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Phenomenological studies of rank-5 Calabi-Yau models

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Abstract

We study the phenomenology of heterotic superstring models compactified on Calabi-Yau manifolds with non-Abelian fundamental groups which can yield the rank-5 gauge group $SU(3)_C \times SU(2)_L \times U(1)^2$ in four dimensions. We examine one 2-generation model and several 4-generation models. They all have fatal problems with proton decay and/or neutrino masses. The most interesting 4-generation model avoids rapid proton decay at the expense of a neutrino weighing more than the proton and an unrealistic pattern of Cabibbo mixing.

I. Introduction

The enthused optimism that some superstring theory ^[1,2] (SST) may lead to the Standard Model has abated due to persistent phenomenological difficulties in realizing such a scenario. To extract low energy physics from SSTs, various avenues have been pursued, notably compactification of 10-dimensional SST ^[3] and SSTs formulated directly in four dimensions ^[4]. It is generally believed that the formulation of 4-dimensional SSTs may contain the 10-dimensional SST compactified on K_6 where K_6 is a compact six-dimensional space. However, at present the direct connection between these two approaches is ill-understood.

On the other hand, in the former approach, under some very reasonable assumptions about the resultant low-energy theory, a special compactification scheme is almost uniquely picked out [3]. If we start with the heterotic string theory [5] with an $E_8 \times E_8$ gauge group in 10 space-time dimensions, then under the assumption of $K_6 \times M_4$ compactification, $N = 1$ supersymmetry and chiral fermions, the compactification scheme which is greatly favoured is the one in which K_6 is a Kahler manifold with $SU(3)$ holonomy [3] (referred to as a Calabi-Yau manifold). Embedding the holonomy simply in the gauge group leads to an anomaly-free theory with an $E_6 \times E_8$ gauge group and is also a solution of string field equations. In addition to $N = 1$ supersymmetry and chiral fermions in 27 representations of E_6 (which has the right quantum numbers to incorporate all the quarks and leptons in a generation, and then some), on multiply-connected Calabi-Yau spaces it can happen that not all gauge fields can be gauged to zero, whereby E_6 breaks down to a smaller rank-5 or rank-6 gauge group^[6]. The Standard Model could then be obtained through additional symmetry breaking

by the conventional Higgs mechanism. Due to this rich structure, a great deal of attention has rightly been focused on the phenomenological implications of Calabi-Yau compactifications^[7,8,9,10]. However, the possibility of obtaining the Standard Model through Calabi-Yau compactification has been frustrated by the lingering problems of baryon number violation, ν masses, $\sin^2 \theta_W$, flavour-changing neutral currents (FCNC) etc.. In two of the 'true' superstring models analyzed so far (based on CP^4 and $CP^3 \times CP^3$ respectively^[8,9]) one or more of the above mentioned problems seems unsurmountable. In both of these models Calabi-Yau spaces have Abelian fundamental groups and hence E_6 is broken to one of its rank-6 subgroups through nontrivial Wilson loops. Generically such models are known to have phenomenological difficulties^[9]. It has been argued in the literature^[11,12] that a manifold with a non-Abelian fundamental group which can lead to a rank-5 low-energy gauge group may have desirable properties and may lead to agreeable phenomenology.

In this paper we explore this possibility in 'true' superstring models based on the space $Y_{7:2,2,2,2}$, i.e. CP^7 with four quadratics set to zero. In particular, we examine two specific models with non-Abelian fundamental groups, deduce their symmetry properties and analyze their phenomenology.

The organization of this paper is as follows. In Section II we discuss general features of rank-5 models indicating the problems likely to be encountered and discrete symmetries which could avoid them. In Section III we discuss a specific 2-generation toy model, which serves as a warm-up for the 4-generation models introduced in Section IV. A detailed analysis of the symmetries of these models reveals one interesting possibility, whose phenomenology is discussed in Section V. We find that it can avoid rapid

proton decay, but only at the expense of unrealistic Cabibbo-Kobayashi-Maskawa mixing and neutrino masses. Finally, Section VI summarizes our conclusions and indicates some directions for future research.

