QUANTUM SPINNING PARTICLE IN A CURVED METRIC

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A B S T R A C T

We show that the combined use of $N=1$ supersymmetry and general co-ordinate invariance allows to completely fix the operator ordering ambiguity for a non-relativistic interacting spinning particle.

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1.

The problem of giving an unambiguous quantum mechanical description of a particle in a curved metric has been repeatedly investigated

The advent of supersymmetry has provided a new way of approaching this question. The reason is that now the ordering of operators in the hamiltonian turns out to be completely fixed - via the SUSY algebra - by the knowledge of the SUSY charges. Since those charges are linear in the canonical momentum, the ordering ambiguity at this level is much easier to handle. Moreover, one is naturally led to imagine that the residual ambiguity can be completely removed by invoking general arguments, such as invariance under general coordinate transformations.

This programme has been successfully applied in a previous paper [2] to the N=2 supersymmetric quantum mechanics in a curved metric.

The N=2 model is interesting in its own right and exhibits beautiful algebraic properties, however the N=1 case is closer to the physical description of the Pauli spinning particle [3]. In fact, the N=1 SUSY theory contains a single irreducible set of real Grassmann variables Ψ^μ , which upon quantization are to be identified with the Pauli matrices σ^μ . Moreover the N=1 case allows the coupling with an (external) electromagnetic field (this is forbidden in the N=2 model). Therefore, even if most of the treatment is common to both cases, we wish to offer a separate discussion of the N=1 model because of its important distinguishing features.

2.

The model under consideration is described in n space dimension by n bosonic coordinates $x^\mu(t)$ and n real Grassmann variables $\psi^\mu(t)$. The lagrangian is

$$\begin{aligned} L = & \frac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + \frac{i}{2} g_{\mu\nu}(x) \psi^\mu \dot{\psi}^\nu + \\ & + \frac{i}{2} \Gamma_{\mu\nu\lambda}(x) \psi^\mu \psi^\nu \dot{x}^\lambda - \dot{x}^\mu A_\mu(x) + \frac{i}{2} F_{\mu\nu}(x) \psi^\mu \psi^\nu \end{aligned} \quad (1)$$

This is seen to describe a nonrelativistic spinning particle in a curved metric $g_{\mu\nu}(x)$ interacting with a vector potential $A_\mu(x)$. The indices μ and ν are space indices, and $A_\mu(x)$ and $g_{\mu\nu}(x)$ do not depend explicitly on time. Although space and time intervene in a completely nonsymmetric way, we hope that the structure of this model would guide us toward a physical space-time treatment of a relativistic spinning particle. The action following from the lagrangian (1) is invariant under the SUSY transformations

$$\begin{aligned} \delta x^\mu &= -i \varepsilon \psi^\mu, \\ \delta \psi^\mu &= \varepsilon \dot{x}^\mu \end{aligned} \quad (2)$$

and the corresponding generator is

$$Q = g_{\mu\nu}(x) \dot{x}^\mu \psi^\nu \quad (3)$$

Owing to the presence of $g_{\mu\nu}(x)$, the canonical formulation requires a little care. The appropriate way to proceed has been discussed thoroughly by many authors [4]. It amounts to introduce the "vielbeine" $e^a_\mu(x)$ such that

$$g_{\mu\nu}(x) = e^a_\mu(x) e^a_\nu(x), \quad e^a_\mu(x) e^b_\nu(x) = \delta^a_b \quad (4)$$

By defining the new fermionic variables

$$\psi^a = e_\mu^a(x) \psi^\mu \quad (5)$$

the lagrangian (1) can be written in the form

$$L = \frac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu - \frac{i}{2} \omega_{\mu,ab}(x) \psi^a \psi^b \dot{x}^\mu + \frac{i}{2} \psi^a \dot{\psi}^a - \dot{x}^\mu A_\mu(x) + \frac{i}{2} F_{\mu\nu}(x) e_\mu^a(x) e_\nu^b(x) \psi^a \psi^b, \quad (6)$$

where the spin connection $\omega_{\mu,ab}(x)$ is defined as

$$\omega_{\mu,ab}(x) = -e_{a\lambda}(x) \partial_\mu e_b^\lambda(x) - e_{a\nu}(x) \Gamma_{\mu\lambda}^\nu(x) e_b^\lambda(x). \quad (7)$$

Indeed, we see explicitly from the lagrangian (6) that the fermionic momenta $p_a = \partial L / \partial \dot{\psi}^a$ coincide now (up to an inessential factor) with the fermionic coordinates ψ^a . We have a $2n$ -dimensional bosonic phase-space but a n -dimensional fermionic phase-space, as it is appropriate to real Grassman variables [3].

