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ON THE VANISHING OF THE GENUS TWO SUPERSTRING VACUUM AMPLITUDE

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ABSTRACT

We use results of Verlinde and Verlinde [1] to explicitly evaluate the two-loop partition function for ten-dimensional closed superstrings in a BRST-invariant manner. By taking picture-changing operators to a ramification point on the surface and using the Riemann identity we obtain a simple expression in terms of theta functions which is known to vanish.

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Recently there has been considerable interest in probing superstrings beyond the one-loop level. Although proposals have been made and some formal results were obtained by indirect methods [1-4], their confirmation by explicit calculations is still necessary, at first for genus two [5,6]. In this letter we intend to clarify the situation by computing the two-loop partition function directly, keeping BRST-invariance manifest. We shall use the notation and conventions of refs.[1,7].

After superconformal gauge-fixing, the two-loop vacuum amplitude for the closed oriented superstring in D=10 dimensions is believed to be an integral over supermoduli space of almost an absolute square [8]:

$$\Gamma_2 = \int (det \operatorname{Im} \tau)^{-5} |W(m_i, \zeta_a) \prod_i dm_i \prod_a d\zeta_a|^2 \quad (1)$$

where $\tau = \tau(m_i)$ is the 2×2 period matrix, and $m_i, i=1,2,3$, and $\zeta_a, a=1,2$, denote commuting and anticommuting complex moduli parameters, respectively. To paraphrase ref.[1] we choose a set of super Beltrami differentials $\eta_i, \chi_a = \delta(z - z_a)$ with $\frac{\partial \chi_a}{\partial m_i} = 0$ and perform the integration over ζ_a to arrive at $W[m_i; \eta_i, \chi_a]$. Suppressing the m_i -dependence we rewrite

$$W(w_i, z_a) = \prod_i \frac{\delta}{\delta \eta_i(w_i)} W[\eta_i, \chi_a = \delta(z - z_a)] \quad (2)$$

and trade the non-holomorphicity for integrations over loop momenta $p_k^\mu, k=1,2, \mu=1,\dots,10$ [1]:

$$|W(w_i, z_a)|^2 = (det \operatorname{Im} \tau)^5 \int \prod_{k,\mu} dp_k^\mu |e^{-\pi p^\mu \tau p_\mu} W_p(w_i, z_a)|^2 \quad (3)$$

Decomposing with respect to spin structures $\delta = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}, \delta_k = \begin{bmatrix} \delta_{k1} \\ \delta_{k2} \end{bmatrix}$, we have

$$W_p(w_i, z_a) = \sum_\delta \epsilon_\delta W_\delta(w_i, z_a) \quad (4)$$

with [1,3]

$$W_\delta(w_i, z_a, x) = \langle \xi(x) X(z_1) X(z_2) b(w_1) b(w_2) b(w_3) \rangle_\delta \quad (5)$$

At this point the problem is reduced to computing a correlation function depending on the insertion points x, z_a and integration variables w_i . The discussion of phases ϵ_δ is deferred to a later stage. The correlation

$$\langle O \rangle_\delta = \int D[X^\mu, \psi^\mu, b, c, \phi, \eta, \xi]_\delta O e^{-S[X^\mu, \psi^\mu, b, c, \phi, \eta, \delta \xi]} \quad (6)$$

refers to the spin structure δ and depends implicitly on p via the zero-mode of X^μ . Due to bosonization of the spinor ghosts [2]

$$\beta = \partial \xi e^{-\phi} , \quad \gamma = e^{\phi} \eta \quad (7)$$

an insertion $\xi(x)$ had to be introduced in Eq.(5) to absorb the zero-mode of ξ . A change of insertion points x or z_a modifies W only by a boundary term in moduli space as may be shown using BRST-invariance [1,2]. We make use of this freedom below.