II. Rank-5 models: some generalities.

In this section we discuss some general features of rank-5 models^[11] and list the constraints on the symmetry breaking, properties of the manifold, Yukawa couplings etc.

In a compactification scheme in which K_6 is a Calabi-Yau manifold and holonomy is simply embedded in the gauge group, the massless spectrum of the effective theory consists of $n_g(\underline{27}) + n_{\bar{g}}(\overline{\underline{27}}) + n_s(\underline{1})$ where the numbers in the brackets denote the transformation properties of the fields under E_6 . Since the massless fields in four dimensions are in one-to-one correspondence with the appropriate zero modes on K_6 , n_g , $n_{\bar{g}}$ and n_s can be obtained from topological considerations^[3,13]. The numbers of gauge non-singlet fields are given by $n_g = b_{21}$ and $n_{\bar{g}} = b_{11}$, where b_{nm} are the Betti-Hodge numbers. The net number of generations $n_F = n_g - n_{\bar{g}} = b_{21} - b_{11} = \frac{1}{2}|\chi|$, where χ is the Euler characteristic of the Calabi-Yau manifold. The field content of the $\underline{27}$ of E_6 in representations of the Standard Model is given in Table I. Note that in addition to a generation of quarks and leptons, the $\underline{27}$ also contains a pair of Higgs ($H^a, \bar{H}^a$) with conventional Higgs quantum numbers which may be used to effect weak gauge symmetry breaking.

On multiply-connected Calabi-Yau spaces $K = M/G$, and depending whether G is Abelian or non-Abelian, E_6 can be broken to rank-6 or rank-5 subgroups by non-trivial Wilson loops^[6]. The possible rank-6 subgroups which are accessible through this mechanism have been enumerated elsewhere^[14]. To arrive at the Standard Model from these subgroups, one needs to reduce the rank by 2, which is generally achieved by non-zero expectation values (V.E.V.'s) of ν^c and N fields at some intermediate scale M_I . However, to

sufficiently suppress baryon-number-violating interactions and to obtain an acceptable value of $\sin^2 \theta_W$, the intermediate scale M_I of symmetry breaking needs to be $\geq 10^{16}$ GeV^[9]. Such a large intermediate scale of symmetry breaking is difficult to generate^[15]. Even when the theory has a flat direction up to some order, the conventional mechanism which drives $m_{\nu^c}^2 < 0$ or $m_N^2 < 0$ is either nonexistent or is in conflict with various phenomenological constraints. In addition, models with $\langle \nu^c \rangle \neq 0$ have 'prima facie' problems with lepton number violation, ν masses, etc.^[16]. Moreover, with $M_I \geq 10^{16}$ GeV, the actual models considered so far do not have a light Higgs among the particle spectrum, but rather the mass of the Higgs turns out to be $O(10^6)$ GeV or higher. Hence the conventional breakdown of $SU(2) \times U(1)$ is not possible^[17]. As has been noted before, both of these problems can be circumvented by going directly to a rank-5 subgroup of E_6 which is only accessible on a manifold with non-Abelian G .

The representation of G embedded in the maximal $SU(3) \times SU(3) \times SU(3)$ subgroup of E_6 takes the form

$$U = [1] \otimes \begin{bmatrix} \alpha & & \\ & \alpha & \\ & & \alpha^{-2} \end{bmatrix} \otimes \begin{bmatrix} \beta^{-2} & & \\ & & \\ & & \beta^{V_g} \end{bmatrix}$$

where V_g is a two-dimensional representation of the discrete group G . The low-energy gauge group is uniquely $SU(3)_c \times SU(2)_L \times U(1)_Y \times U(1)$. Note that the embedding of the extra $U(1)$ is also fixed leading to well-defined neutral current phenomenology.

