.

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Recently, increasingly more accurate data on the quark mixing matrix have been accumulated [1]. This wealth of experimental results motivates the search for a suitable parametrization of the Kobayashi-Maskawa (KM) matrix [2]. In this letter, we suggest that within the context of the standard model with three generations, the most convenient way of parametrizing the KM matrix is through four independent moduli of its matrix elements. This parametrization has the advantage of being obviously invariant under a rephasing of the quark fields and furthermore its independent parameters are directly and easily measurable quantities. This is certainly not the case for traditional parametrizations of the KM matrix through generalized Euler angles and CP violating phases. In general, one does not measure directly either Euler angles or KM phases, but rather certain rephasing invariant products of the KM matrix elements. We will show that in the standard model with three generations, the knowledge of four independent moduli completely determines the quark mixing matrix, and in particular the strength of CP violation. At this point, the reader may wonder how it is possible to fix the strength of CP violation through the knowledge of non-CP-violating quantities like the moduli. It will be seen that this connection arises as a consequence of the unitarity of the quark mixing matrix.

We will first present the parametrization of the three generation KM matrix through independent moduli, giving special emphasis to unitarity constraints. We then illustrate the rôle played by this parametrization in the search for a minimum set of weak-basis invariants which completely determine the physical content of the quark mass matrices. Finally we discuss the extension of our parametrization to a larger number of generations, as well as to models beyond the standard one.

INVARIANT PARAMETRIZATION FOR THREE GENERATIONS

It is well known that the KM matrix arises from the bi-diagonalization of the quark mass matrices which leads to a non-diagonal charged current. Under a change of phases of the quark fields, the diagonalized quark mass matrices remain invariant, while the KM matrix V transforms as:

$$V_{ij} \longrightarrow V_{ij} \exp \left[i (\beta_j - \alpha_i) \right] \quad (1)$$

where α_i, β_j denote the arbitrary changes of phase in the up and down quark fields. Only functions of the V_{ij} which remain invariant under the rephasing (1) are observable [3]. Obviously, the simplest invariants are the moduli $|V_{ij}|$ and the quartic products $V_{ij} V_{kl} V_{il}^* V_{kj}^*$ which will be designated "quartets". Moreover, any invariant of higher order can be written as a function of the quartets and the moduli:

$$V_{ij} V_{kl} V_{mn} \dots V_{op} V_{rs} V_{il}^* V_{kn}^* \dots V_{os}^* V_{rj}^* = \\ |V_{kj} V_{mj} \dots V_{oj}|^{-2} (V_{ij} V_{kl} V_{il}^* V_{kj}^*) (V_{kj} V_{mn} V_{kn}^* V_{mj}^*) \dots (V_{oj} V_{rs} V_{os}^* V_{rj}^*) \quad (2)$$

An invariant parametrization of the KM matrix, for a given number of generations, consists of a minimum set of basic invariants which completely determine all other invariants. Next we will show that for three generations a very convenient choice of basic invariants are four-independent squared moduli $U_{ij} \equiv |V_{ij}|^2$. There are, of course, many possible choices for the four independent U_{ij} , and their choice should be dictated by the availability and precision of the experimental data. For definiteness, let us first take as basic invariants U_{11} , U_{12} , U_{21} and U_{22} . The remaining U_{ij} are trivially deduced from these, through unitarity. The quartets, too, are related among themselves and with the moduli by unitarity [3]. For example, the unitarity relation $V_{13}^* V_{23} = -V_{11}^* V_{21} - V_{12}^* V_{22}$ implies $V_{11} V_{23} V_{13}^* V_{21}^* = -V_{11} V_{22} V_{12}^* V_{21}^* - U_{11} U_{21}$, so that out of the nine quartets only one, say $V_{11} V_{22} V_{12}^* V_{21}^*$ is independent. In particular, the imaginary parts of the quartets are all equal [3] up to a sign, as a consequence of the relations among quartets implied by unitarity. In order to obtain $\text{Im}(V_{11} V_{22} V_{12}^* V_{21}^*)$, which controls the strength of CP violation, in terms of our four independent moduli, we start by noting that the real part of this quartet is related to the moduli. Indeed squaring the unitarity relation $V_{11} V_{21}^* + V_{12} V_{22}^* = -V_{13} V_{23}^*$, and using unitarity to express U_{13} , U_{23} in terms of the basic moduli, one gets:

$$R \equiv \text{Re}(V_{11} V_{22} V_{12}^* V_{21}^*) = \frac{1}{2} (1 - U_{11} - U_{22} - U_{12} - U_{21} + \\ + U_{11} U_{22} + U_{12} U_{21}) \quad (3)$$