We now evaluate W_δ in terms of genus two theta functions, Θ_δ . The correlation decomposes into an X^μ , ψ^μ , (b, c) - and (ϕ, η, ξ) -part. Whereas the first three can be computed by standard methods [3,5,7,9,10], the spinor ghost correlation requires a different treatment. We shall make use of Eq.(36) in ref.[1]. It is natural to split the picture-changing operator [2] according to the ϕ -charge

$$\begin{aligned} X &= X_0 + X_1 + X_2 \\ X_0 &= c \partial \xi \\ X_1 &= -\frac{1}{2} e^{\phi} \partial X \cdot \psi \\ X_2 &= -\frac{1}{4} \partial \eta e^{2\phi} b - \frac{1}{4} \partial (\eta e^{2\phi} b) \end{aligned} \quad (8)$$

To ensure BRST-invariance the full X is needed. At tree-level the reader may easily check explicitly that, for example, the equality

$$\langle \xi \ cV_{-1}(z_1) cV_{-1}(z_2) cV_{-2}(z_3) X(x) X(y) \rangle = \langle \xi \ cV_{-1}(z_1) cV_{-1}(z_2) cV_0(z_3) \rangle$$

requires all three terms in Eq.(8). The two-loop background charge of the ϕ -system enforces a projection onto $X_1 X_1 + X_0 X_2 + X_2 X_0$ in Eq.(5), leaving us with

$$\begin{aligned} W_\delta &= Z_1^{-7} (W_\delta^{11} + W_\delta^{02} + W_\delta^{20}) \\ W_\delta^{11}(z_a, x) &= Z (\sum w_i - 3\Delta) \frac{\Theta_\delta(0)^4 \Theta_\delta(z_1 - z_2)}{\Theta_\delta(z_1 + z_2 - 2\Delta)} \sigma(z_1)^{-2} \sigma(z_2)^{-2} E(z_1, z_2)^{-2} \\ &\quad \times \left[D \partial_1 \partial_2 \ln E(z_1, z_2) + 2\pi p_i \cdot p_j \omega_i(z_1) \omega_j(z_2) \right] \\ W_\delta^{02}(z_a, x) &= -(\partial_y f)|_{y=z_2} - \partial_{z_2} (f|_{y=z_2}) \\ W_\delta^{20}(z_1, z_2, x) &= W_\delta^{02}(z_2, z_1, x) \\ f &= Z (z_2 - z_1 + \sum w_i - 3\Delta) \frac{\Theta_\delta(0)^5}{\Theta_\delta(2z_2 + x - y - 2\Delta)} E(x, y)^{-1} \sigma(z_2)^{-4} \partial_{z_1} \frac{\Theta_\delta(z_1 + 2z_2 + x - 2y - 2\Delta) E(z_1, x)}{\Theta_\delta(z_1 + 2z_2 - y - 2\Delta) E(z_1, y)}. \end{aligned} \quad (9)$$

Here and below, Z_1 and $Z(\dots)$ denote partition functions of the X^μ and (b, c) systems, respectively. For a definition of these and the holomorphic 1-forms ω_i and σ , the prime form E and the Riemann class Δ we refer the reader to ref.[7]. W_δ^{11} is trivially independent of the insertion

point x . Consider the limit $x \rightarrow z_1$. Apparently W^{02} is finite but W^{20} develops poles which, however, cancel after closer inspection. Nevertheless this limit breaks the symmetry between W^{02} and W^{20} and leads to