The embedding of U in E_6 also causes different components of the 27 representations to transform differently under G . The components of (27 +

27) which survive this process are the ones which remain invariant under G [18,19]. For certain choices of parameters different possibilities can be achieved, which are

	Parameters	Light fields	Constraints	
(1)	$\alpha = \beta$	$e^c + \bar{e}^c$	$\alpha \neq \beta^{-1}$	(1a)
(2)	$\alpha = \beta^{-2}$	$H + \bar{H}^c$	$\alpha \neq \beta^2$	(1b)
(3)	$\alpha^{-2} = 1$	$D + \bar{D}$	$\beta^{-2} \neq 1$	(1c)
(4)	$\beta^{-2} = 1$	$u^c + \bar{u}^c$	$\alpha^{-2} \neq 1$	(1d)

where $\bar{e}^c$, $\bar{H}^c$, $\bar{D}$ and $\bar{u}^c$ denote the mirrors of e^c , H , D and u^c . From a phenomenological point of view, none of these mirror fields serve any useful purpose in the low-energy theory. Hence, if the freedom in G allows, one may choose an embedding in which the constraints (1) are never satisfied.

In addition to the conventional Yukawa couplings of the Standard Model, $h^D Q H u^c + \bar{h}^D Q \bar{H} \bar{d} + h^c L \bar{H} e^c$, the low energy theory may contain the following sets of couplings

$$\begin{aligned}
 & (\lambda_L H L \nu^c + \lambda_D D d^c \nu^c) \quad ; \quad (\lambda H \bar{H} N + k D \bar{D}^c N) \\
 & (\lambda' D^c L Q + \lambda'' D e^c u^c) \quad ; \quad (\lambda''' D Q Q + \lambda^{iv} D^c u^c d^c)
 \end{aligned} \tag{2}$$

In rank-5 models the N field can acquire a V.E.V. of $O(m_W)$ through radiative symmetry breaking made possible by its couplings $\lambda H \bar{H} N$ and $k D D^c N$. Hence in these models the D and D^c fields remain relatively light. The problem of rapid proton decay can be addressed by noting that the sets of couplings (λ', λ'') and $(\lambda''', \lambda^{iv})$ do not violate baryon number individually [11,12]. Therefore a feasible theory may contain either set (but not both) of couplings without creating many problems with baryon-number-violating decays. Nonexistence of these couplings can occur either because of the symmetries of the manifold or through topological arguments or both.

For example, simple Z_2 symmetries which achieve this goal are

$$Z_2: \quad (3, \bar{3} \text{ of } SU(3)) \rightarrow (-1) (3, \bar{3} \text{ of } SU(3)) \quad (3a)$$

$$\nu^c \rightarrow -\nu^c$$

or

$$Z_2 \quad (L, e^c, \nu^c) \rightarrow (-1) (L, e^c, \nu^c) \quad (3b)$$

Note that we have assumed a family-independent symmetry which is not generally the case in Calabi-Yau compactification. In a generic case some of the baryon-number violating couplings may be harmless thanks to phase-space constraints (eg. $p \rightarrow \tau X$). Incorporating these would weaken the requirements on the kind of discrete symmetry which is necessary.

In what follows, we will examine different Calabi-Yau spaces with non-Abelian fundamental groups and inquire whether it is possible to find examples of rank-5 models which have discrete symmetries suitable for preventing rapid proton decay or any other phenomenological disaster.

III. A two-generation toy model.

Here we analyze a simple two-generation toy model based on a Calabi-Yau space with a non-Abelian fundamental group. Through this exercise we establish our notation, clarify some of the subtle points about the techniques involved and probe the nature of the global symmetries of the manifold.

Consider the $Y_{7:2,2,2,2}$ space which is defined as a subspace of CP^7 with four transversly intersecting^[20] quadratics set to zero. The Euler characteristic of this manifold is $\chi = -128$ corresponding to 64 generations. A two-generation manifold, first constructed by Strominger and Witten^[20], can be obtained by dividing the above manifold by the action of a freely acting discrete symmetry of 32 elements.