The canonical momentum corresponding to the bosonic coordinate x^μ is

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = g_{\mu\nu}(x) \dot{x}^\nu - \frac{i}{2} \omega_{\mu,ab}(x) \psi^a \psi^b - A_\mu(x) \quad (8)$$

We thus have the "classical" Poisson brackets

$$\begin{aligned} \{x^\mu, p_\nu\}_{PB} &= i \delta^\mu_\nu, \\ \{\psi^a, \psi^b\}_{PB} &= \delta_{ab} \end{aligned} \quad (9)$$

(all the others vanish).

It is useful to define

$$\pi_\mu = p_\mu + \frac{i}{2} \omega_{\mu,ab}(x) \psi^a \psi^b + A_\mu(x) \quad (10)$$

so that

$$Q = \pi_\mu \Psi^\mu \quad (11)$$

We then obtain for the hamiltonian the following expression

$$\begin{aligned} H = & \frac{1}{2} g^{\mu\nu}(x) \pi_\mu \pi_\nu + \pi_\mu A^\mu(x) + \frac{1}{2} A_\mu(x) A^\mu(x) - \\ & - \frac{i}{2} F_{\mu\nu}(x) e^\mu_a(x) e^\nu_b(x) \psi^a \psi^b \end{aligned} \quad (12)$$

3.

The quantum version of the model is defined in terms of the independent quantum variables (x^μ, p_μ, Ψ^a) , which obey the following canonical commutation and anticommutation relations

$$[x^\mu, p_\nu] = i \delta^\mu_\nu \quad (13)$$

$$\{\Psi^a, \Psi^b\} = \delta_{ab}$$

(all the others vanish). We work with a Hilbert space with scalar product defined as *)

$$\langle \varphi_1, \varphi_2 \rangle = \sum_\xi \int d^n x \varphi_1^*(x, \xi, t) \varphi_2(x, \xi, t). \quad (14)$$

*) If one wishes to use the invariant measure $\sqrt{g} d^n x$ in the definition of the scalar product, one is led to operators $\hat{\Omega}$ which are related to ours Ω by the formula $\hat{\Omega} = g^{-1/4} \Omega g^{1/4}$.

Let us examine the ordering properties of the relevant operators.

a) The quantum counterpart of eq.(10) i.e.

$$\pi_\mu = p_\mu + \frac{1}{2} \omega_{\mu,ab}(x) \Sigma_{ab} + A_\mu(x), \quad (15)$$

$$\Sigma_{ab} = \frac{i}{2} [\psi_a, \psi_b],$$

is uniquely defined thanks to the $a \leftrightarrow b$ antisymmetry of $\omega_{\mu,ab}(x)$. Its

commutation relations are easily computed

$$\begin{aligned} [x^\mu, \pi_\nu] &= i \delta^\mu_\nu, \quad [\pi_\mu, \psi^\nu] = i T_{\mu\lambda}^\nu \psi^\lambda, \\ [\pi_\mu, \pi_\nu] &= -\frac{i}{2} R_{\mu\nu,ab} \Sigma_{ab} - \frac{i}{4} F_{\mu\nu}, \end{aligned} \quad (16)$$

with

$$R_{\mu\nu,ab} = \partial_\mu \omega_{\nu,ab} - \partial_\nu \omega_{\mu,ab} - \omega_{\mu,ac} \omega_{\nu,cb} + \omega_{\nu,ac} \omega_{\mu,cb}. \quad (17)$$

b) We now come to the problem of giving a unique quantum expression for the SUSY charge Q . The natural choice is the hermitian expression

$$Q = 1/2 \{ \pi_\mu, \Psi^\mu \} \quad (18)$$

It is to be noticed that on purely hermiticity grounds a term

$$i [\pi_\mu, \Psi^\mu] = -\partial_\mu \ln \sqrt{g} \Psi^\mu \quad (19)$$

cannot be excluded. It is rewarding to verify, as it will be done in the Appendix, that the choice (18) guarantees the invariance of Q (and hence of H) under general coordinate transformations and local rotations.

c) The quantum hamiltonian H is now uniquely defined through the SUSY generator Q as

$$H = Q^2 \quad (20)$$

A little algebra yields

$$H = \frac{1}{2} g^{-1/4} \pi_\mu g^{\mu\nu} g^{1/2} \pi_\nu g^{-1/4} - \frac{1}{8} R - \frac{1}{2} F_{\mu\nu} \sum_{ab} e_a^\mu e_b^\nu \quad (21)$$

with R the curvature scalar. It is amusing to notice that SUSY leads to a magnetic interaction with the "correct" gyromagnetic ratio. This nonminimal term in H follows, through eq.(20), from the form of Q , where $A_\mu(x)$ appears only through the minimal coupling $p_\mu \rightarrow p_\mu - A_\mu(x)$ (see eq.(15)).