$$\begin{aligned}
 W_{\delta}^{02}(z_a) &= -Z(z_2 - z_1 + \sum w_i - 3\Delta) \frac{\Theta_{\delta}(0)^5 \Theta_{\delta}(2z_1 - 2\Delta)}{\Theta_{\delta}(z_1 + z_2 - 2\Delta)^2} E(z_1, z_2)^{-2} \sigma(z_2)^{-4} \\
 &\quad \times \left\{ -2\partial_2 \ln \Theta_{\delta}(2z_1 - 2\Delta) + \partial_2 \ln \Theta(z_2 - z_1 + \sum w_i - 3\Delta) + \sum \partial_1 \ln E(z_2, w_i) - 5\partial_1 \ln E(z_2, z_1) - \partial \ln \sigma(z_2) \right\} \\
 W_{\delta}^{20}(z_a) &= +Z(z_1 - z_2 + \sum w_i - 3\Delta) \frac{\Theta_{\delta}(0)^5}{\Theta_{\delta}(2z_1 - 2\Delta)} \sigma(z_1)^{-4} \\
 &\quad \times \left\{ \partial_1 \partial_1 \partial_2 \ln \Theta_{\delta}(z_1 + z_2 - 2\Delta) - \partial_1 \partial_2 \partial_2 \ln E(z_2, z_1) + \left[\partial_1 \partial_2 \ln \Theta_{\delta}(z_1 + z_2 - 2\Delta) + \partial_1 \partial_2 \ln E(z_2, z_1) \right] \right. \\
 &\quad \times \left. \left[2\partial_1 \ln \Theta_{\delta}(z_1 + z_2 - 2\Delta) - \partial_1 \ln \Theta(z_1 - z_2 + \sum w_i - 3\Delta) - \sum \partial_1 \ln E(z_1, w_i) + 3\partial_1 \ln E(z_1, z_2) + \partial \ln \sigma(z_1) \right] \right\}
 \end{aligned} \tag{10}$$

where we abbreviate

$$\partial_a = \omega_i(z_a) \partial_i$$

$$\partial_a \partial_b = \omega_i(z_a) \omega_j(z_b) \partial_i \partial_j + \delta_{ab} \omega_i'(z_a) \partial_i \quad \text{etc.}$$

acting on theta functions Θ_{δ} or $\Theta \equiv \Theta_0$, and $\partial_i E(\dots)$ denotes a derivative with respect to the i th argument of the prime form. Notice that Eq.(10) differs from Eq.(38) of ref.[1] in the expression for W_{δ}^{20} .

To get more insight we shall analyse W_{δ} in the limit $z_1 \rightarrow z_2$ which corresponds to taking the picture-changing operators on top of each other [5]. Writing

$$W_{\delta}(z_a) = Z(\sum w_i - 3\Delta) \sum_{k=-4}^{\infty} (z_1 - z_2)^k W_{\delta, -k}(z_2) \tag{11}$$

we find ($z \equiv z_2$):

$$W_{\delta, 4} = \Theta_{\delta}(0)^5 \Theta_{\delta}(2z - 2\Delta)^{-1} \sigma(z)^{-4} (-D + 10)$$

$$W_{\delta, 3} = \Theta_{\delta}(0)^5 \Theta_{\delta}(2z - 2\Delta)^{-1} \sigma(z)^{-4} (-D + 10) \left\{ -\partial \ln \Theta_{\delta}(2z - 2\Delta) - 2\partial \ln \sigma(z) \right\} \tag{12}$$

$$\begin{aligned}
 W_{\delta, 2D=10} &= \Theta_{\delta}(0)^5 \Theta_{\delta}(2z - 2\Delta)^{-1} \sigma(z)^{-4} \times \left\{ -5\partial_2 \partial_2 \ln \Theta_{\delta}(0) \right. \\
 &\quad + (5\partial_1 - 4\partial_2) \partial_2 \ln \Theta_{\delta}(2z - 2\Delta) + (4\partial_2 - \partial_1) \partial_2 \ln \Theta - \sum \partial_1 \partial_1 \ln E(z, w_i) + \partial^2 \ln \sigma(z) - 2\pi p_i \cdot p_j \omega_i \omega_j(z) \\
 &\quad \left. + \left[\partial \ln \Theta_{\delta}(2z - 2\Delta) - 2\partial \ln \Theta - 2 \sum \partial_1 \ln E(z, w_i) - 4\partial \ln \sigma(z) \right]^2 - \left[\partial \ln \Theta + \sum \partial_1 \ln E(z, w_i) + 3\partial \ln \sigma(z) \right]^2 \right\}_{z_a=z}
 \end{aligned}$$

etc.