Labelling the eight complex co-ordinates of CP^7 as $(Z_1, Z_2, Z_3, Z_4, W_1, W_2, W_3, W_4)$, the defining polynomials are given by

$$p^1 = \frac{1}{2} (Z_0^2 + Z_2^2) + \epsilon Z_1 Z_3 + \delta W_0 W_2 \quad (4a)$$

$$p^2 = \frac{1}{2} (Z_1^2 + Z_3^2) + \epsilon Z_0 Z_2 + \delta W_1 W_3 \quad (4b)$$

$$p^3 = \delta Z_0 Z_2 + \epsilon' W_1 W_3 + \frac{1}{2} (W_0^2 + W_2^2) \quad (4c)$$

$$p^4 = \delta Z_1 Z_3 + \epsilon' W_0 W_2 + \frac{1}{2} (W_1^2 + W_3^2) \quad (4d)$$

*This means that the four form $\gamma = dp^1 \wedge dp^2 \wedge dp^3 \wedge dp^4$, where p^i are the quadratics, is never zero. This guarantees that the manifold is non-singular.

whereas the symmetry G is generated by

$$\begin{aligned}
 X(Z_k) &= Z_k + 1 & X(W_k) &= W_k + 1 \\
 V(Z_k) &= \beta^k Z_k & V(W_k) &= i\beta^{-k} W_k \\
 T(Z_k) &= Z_k & T(W_k) &= -W_k
 \end{aligned} \tag{5}$$

Since $XV = TVX$, G is non-Abelian. The 32-element group G has 16 one-dimensional representations corresponding to $X = i^m$, $V = i^n$, $T = 1$, and four two dimensional representations with $T = -1$ ^[21]. A specific two-dimensional representation can be easily constructed and is given by:

$$V_g: \quad X = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}; \quad V = \begin{bmatrix} i & \\ & -i \end{bmatrix}; \quad W = \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} \tag{6}$$

To accomplish the E_6 breaking to $SU(3)_c \times SU(2)_L \times U(1)_Y \times U(1)$ the following embedding of G in the $SU(3)^3$ subgroup of E_6 is employed:

$$U = \begin{bmatrix} 1 \end{bmatrix} \otimes \begin{bmatrix} \alpha & & \\ & \alpha & \\ & & \alpha^{-2} \end{bmatrix} \otimes \begin{bmatrix} \beta^{-2} & & \\ & \beta V_g & \\ & & \beta V_g \end{bmatrix}$$

Although both $\alpha^4 = 1$ and $\beta^4 = 1$, they differ from each other due to the fact that they correspond to different representations of the group G . If the 16 one-dimensional representations of G are labelled by $R_{m,n}$, corresponding to

$U_X = i^m$, $U_V = i^n$ then α and β correspond to representations $R_{1,0}$ and $R_{0,1}$ respectively. Explicitly the embedding (6,7) implies

$$U_X = [1] \otimes \begin{bmatrix} -i & & \\ & i & \\ & & i^2 \end{bmatrix} \otimes \begin{bmatrix} 1 & & \\ & 0 & i \\ & i & 0 \end{bmatrix} \quad (8a)$$

$$U_V = [1] \otimes [1] \otimes \begin{bmatrix} -1 & -1 & \\ & & 1 \end{bmatrix} \quad (8b)$$

$$U_T = [1] \otimes [1] \otimes \begin{bmatrix} -1 & -1 & \\ & & 1 \end{bmatrix} \quad (8c)$$

After this embedding, the $\underline{27}$ of E_6 splits up into various irreducible representations of the unbroken subgroup of E_6 , namely $SU(3) \times SU(2) \times U(1) \times U(1)$, and each irreducible representation transforms nontrivially under G . The transformation properties of these fields under the action of G are given in Table II.

The next step is to deduce the global properties of the manifold and determine how different fields transform under these symmetries. The manifold M is endowed with the following global symmetries^[21],

$$A: Z_i \rightarrow Z_i ; W_0, W_2 \rightarrow -W_0, -W_2 ; W_1, W_3 \rightarrow W_1, W_3 \quad (9a)$$

$$B: Z_0, Z_2 \leftrightarrow Z_1, Z_3 ; W_i \rightarrow W_i \quad (9b)$$

$$C: Z_0 \leftrightarrow Z_2 ; Z_1 \leftrightarrow Z_3 ; W_i \rightarrow W_i \quad (9c)$$

In general, the symmetries of M do not survive the passage from M to M/G . For a symmetry Π of M to be also a symmetry of M/G , it must satisfy^[18]

$$\Pi g \Pi^{-1} = g' \quad g, g' \in G \quad (10)$$

In our case it is straightforward to confirm that A , B and C are also symmetries of M/G .