We now examine the ordering properties of the hamiltonian (21) in the basic set of canonical variables (x^μ, p_μ, Ψ^a) . In particular, we are interested in the relation with Weyl ordering ^{*)}, whose special rôle is due to the link with the mid-point prescription of the path-integral approach [5]. With a slightly cumbersome calculation, which parallels the one presented in [2], one obtains

$$H = \frac{1}{2} (\pi_\mu g^{\mu\nu} \pi_\nu - F_{\mu\nu} \sum_{ab} e_a^\mu e_b^\nu)_{\text{W.O.}} + \Delta V \quad (22)$$

where the effective potential ΔV is

$$\Delta V = \frac{1}{8} g^{\mu\nu} (\Gamma_{\mu\sigma}^\lambda \Gamma_{\nu\lambda}^\sigma + \frac{1}{2} \omega_{\mu,ab} \omega_{\nu,ab}) \quad (23)$$

^{*)} The Weyl ordering with respect to bosonic and fermionic variables is defined in [2], where the main properties are discussed.

At this point, it is interesting to make a comparison with the $N=2$ case [2], where the similar vielbein basis $(x^\mu, p^\mu, \Psi^a, \bar{\Psi}^a)$ exists. The main difference is that in the $N=2$ model there are $2n$ fermionic variables. This allows the introduction of two further canonical sets of basic quantum variables $(x^\mu, p_\mu^{(1)}, \Psi^\mu, i\bar{\Psi}_\mu)$ and $(x^\mu, p_\mu^{(2)}, \bar{\Psi}^\mu, -i\Psi_\mu)$, in which spinors have still world indices. In the Table we list the various bosonic and fermionic contributions to the effective potential for both models and the purely bosonic case. We see that a partial cancellation between these two contributions always occurs. In order to have a complete cancellation, it turns out that the two following requirements should be met.

a) Equal number of bosonic and fermionic degrees of freedom. This is true in the $N=2$ case ($2n$ bosons and $2n$ fermions) but not in the $N=1$ case ($2n$ bosons and n fermions).

b) Quantization should treat - in the spirit of SUSY - bosonic and fermionic variables on the same footing. In the $N=2$ model this is the case in the two bases $(x^\mu, p_\mu^{(1)}, \Psi^\mu, i\bar{\Psi}_\mu)$ and $(x^\mu, p_\mu^{(2)}, \bar{\Psi}^\mu, -i\Psi_\mu)$, and we indeed get a complete cancellation, but this fails to happen in the basis $(x^\mu, p_\mu, \Psi^a, \bar{\Psi}^a)$. In the $N=1$ model fermions must be quantized in the vielbein basis (x^μ, p_μ, Ψ^a) , and the effective potential has the same form as in the $N=2$ case for the analogous basis.

The general case is shown in the Table. In all SUSY cases the terms in the curvature always cancel. On the other hand, there is a mismatch in the contributions containing the connections: the $1/2$ factor in the $N=1$ contribution reflects the fact that in this case there is half of the fermionic degrees of freedom of the $N=2$ model.

4.

We have seen that the programme to use supersymmetry to solve the problem of operator ordering in a curved metric can be successfully carried out both for the $N=1$ and $N=2$ models.

In the example of the present paper ($N=1$ case), where a single SUSY charge Q is present, it is possible (unlike the $N=2$ case) to introduce the interaction with an (external) electromagnetic field, with a magnetic interaction which yields the "correct" gyromagnetic ratio.

If we wish to proceed toward the treatment of a relativistic string interacting with external fields, we are led - as an intermediate step - to consider the theory of a relativistic interacting spinning particle. This would require the extension of our approach to the case of constrained systems. Work is in progress in this direction.

APPENDIX

We want now to show that the hamiltonian (21) is invariant under both general coordinate transformation (GCT) and local rotations (LR). Obviously - thanks to SUSY - it is enough to prove these invariances for the SUSY charge Q as given by eq. (18).

i) An infinitesimal GCT

$$x^\mu \rightarrow x'^\mu = x^\mu + f^\mu(x) \quad (A.1)$$

induces in any dynamical variable A a total variation

$$\Delta_{\text{GCT}} A = \delta_{\text{GCT}} A + i [G_{\text{GCT}}, A] \quad (A.2)$$

where $\delta_{\text{GCT}} A$ is the variation brought about by (A.1) in the functional form of A , and G_{GCT} is the self-adjoint generator of the unitarily implemented GCT

$$G_{\text{GCT}} = \frac{1}{2} \{ p_\lambda, f^\lambda(x) \} \quad (A.3)$$

which is uniquely defined [1]. Standard calculations yield

$$\Delta_{\text{GCT}} \psi^\mu = \partial_\lambda f^\mu(x) \cdot \psi^\lambda, \quad (A.4)$$

$$\Delta_{\text{GCT}} \pi_\mu = -\frac{1}{2} \{ \pi_\lambda, \partial_\mu f^\lambda \}.$$