Since

$$I \equiv \text{Im}(V_{11} V_{22} V_{12}^* V_{21}^*) = \pm \left[U_{11} U_{22} U_{12} U_{21} - R^2 \right]^{1/2} \quad (4)$$

the value of $|I|$ is determined by the four independent moduli, with an ambiguity in the sign of I only remaining.

UNITARITY CONSTRAINTS

In our parametrization, the quark mixing matrix is not manifestly unitary. It is therefore important to analyze the constraints on the basic invariants, implied by unitarity. For definiteness, let us choose again U_{11} , U_{22} , U_{12} and U_{21} as basic invariants. There are obvious constraints obliging $0 < U_{ij} < 1$. Next we consider

the constraints implied by the orthogonality between two different rows or columns of V . For example, the orthogonality relation $V_{11}V_{21}^* + V_{12}V_{22}^* + V_{13}V_{23}^* = 0$ implies that the three complex numbers $(V_{11}V_{21}^*)$, $(V_{12}V_{22}^*)$ and $(V_{13}V_{23}^*)$ should form a triangle (see Fig. 1) in the complex plane. It is worth noticing that the area of this triangle has physical significance, since it equals $\frac{1}{2}|I|$. Rephasing of the KM matrix corresponds to rotate the triangle as a whole in the complex plane which, of course, does not change its area. By considering other orthogonality relations, involving other rows or columns, one gets other triangles, all of them with the same area. It is well known that in order for three segments to be able to form a triangle, the length of each segment should not exceed the sum of the other two. This constraint is equivalent to the inequality:

$$2(l_1^2 l_2^2 + l_2^2 l_3^2 + l_1^2 l_3^2) \geq l_1^4 + l_2^4 + l_3^4 \quad (5)$$

where l_i denote the lengths of the segments. If we identify

$$l_1 \equiv |V_{11} V_{21}^*|, \quad l_2 \equiv |V_{12} V_{22}^*|, \quad l_3 \equiv |V_{13} V_{23}^*|$$

one obtains

$$4(U_{11} U_{21} U_{12} U_{22}) \geq (U_{13} U_{23} - U_{11} U_{21} - U_{12} U_{22})^2 \quad (6)$$

which in terms of the chosen basic invariants can be written as:

$$4U_{11} U_{12} U_{21} U_{22} \geq (1 - U_{11} - U_{22} - U_{12} - U_{21} + U_{11} U_{22} + U_{12} U_{21})^2 \quad (7)$$

Now, in a 3×3 unitarity matrix, one has six orthogonality relations which in turn lead to six orthogonality constraints analogous to Eq. (6). However, it turns out that, when expressed in terms of the independent moduli all these constraints are equivalent to Eq. (7). At this point, we should point out that Eq. (7) can be trivially obtained from eq. (3) by requiring that $[\text{Re}(V_{11}V_{22}V_{12}^*V_{21}^*)]^2$ does not exceed $|V_{11}V_{22}V_{12}^*V_{21}^*|^2$. However, it will be seen that the above derivation, using an analogy with a triangle, has the advantage of being easily generalizable to four generations, where the unitarity constraints are much more complex.

We close our discussion of the unitarity constraints for three generations by noting that: (i) the condition of Eq. (7) further restricts the four basic moduli, beyond the trivial constraints resulting from $0 < U_{ij} < 1$; (ii) we have verified that these trivial constraints, together with Eq. (7), are necessary and sufficient conditions for the unitarity of the quark mixing matrix. Needless to say, equivalent unitarity conditions are obtained, if a different set of basic moduli is chosen.