To infer the transformation properties of various fields under A , B and C , there are two methods at our disposal: (a) to use Lefschetz index theorems^[21, 8] and (b) to use deformation theory to identify fields with the polynomials corresponding to deformations of the manifold^[22,23].

In the first approach for a transformation Π one starts with the Lefschetz number

$$L(\Pi) = \sum_{\substack{p=0 \\ q=0}}^n (-1)^{p+q} \text{Tr } H^{p,q}(\Pi) \quad (11)$$

where $H^{p,q}$ is the vector space spanned by (p, q) forms and $\text{Tr } H^{p,q}$ denotes the trace in the vector space $H^{p,q}$ of the action of Π . One then supplements the above equation with the Lefschetz fixed-point theorem.

$$L(\Pi) = \sum \chi(\mu_j) \quad (12)$$

where the μ_j are the submanifolds of M left fixed by Π and the $\chi(\mu_j)$ are the Euler characteristics of μ_j .

Knowing the transformation properties of (1,1) and (0,3) forms under Π , it is possible to infer the properties of (2,1) forms from eqs. (11, 12). In our case both the (1,2) form and the (0,3) form are invariant under Π since they correspond to the Kahler form and the unique holomorphic three-form respectively. Thus the transformation properties of the 27 can be indirectly extracted (for further details, see [21, 8])

In the present case we will use the second method to derive the transformation properties of different fields, partly because it is less tedious and partly because (in principle) it allows us to calculate Yukawa couplings explicitly^[22,23]. To briefly recapitulate the salient features of this method, we recall that the massless 27's are given by nontrivial (2,1) forms $W_{k\lambda\rho}$. Using the unique (anti) holomorphic three form $\bar{\Omega}$, of the Calabi-Yau space they can be shown to be in one-to-one correspondence with the elements of $H_1(T)$, i.e. closed but non-exact one-forms with values in the tangent space^[18];

$$A^\mu = \bar{\Omega}^{\mu k\lambda} W_{k\lambda\rho} d x^{\bar{\rho}} \in H_1(T). \quad (13)$$

an explicit representation of the elements of $H_1(T)$ can be obtained from deformation theory. The deformations of the manifold M may be realized as deformations of the polynomials p^α that define M ^[22]:

$$p^\alpha \rightarrow p^\alpha + \delta t q^\alpha. \quad (14)$$

The element of $H_1(T)$ that corresponds to the deformation q^α is

$$a^\mu = q^\alpha \chi_{\bar{\rho}}^\mu(p^\alpha) dx^{\bar{\rho}} \quad (15)$$

where $\chi_{\bar{\rho}}^\mu(p^\alpha)$ is the extrinsic curvature of M embedded in $\mathbb{CP}^7$. However not all of the one-forms with values in the tangent space obtained by eq. (15) are elements of $H_1(T)$, since some of them may be exact. The exact forms correspond to q^α for which deformation of M is trivial. They correspond to the deformations arising from the linear transformations of the coordinates or from the interchanges of defining polynomials.

For example the $Y_{7:2,2,2,2}$ space has $\chi = -128$, $b_{11} = 1$ and $b_{21} = 65$ leading to 64 generations. We can use deformation theory to obtain these numbers in the following way. The most general quadratics which define the manifold are

$$p^\alpha = \sum A_{ij}^{(\alpha)} \xi_i \xi_j = 0 \quad \alpha = 1, \dots, 4 \quad (16)$$

$$i, j = 1, \dots, 8$$

The total number of parameters in the above eq. is $4 \times 36 = 144$. However, some of the parameters are redundant. They can be eliminated by noting:

(a) in $\mathbb{CP}^{N-1}$, $N^2 - 1$ parameters can be removed by a change of basis-for our case $8^2 - 1 = 63$; (b) the overall scale of each polynomial is superfluous, and (c) if p^1 , p^2 , p^3 and p^4 are four transversely intersecting polynomials then

$$p^1 + \lambda_1 p^2 + \lambda_2 p^3 + \lambda_3 p^4 = 0$$

$$p^2 + \mu_1 p^1 + \mu_2 p^3 + \mu_3 p^4 = 0$$

$$p^3 + \nu_1 p^1 + \nu_2 p^2 + \nu_3 p^4 = 0$$

$$p^4 + \rho_1 p^1 + \rho_2 p^2 + \rho_3 p^3 = 0 \quad (17)$$

where λ_i , ν_i , μ_i , and ρ_i are complex constants, defines the same manifold. This freedom can be used to remove 12 more parameters. Thus we have $144 - 63 - 4 - 12 = 65$ independent parameters, as we must since $b_{21} = 65$.

Now we turn our attention to the polynomial representation of fields transforming in a definite fashion under G . The first point to note is that the defining polynomials $p^{(\alpha)}$: $\alpha = 1 \dots 4$ are individually not invariant under the transformations X , V and T of (5), even though the manifold is. Under X, V and T ,

$$\begin{array}{lll} X: & p^1 \rightarrow p^2 & V: & p^1 \rightarrow p^1 & T: & p^1 \rightarrow p^1 \\ & p^2 \rightarrow p^1 & & p^2 \rightarrow p^2 & & p^2 \rightarrow p^2 \\ & p^3 \rightarrow p^4 & & p^3 \rightarrow p^3 & & p^3 \rightarrow p^3 \\ & p^4 \rightarrow p^1 & & p^4 \rightarrow p^4 & & p^4 \rightarrow p^4 \end{array} \quad (18)$$

Generally it is sufficient to write the polynomial representation corresponding to a field ϕ as a column vector q^α . However, to ensure that the transformation properties of the $p^{(\alpha)}$ are taken into account, we must also take account of $\chi(p^\alpha)$ in specifying the representation. To illustrate, consider a typical field ϕ whose polynomial representation is given by

$$\phi = \begin{bmatrix} z_0 z_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (19)$$

Naively, ϕ would seem to be invariant under the symmetry $z_0 \leftrightarrow z_1$, $z_2 \leftrightarrow z_3$, $w_0 \leftrightarrow w_1$, $w_2 \leftrightarrow w_3$. However, under $z_0 \leftrightarrow z_2$, $p^1 \leftrightarrow p^2$ and

$$\phi \rightarrow \phi' = \begin{bmatrix} 0 \\ z_0 z_1 \\ 0 \\ 0 \end{bmatrix} \quad (20)$$

i.e. ϕ is not invariant under the above mentioned symmetry. The $\bar{\phi}$ invariant under the symmetry is

$$\bar{\phi} = \begin{bmatrix} z_0 z_1 x(p^1) \\ z_0 z_1 x(p^2) \\ 0 \\ 0 \end{bmatrix} \quad (21)$$

After incorporating this feature, the construction of the polynomial representation is a tedious but straightforward exercise. Note that the fields transforming as a 2-dimensional representation of G are given by a set of two polynomials with each polynomial being a column vector. This construction will become clearer with the following example.

$$\begin{array}{ccc}
& A & B & C \\
u^c: & \text{Transforms as } \beta^2 & & \\
u_1^c: & \begin{bmatrix} Z_0 Z_2 \chi(p^1) \\ Z_1 Z_3 \chi(p^3) \\ 0 \\ 0 \end{bmatrix} & + & + & + \\
u_2^c: & \begin{bmatrix} 0 \\ 0 \\ W_0 W_2 \chi(p^3) \\ W_1 W_3 \chi(p^4) \end{bmatrix} & + & + & +
\end{array} \tag{22}$$