It is then easy to conclude that

$$\Delta_{\text{GCT}} Q = 0. \quad (A.5)$$

ii) An infinitesimal LR is defined by

$$e_{\mu}^a(x) \rightarrow e_{\mu}^{'a}(x) = e_{\mu}^a(x) + \lambda_{ab}(x) e_{\mu}^b(x), \quad \lambda_{ba} = -\lambda_{ab}, \quad (A.6)$$

and induces the following variation in the spin connection defined in eq.(7)

$$\delta_{LR} \omega_{\mu,ab} = \lambda_{ac} \omega_{\mu,cb} + \lambda_{bd} \omega_{\mu,ad} + \partial_{\mu} \lambda_{ab} \quad (A.7)$$

The total variation of any dynamical variable A under an infinitesimal LR is given by

$$\Delta_{LR} A = \delta_{LR} A + i [G_{LR}, A], \quad (A.8)$$

where now $\delta_{LR} A$ is the variation in the functional dependence of A on the spin connection $\omega_{\mu,ab}$ and G_{LR} is the self-adjoint generator of unitarily implemented LR. Now, x^{μ} and Ψ^{μ} are obviously invariant under LR. By using the definition of Ψ^a (eq.(5)), eq.(A.6) then yields

$$\Delta_{LR} \Psi^a = \lambda_{ab}(x) \Psi^b \quad (A.9)$$

Yet, we can apply eq.(A.8) to Ψ^a . Consistency with eq.(A.9) gives the explicit form of G_{LR}

$$G_{LR} = \frac{1}{2} \lambda_{ab} \sum^{ab} \quad (A.10)$$

The case of p_μ is different, due to the x -dependence of λ_{ab} . Application of eqs.(A.8)

and (A.9) to p_μ implies

$$\Delta_{LR} p_\mu = -\frac{1}{2} \partial_\mu \lambda_{ab} \sum^{ab} \quad (A.11)$$

Next, we want to compute $\Delta_{LR} \pi_\mu$. To this end, we apply eqs. (A.8) and (A.9) to

$\omega_{\mu,ab} \sum^{ab}$. Use of the identity

$$[\sum^{ab}, \sum^{cd}] = i(\delta^{ac} \sum^{db} - \delta^{ad} \sum^{cb} + \delta^{bc} \sum^{ad} - \delta^{bd} \sum^{ac}) \quad (A.12)$$

entails

$$\Delta_{LR} (\omega_{\mu,ab} \sum^{ab}) = \partial_\mu \lambda_{ab} \sum^{ab} \quad (A.13)$$

and combining eqs.(A.11) and (A.13) we obtain

$$\Delta_{LR} \pi_\mu = 0 \quad (A.14)$$

We thus conclude

$$\Delta_{LR} Q = 0 \quad (A.15)$$

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TABLE

	ΔV BOSONIC	ΔV FERMIONIC
$N = 1$ SUSY (x^μ, p_μ, Ψ^a)	$\frac{1}{8} (g^{\mu\nu} \Gamma_{\mu\sigma}^\lambda \Gamma_{\nu\lambda}^\sigma + R)$	$\frac{1}{8} \left(\frac{1}{2} g^{\mu\nu} \omega_{\mu,ab} \omega_{\nu,ab} - R \right)$
$N = 2$ SUSY $(x^\mu, p_\mu, \Psi^a, \bar{\Psi}^a)$	$\frac{1}{8} (g^{\mu\nu} \Gamma_{\mu\sigma}^\lambda \Gamma_{\nu\lambda}^\sigma + R)$	$\frac{1}{8} (g^{\mu\nu} \omega_{\mu,ab} \omega_{\nu,ab} - R)$
$N = 2$ SUSY $(x^\mu, p_\mu^{(1)}, \Psi^\mu, i \bar{\Psi}_\mu)$ $(x^\mu, p_\mu^{(2)}, \bar{\Psi}^\mu, -i \Psi_\mu)$	$\frac{1}{8} (g^{\mu\nu} \Gamma_{\mu\sigma}^\lambda \Gamma_{\nu\lambda}^\sigma + R)$	$-\frac{1}{8} (g^{\mu\nu} \Gamma_{\mu\sigma}^\lambda \Gamma_{\nu\lambda}^\sigma + R)$