CONFRONTING EXPERIMENT

In this section we will take as basic invariants $|V_{12}|$, $|V_{13}|$, $|V_{21}|$ and $|V_{23}|$. At present, our experimental knowledge of these quantities is [1]:

$$\begin{aligned} |V_{11}| &= .9747 \pm .0010 ; |V_{12}| = .220 \pm .002 ; |V_{13}| \leq .010 \\ |V_{21}| &= .207 \pm .024 ; |V_{22}| = .95 \pm .14 ; |V_{23}| = .045 \pm .010 \end{aligned} \quad (8)$$

Unitarity further restricts the allowed range for the basic invariants. For example, positivity of $|V_{31}|^2$ and $|V_{32}|^2$ constrains $|V_{21}|^2$ to be in interval:

$$|V_{12}|^2 - |V_{23}|^2 \leq |V_{21}|^2 \leq |V_{12}|^2 + |V_{13}|^2 \quad (9)$$

Next we will consider the unitarity constraint of Eq. (7) which, when expressed in terms of the new set of independent moduli, reads:

$$P(U_{21}) \equiv c_1 U_{21}^2 + c_2 U_{21} + c_3 \leq 0 \quad (10)$$

where the c_i are functions of U_{12} , U_{13} and U_{23} :

$$\begin{aligned} c_1 &= (1 - U_{13})^2 \\ c_2 &= 2 \left[U_{12} (U_{23} - 1) + U_{13} (U_{12} - U_{23} + U_{12} U_{23} + U_{13} U_{23}) \right] \\ c_3 &= \left[U_{12} (U_{23} - 1) + U_{13} U_{23} \right]^2 \end{aligned} \quad (11)$$

Since $P(U_{21})$ is a quadratic polynomial in $|V_{21}|^2$, the inequality (10) implies that

$$r_1 \leq |V_{21}|^2 \leq r_2 \quad (12)$$

where r_i are the roots of P , which are functions of U_{12} , U_{13} , U_{23} . Given the experimentally allowed range for U_{12} , U_{13} and U_{23} , it turns out that this constraint on $|V_{21}|^2$ in Eq. (12) is more restrictive than that of Eq. (9). In Fig. 2, we give the allowed range of $|V_{21}|$. If one keeps $|V_{12}|$ fixed at (0.220), $|V_{21}|$ may vary between the solid lines if $|V_{13}| = 0.010$, or between the dashed lines if $|V_{13}| = 0.003$. When $|V_{13}| = 0$, $|V_{21}|$ is completely determined by $|V_{23}|$ and $|V_{12}|$:

$$U_{21} = U_{12} (1 - U_{23}) \quad (13)$$

which is marked as a dotted line in Fig. 2. We have kept $|V_{12}|$ fixed at $|V_{12}| = 0.220$, but other values of $|V_{12}|$ can be easily studied from the graph of Fig. 2. The reason is that we have verified that if one considers another value for $|V_{12}|$, say $|V_{12}| = 0.220 + \Delta|V_{12}|$, the allowed range for $|V_{12}|$ is, to a great accuracy, simply displaced by the same amount $\Delta|V_{12}|$.

We have seen that the knowledge of four independent moduli is sufficient to determine the strength of CP violation in the standard model with three generations. For our choice of basic moduli, we have

$$I = \pm (-P)^{1/2} / 2 \quad (14)$$

where P is given by Eq. (10). Clearly, for given values of $|V_{12}|$, $|V_{13}|$, $|V_{23}|$, the maximum value of $|I|$ is reached when:

$$U_{21} = - \frac{C_2}{2C_1} \quad (15)$$

It turns out that, due to the smallness of $|V_{13}|$, this value of $|V_{21}|$ coincides, to great accuracy, with the value given by Eq. (13). Of course, if one wants to calculate the maximum value of $|I|$, one has to vary the four independent moduli, within the ranges allowed by experiment and unitarity. In practice, the best way to look for this maximum is to constrain U_{21} to obey Eq. (15) and then to vary U_{12} , U_{13} , U_{23} . We have thus found:

$$|I_{\max.}| = 1.08 \times 10^{-4} \quad (16)$$

which arises when $|V_{12}|$, $|V_{13}|$ and $|V_{23}|$ are all put in their upper experimental limits.