The polynomial representations of other fields are given in Table III, where we have also listed the transformation properties of fields under A, B and C, established by examining the corresponding polynomial. Note that the symmetries A, B and C are generation-dependent symmetries, i.e. they are different for different generations. The superpotential consistent with all symmetries of the manifold can be easily constructed and is given by

$$\begin{aligned}
W \rightarrow & \lambda_1 \bar{H}_2 N_1 H_2 + \lambda_2 \bar{H}_2 N_1 H_1 + \lambda_3 \bar{H}_1 N_2 H_1 + \lambda_4 \bar{H}_1 N_2 H_2 \\
& + \rho_1 \bar{H}_1 d_2^c Q_1 + \rho_2 \bar{H}_1 d_2^c Q_2 + \rho_3 \bar{H}_2 d_1^c Q_1 + \rho_4 \bar{H}_2 d_1^c Q_2 \\
& + \mu_1 L_1 \nu_2^c H_1 + \mu_2 L_1 \nu_2^c H_2 + \mu_3 L_2 \nu_1^c H_1 + \mu_4 L_2 \nu_1^c H_2 \\
& + \sigma_1 D_1 d_1^c \nu_2^c + \sigma_2 D_2 d_1^c \nu_2^c + \sigma_3 D_1 d_2^c \nu_1^c + \sigma_4 D_2 d_2^c \nu_1^c
\end{aligned}$$

$$+ \sigma_5 D_1^C L_2 Q_1 + \sigma_6 D_1^C L_2 Q_2 + \sigma_7 D_2^C L_1 Q_1 + \sigma_8 D_2^C L_1 Q_2 \quad (23)$$

where λ_i , ρ_i , μ_i and σ_i are the corresponding Yukawa couplings.

Now we are ready to examine the phenomenology of the model. As noted in Section II, the presence of $\lambda N \bar{H} \bar{H}$ couplings will allow the extra $U(1)$ symmetry to be broken by radiative effects at $O(M_W)$. The disastrous problem seems to be proton decay mediated by D , D^C fields, the source of which can be traced back to the fact that the symmetries put constraints on the couplings σ_i which are not nearly strong enough to eliminate baryon-number-violating interactions. There also seem to be difficulties with ν masses, specifically the couplings μ_i , $i = 1 \dots 4$, lead to large ν masses.

To seek to salvage the situation one might choose to embed G differently in E_6 , i.e. choosing different U_G (eqn. (8)) which would lead to different transformation properties and hence different couplings. However, the case we have considered here is very generic and no other embeddings appear to give substantially different results. Another possibility one may explore is to choose the values of ϵ , ϵ' and δ in $p^{(\alpha)}$ in such a way that the global symmetry of the manifold is enlarged. Two such constructions are

$$\epsilon' = i\epsilon \quad (24)$$

which leads to an additional symmetry [20]

$$Z_k \rightarrow i^{1/2} \beta^{k^2/2} W_k \quad (25)$$

$$W_k \rightarrow i^{1/2} \beta^{-k^2/2} Z_k$$

or choose

$$\epsilon' = \epsilon \tag{26}$$

which has $Z_i \leftrightarrow W_i$ symmetry. Unfortunately, neither of these symmetries puts any new constraints on the couplings, and consequently they are of little help.

Various double covers to this manifold will lead to 4-generation models. However, all the 16-element subgroups of G are Abelian. The non-Abelian property needed to get a rank-5 gauge group is lost in the double cover. The Abelian part that remains is nothing but $Z_4 \times Z_2$. As noted earlier, such models are of limited phenomenological interest.

IV. Four generation examples:

Strominger and Witten^[20] have constructed various examples of Calabi-Yau spaces with $\chi = -8$, although very few of them have non-Abelian fundamental groups. They are (1) M_1 , based on the $Y_{7:2,2,2,2}$ space with a freely acting group G' (to be defined later), (2) M_2 based on the $Y_{7:2,2,2,2}$ space with a freely acting group $G'' \neq G'$, and (3) M_3 based on the $Y_{5:3,3}$ space. The last example is constructed by considering the quotient of $Y_{5:3,3}$ by a non-freely acting symmetry and then patching up the singularities. Such a manifold is not accessible to any detailed analysis with presently known techniques. Furthermore, as we will show, for M_1 the structure of G' is such that E_6 breaking to $SU(3) \times SU(2) \times U(1) \times U(1)$ is not possible and hence we will concentrate on the manifold M_2 .