where the argument of Θ is $\sum w_i - 3\Delta$, when not indicated. Of course we are ultimately interested in $W_{\delta,0}$. Eq.(12) can be verified alternatively by taking the expectation value of the operator product expansion

$$X(z_1) X(z_2) \sim (z_1 - z_2)^{-4} (-D + 10) e^{2\phi(z_2)} + (z_1 - z_2)^{-3} (-D + 10) \partial\phi e^{2\phi(z_2)} \\ + (z_1 - z_2)^{-2} 2 [T^X + T^\psi + T^{bc} - T^{\eta\xi} + T^\phi - 2c \partial b + 2bc \partial\phi] e^{2\phi(z_2)} + O((z_1 - z_2)^{-1}) \quad (13)$$

where T^* denote the stress-energy tensors of the various systems. It is worth noting that the two leading singularities are absent for $D=10$ as a result of BRST-invariance. This is different from ref.[5] where only $W_{\delta^{11}}$ was taken into account.

Since $W_{\delta,2}$ and $W_{\delta,1}$ are non-zero, we have to perform the sum over spin structures to see the remaining poles disappear. To achieve this the phases ϵ_δ in Eq.(4) have to be specified. For $D=10$ superstrings we can treat left- and right-movers separately. A unique choice

$$\epsilon_\delta = +1 \quad \text{for all } \delta \quad (14)$$

emerges from demanding that physical correlations involving fermion emission vertices $V_f(z)$ are periodic under shifts of z around homology cycles [5]. For example it is relatively easy to look at the spin structure dependent part of the matter contribution to the two-fermion amplitude. In other words, Eq.(14) implements the GSO projection and is necessary to render amplitudes well-defined. As the latter factorize on the vacuum amplitude, Eq.(14) should also be imposed here.

To evaluate the spin structure sum it is very useful to move the insertion point $z \equiv z_2$ to one of the six ramification points r_α on the surface given by

$$r_\alpha = \Delta + \tau\alpha_1 + \alpha_2 \quad (15)$$

where $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ denotes an arbitrary odd reference spin structure [11,5,7]. Note that Δ drops out of the arguments of all Θ_δ in Eq.(12). With the use of

$$\Theta_\delta(2(\tau\alpha_1 + \alpha_2)) = e^{-4\pi i \alpha_1 \tau \alpha_1} \eta_\delta[\alpha] \Theta_\delta(0) \quad (16)$$

$$\eta_\delta[\alpha] = \exp \{4\pi i (\delta_1 \cdot \alpha_2 - \delta_2 \cdot \alpha_1)\}$$

some theta functions originating from the spinor ghost systems are cancelled in Eq.(12) by longitudinal ψ^μ degrees of freedom. At this point we recover the usual α -dependent phase [5]. This allows the application of the Riemann identity [12]

$$\sum_{\delta} \eta_{\delta}[\alpha] \Theta_{\delta}(\vec{x}) \Theta_{\delta}(\vec{y}) \Theta_{\delta}(\vec{u}) \Theta_{\delta}(\vec{v}) \quad (17)$$

$$= 4 \Theta_{\alpha}(\frac{\vec{x}+\vec{y}+\vec{u}+\vec{v}}{2}) \Theta_{\alpha}(\frac{\vec{x}+\vec{y}-\vec{u}-\vec{v}}{2}) \Theta_{\alpha}(\frac{\vec{x}-\vec{y}+\vec{u}-\vec{v}}{2}) \Theta_{\alpha}(\frac{\vec{x}-\vec{y}-\vec{u}+\vec{v}}{2})$$

and its derivatives. Since all arguments of Θ_{δ} appear to be zero we only have to keep terms proportional to

$$\begin{aligned} \sum_{\delta} \eta_{\delta}[\alpha] \Theta_{\delta}(0)^4 &= 0 \quad \text{in } W_{\delta,4} \cdots W_{\delta,0} \\ \sum_{\delta} \eta_{\delta}[\alpha] \Theta_{\delta}(0)^3 \Theta_{\delta}''(0) &= 0 \quad \text{in } W_{\delta,2} \cdots W_{\delta,0} \\ \sum_{\delta} \eta_{\delta}[\alpha] \Theta_{\delta}(0)^3 \Theta_{\delta}'''(0) &= -2 \Theta_{\alpha}'(0)^4 \quad \text{in } W_{\delta,0} \\ \sum_{\delta} \eta_{\delta}[\alpha] \Theta_{\delta}(0)^2 \Theta_{\delta}''(0)^2 &= +6 \Theta_{\alpha}'(0)^4 \quad \text{in } W_{\delta,0} \end{aligned} \quad (18)$$