A MINIMAL SET OF MASS MATRIX INVARIANTS

We have seen that a set of four independent moduli of the KM matrix elements, the sign of I , together with the six quark masses, exhaust the physical information contained in the up and down quark mass matrices. Obviously, this information can be obtained directly in any weak basis, without the need to explicitly diagonalize those matrices. We will show next that the parametrization through independent moduli of KM matrix elements, which we have presented, plays an important rôle in the proof that a particular set of weak-basis invariants contain sufficient information to completely fix the value of quark masses and mixings. For simplicity, let us restrict ourselves for the moment to the case of three generations and consider the following weak-basis invariants [4]:

$$\text{tr}(H_u H_d) ; \text{tr}(H_u H_d^2) ; \text{tr}(H_u^2 H_d) ; \text{tr}(H_u^2 H_d^2) \\ H_i \equiv M_i M_i^\dagger \quad (17)$$

together with the coefficients of the characteristic equations for H_d , H_u . We will show that these ten invariants completely determine the quark masses and mixings, leaving only a twofold ambiguity in the sign of I . First notice that for an arbitrary number of generations:

$$\text{tr}(H_u^a H_d^b) = \sum_{i,j=1}^n U_i^a D_j^b |V_{ij}|^2 \quad (18)$$

where a, b are arbitrary integers and U_i , D_i denote the square of the up and down quark masses. Using unitarity to relate the different moduli, one obtains:

$$\text{tr}(H_u^a H_d^b) = \sum_{i,j=1}^{n-1} (U_i^a - U_n^a)(D_j^b - D_n^b) |V_{ij}|^2 + \\ + U_n^a \text{tr}(H_d^b) + D_n^b \text{tr}(H_u^a) - 3 U_n^a D_n^b \quad (19)$$

Considering now the case of three generations, one can use Eq. (19) to express the invariants of Eq. (17) in terms of quark masses and the four independent moduli $|V_{ij}|^2$ ($i, j=1, 2$). One obtains then a set of four linear equations for the $|V_{ij}|^2$, which has a unique solution, provided that the up and down quark masses are non-degenerate. Since we have seen that four independent moduli completely determine the KM matrix (apart from an ambiguity in the sign of I), the proof is completed. This result is trivially generalized for n generations and one can easily show using again Eq. (19) that the $(n-1)^2$ invariants $\text{tr}(H_u^a H_d^b)$ (a, b from 1 to $n-1$) uniquely determine all the $|V_{ij}|^2$, provided the quark masses are non-degenerate. A crucial question is then whether the knowledge of the $|V_{ij}|^2$ completely fixes the KM matrix, for an arbitrary number of generations. We will see that this is not a trivial question for more than three generations.

FOUR GENERATIONS

Inspired by the three-generation case, we will try to apply the idea of parametrizing the KM matrix by independent moduli of its matrix elements, to the case of four generations. Let us write the unitary 4×4 mixing matrix V as:

$$V = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & p & q \end{pmatrix} \quad (20)$$

and adopt the convention that a capital letter stands for the squared modulus of the corresponding matrix element: $A = |a|^2$, $B = |b|^2$, etc. Our aim is to find all quartets in terms of a set of nine independent moduli, say those of the 3×3 upper left-hand submatrix: $A, B, \dots, K$. First we note that there are only four independent quartets. One can, for example, express all quartets involving matrix elements either of the last row or of the last column of V , in terms of quartets not involving such matrix elements, by using expressions like:

$$g l h^* k^* = - (G K + g j f^* k^* + g i e^* k^*) \quad (21)$$

One thus reduces to $({}^3C_2)^2 = 9$ the number of potentially independent quartets. A further reduction is achieved by noting that any of these quartets may be written as a product of other two quartets, divided by some moduli. For example:

$$b g c^* f^* = \frac{(a^* f^* b e)(a g c^* e^*)}{A E} \quad (22)$$

Therefore, our task would be completed if we could express the arguments of four independent quartets, say:

$$\begin{aligned} \alpha &= \arg(a f b^* e^*) & ; & \beta = \arg(a g c^* e^*) \\ \gamma &= \arg(a j b^* i^*) & ; & \delta = \arg(a k c^* i^*) \end{aligned} \quad (23)$$

in terms of a set of nine independent moduli. Now, the orthogonality of the first two rows of V leads to:

$$\operatorname{Re}(a f b^* e^* + a g c^* e^* + b g c^* f^*) = d_1 \quad (24a)$$

where $d_1 = (DH - AE - BF - CG)/2$. If one considers all similar equations for any pair of rows or columns of V and uses the relations among quartets and moduli implied by unitarity - as in Eq. (21) - one finds another four equations similar to Eq. (24a):

$$\operatorname{Re}(a j b^* i^* + a k c^* i^* + b k c^* j^*) = d_2 \quad (24b)$$

$$\text{Re} (e j f^* i^* + e k g^* i^* + f k g^* j^*) = d_3 \quad (24c)$$

$$\text{Re} (a f b^* e^* + a j b^* i^* + e j f^* i^*) = d_4 \quad (24d)$$

$$\text{Re} (a g c^* e^* + a k c^* i^* + e k g^* i^*) = d_5 \quad (24e)$$

where the d_i are function of the moduli. If one takes now into account the algebraic relations among the nine quartets, like the one of Eq. (22), one can translate Eqs. (24a-e) into equations involving only the phases of the independent quartets:

$$c_1 \cos \alpha + c_2 \cos \beta + c_3 \cos (\beta - \alpha) = d_1 \quad (25a)$$

$$c_4 \cos \gamma + c_5 \cos \delta + c_6 \cos (\delta - \gamma) = d_2 \quad (25b)$$

$$c_7 \cos (\gamma - \alpha) + c_8 \cos (\delta - \beta) + c_9 \cos (\alpha + \delta - \beta - \gamma) = d_3 \quad (25c)$$

$$c_1 \cos \alpha + c_4 \cos \gamma + c_7 \cos (\gamma - \alpha) = d_4 \quad (25d)$$

$$c_2 \cos \beta + c_5 \cos \delta + c_8 \cos (\delta - \beta) = d_5 \quad (25e)$$

where the c_i , d_i are function of the moduli only, e.g., $c_1 = (AFBE)^{\frac{1}{2}}$, $c_2 = (AGCE)^{\frac{1}{2}}$, etc. In principle, one can then use Eqs. (25) to find $\alpha, \beta, \gamma, \delta$ in terms of the independent moduli. But due to the great complexity of Eqs. (25), it is not easy to give a complete answer to questions such as: (i) what conditions should the independent moduli satisfy in order for Eqs. (25) to have a solution? (ii) for a given set of independent moduli, what is the level of degeneracy of the solution? The solutions of Eqs. (25) have a trivial degeneracy: starting from a given solution, one may obtain another one by inverting all phases of all quartets simultaneously. Obviously the two solutions differ only in an overall sign of all the CP violating amplitudes. Although we have not a formal proof, we find plausible that there is no other degeneracy beyond the one described above, and therefore that in the case of four generations the knowledge of a set of nine independent moduli is sufficient to determine the strength of all CP violating amplitudes.

Regarding question (i), one may derive a series of necessary conditions for a set of moduli to be compatible with unitarity. These necessary conditions go beyond the trivial constraints arising from normalization of all the columns and rows of V . For example, from orthogonality of the first two rows, it follows that the four complex quantities $(V_{11}V_{21}^*)$, $(V_{12}V_{22}^*)$, $(V_{13}V_{23}^*)$, $(V_{14}V_{24}^*)$ should form a quadrilateral in the complex plane. For this to be possible, the moduli of each one of these complex numbers should not exceed the sum of the other three. We have verified that these conditions are equivalent to requiring that:

$$2(l_1^2 l_2^2 + l_1^2 l_3^2 + l_1^2 l_4^2 + l_2^2 l_3^2 + l_2^2 l_4^2 + l_3^2 l_4^2) \geq (l_1^4 + l_2^4 + l_3^4 + l_4^4) - 8l_1 l_2 l_3 l_4 \quad (26)$$

where

$$l_1 = |V_{11} V_{21}^*|; \quad l_2 = |V_{12} V_{22}^*|; \quad l_3 = |V_{13} V_{23}^*|; \quad l_4 = |V_{14} V_{24}^*|$$

This is the generalization to four generations of the condition of Eq. (5) in three generations. Similarly, one can derive analogous necessary conditions for unitarity, from orthogonality of other rows and columns. Although we have again no formal proof, we conjecture that the conditions of the type of inequality (26), together with the trivial unitarity constraints arising from normalization of columns and rows, are sufficient conditions for a set of independent moduli to be compatible with unitarity and therefore for Eqs. (25), to have a solution. Just to get an idea of how the condition of Eq. (26) goes beyond the trivial unitarity constraints, we consider a realistic choice of moduli:

$$\begin{aligned} |V_{11}|^2 &= .95 ; \quad |V_{12}|^2 = .0484 ; \quad |V_{13}|^2 = .0001 ; \quad |V_{14}|^2 = .0015 \\ |V_{21}|^2 &= .049 ; \quad |V_{22}|^2 = x ; \quad |V_{23}|^2 = .003 ; \quad |V_{24}|^2 = .948 - x \end{aligned} \quad (27)$$

One can verify that in this case the condition of Eq. (26) implies the bound:

$$.85447 < x < .94721 \quad (28)$$

on $|V_{22}|$, $|V_{24}|$.

The areas of the various quadrilateral that one can form in the way previously indicated have also physical significance. Each one of them equals half the sum of the imaginary parts of two quartets.

We would like to emphasize that the difficulties one faces in solving Eqs. (25) are in no way eliminated by using a parametrization of the 4×4 KM matrix [5] through Euler angles and phases. One can convince oneself of this, by trying to deduce the value of the parameters of a given parametrization, from the knowledge of a set of independent $|V_{ij}|$.

In conclusion, we have presented a parametrization of the quark mixing matrix in terms of independent moduli of the KM matrix elements. This parametrization illustrates the remarkable fact that in the standard model one can deduce the strength of CP violation from the knowledge of quantities like the moduli, which are not directly related to CP violation. At present, one can only derive an upper bound on $|I|$, the invariant which controls the strength of CP violation. The determination of a lower bound on $|I|$ is at present not possible, given the experimental uncertainty in $|V_{21}|$, $|V_{22}|$. The derivation of such a lower bound would be a proof that charged gauge current interactions do indeed violate CP. The present experimental value of the various moduli are still compatible with a real KM matrix and therefore one cannot exclude the possibility that the origin of CP violation lies entirely in physics beyond the standard model. Finally, we would like to point out that the possibility of parametrizing the KM matrix through independent moduli, is a very special feature of the standard model which no longer holds true in most of its extensions. For example, in left-right symmetric models [6], one can easily verify that the total number of independent parameters [7] entering the left-handed and right-handed charged currents exceeds the number of independent moduli in the corresponding quark mixing matrices.

While this work was in progress, we have received an interesting preprint by J.D. Bjorken and I. Dunietz (EFI 87-24) where a different rephasing-invariant parametrization of the Kobayashi-Maskawa matrix is suggested.

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FIGURE CAPTIONS

- Fig. 1 : Unitarity constrains the three complex numbers $V_{11}V_{21}^*$, $V_{12}V_{22}^*$, $V_{13}V_{23}^*$ to form a triangle in the complex plane. The area of the triangle equals $\frac{1}{2}|I|$.
- Fig. 2 : The values of $|V_{21}|$ allowed by unitarity are plotted against $|V_{23}|$, for three values of $|V_{13}|$ and with $|V_{12}|$ fixed at 0.220. The solid and dashed lines limit the unitarity regions for $|V_{13}| = 0.010$ and $|V_{13}| = 0.003$, respectively. The dotted line gives the value of $|V_{21}|$ when $|V_{13}| = 0$.

Fig. 1

Fig. 2