where $\Theta_{\delta}'(0)$ stands for $\partial_i \Theta_{\delta}(\vec{0})$ etc. (indices suppressed). As a side remark, $W^{11} = \sum_{\delta} W_{\delta}^{11}$ from Eq.(9) vanishes already for arbitrary z_1 , provided $z_2 = r_{\alpha}$. Hence, only the spinor ghosts may contribute to the partition function. It follows from Eq.(18) that all poles in $(z_1 - z_2)$ disappear since $W_{\delta,2}$ and $W_{\delta,1}$ now sum to zero. Furthermore, we see at least four derivatives acting on Θ_{δ} are needed to generate a non-zero contribution. After extending the expansion Eq.(11) to the finite part and collecting the relevant terms in $W_{\delta,0}$ we find with Eq.(18)

$$W(w_i, z_a = r_{\alpha}) = Z_1^{-7} Z(\sum w_i - 3\Delta) \sigma(r_{\alpha})^{-4} e^{-4\pi i \alpha_1 \tau \alpha_1} (-4) [\omega_i(r_{\alpha}) \partial_i \Theta_{\alpha}(0)]^4 \quad (19)$$

For arbitrary genus g and odd spin structure α the function $g_{\alpha}(z) := \omega_i(z) \partial_i \Theta_{\alpha}(0)$ is known to have double zeros at all non-trivial zeros $p_1, \dots, p_{g-1}$ of $\Theta_{\alpha}(0)$. The Riemann vanishing theorem [13] yields

$$\sum_{j=1}^{g-1} p_j = \Delta + \tau \alpha_1 + \bar{\alpha}_2, \quad (20)$$

so for $g=2$ this point belongs to the surface and

$$p_1 = r_{\alpha}, \quad g_{\alpha}(r_{\alpha}) = 0 \quad (21)$$

which makes W vanish for our particular choice $x = z_a = r_{\alpha}$ of insertion points. Hence, using an argument of ref.[1], we proved that

$$W(\tau, z_a = r_{\alpha}(\tau_0)) = \frac{\partial}{\partial \tau} Y_{\alpha}(\tau), \quad Y_{\alpha}(\tau_0) = 0 \quad (22)$$

As a result, Γ_2 is locally a total derivative on moduli space. Although the above computation has been done for the type II superstring it applies as well to heterotic superstrings due to

holomorphic factorization. In comparison, the result of ref.[11] seems to be slightly more general ($x = \infty, z_1 = r_\alpha, z_2 = r'_\alpha$). It is not clear to us if the choice $z_\alpha = r_\alpha(\tau_0)$ can be replaced by $z_\alpha = r_\alpha(\tau)$ in the modular integrand Eq.(1). This would yield a point-by-point vanishing: $W(\tau) \equiv 0$ (the so-called unitary gauge); however, it questions the use of Eq.(5) since $\frac{\partial z_\alpha}{\partial \tau} = 0$ was assumed in its derivation (see ref.[1]).

It is straightforward to extend to above calculation to higher loops and higher-point amplitudes. An explicit verification of the non-renormalization theorem [1-3] would naturally be the next subject to investigate. It might also be of interest to examine the two-loop cosmological constant of the $o(16) \times o(16)$ string [14]. It is important to note that the choice of insertion points and the Riemann identity are not modular invariant, so intermediate stages of the calculation appear to partially break modular invariance. However, a modular transformation merely displaces the insertion point r_α which can be undone by adding a total derivative in moduli space. For a vanishing result, of course, this is irrelevant. It would nevertheless be interesting to see this mechanism realized in a case where the final answer is non-zero.

Note added: After completion of this work we learned from J. Atick and G. Moore about ref.[15] where similar results have been obtained.

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