The defining polynomials for M_1 and M_2 are given by ^[20]

$$\begin{aligned}
 p^1 &= \frac{1}{2} (z_8^2 + z_4^2) + \epsilon (z_1 z_7 + z_3 z_5) + \epsilon \lambda (z_1 z_3 + z_5 z_7) + \delta z_2 z_6 \\
 p^2 &= \frac{1}{2} (z_1^2 + z_5^2) + \epsilon (z_8 z_2 + z_4 z_6) + \epsilon \lambda (z_2 z_4 + z_6 z_8) + \delta z_3 z_7 \\
 p^3 &= \frac{1}{2} (z_2^2 + z_6^2) + \epsilon (z_1 z_3 + z_5 z_7) + \epsilon \lambda (z_3 z_5 + z_7 z_1) + \delta z_8 z_4 \\
 p^4 &= \frac{1}{2} (z_3^2 + z_7^2) + \epsilon (z_2 z_4 + z_6 z_8) + \epsilon \lambda (z_4 z_6 + z_8 z_2) + \delta z_5 z_1
 \end{aligned} \tag{27}$$

with G' generated by

$$W: z_i \rightarrow z_{3i+1}$$

$$X = S^2: Z_i \rightarrow Z_i + 2$$

$$V = T^4: Z_i \rightarrow (-1)^i Z_i$$

$$X^2 = W^2; XW = W^3X \quad (28)$$

and G'' generated by

$$W: Z_i \rightarrow Z_{3i+1}$$

$$T: Z_i \rightarrow \beta^i Z_i \quad \beta^4 = 1 \quad (29)$$

Both G' and G'' are clearly non-Abelian.

The 16-element group G' splits up into an 8-element subgroup $\bar{G}$ and Z_2 , i.e. $G' = \bar{G} \times Z_2$ where Z_2 is generated by V . The subgroup $\bar{G}$ has five conjugacy classes consisting of $(1, (W, W^3), (XW, WX), (X, X^3), X^2W^2)$. The character table for $\bar{G}$ is given in Table III. As is clear from examining Table IV, $\bar{G}$ admits four one-dimensional representations corresponding to $W = \pm 1$ and $X = \pm 1$ and a two-dimensional representation with $X^2W^2 = -1$. If we try to embed these representations in the $SU(3)^3$ maximal subgroup of E_6 in the form

$$\begin{bmatrix} 1 \end{bmatrix} \otimes \begin{bmatrix} \alpha & & \\ & \alpha & \\ & & \alpha^{-2} \end{bmatrix} \otimes \begin{bmatrix} \beta^{-2} & & \\ & \beta^V & \\ & & \beta^V \end{bmatrix} \quad (30)$$

then both α^{-2} and β^{-2} must be 1 since $\alpha = \pm 1$ either due to X, W or V. However, if $\alpha^{-2} = \beta^{-2} = 1$ then some of the generators in the coset space $E_6/SU(3)^3$ remain unbroken. Hence $\tilde{G} \times Z_2$ can only break E_6 to $SU(4) \times SU(2) \times U(1)$ if $\alpha = -1$, $\beta = \pm 1$ or $SU(6)$ if $\alpha = 1$, $\beta = \pm 1$, i.e. $E_6 \rightarrow SU(3) \times SU(2) \times U(1)$ cannot be achieved. One of the main reasons why the non-Abelian groups are attractive is that they dispense with the need for intermediate scales which are very difficult to generate^[15]. For this reason $SU(4) \times SU(2) \times U(1)$ and $SU(6)$ are not very appealing and we will not pursue them here.

For the M_2 manifold, G'' admits 8 1-dimensional representations and 2 2-dimensional representations which can successfully effect E_6 breaking in the desirable manner. A possible embedding is

$$U = [1] \otimes \begin{bmatrix} \beta & & \\ & \beta & \\ & & \beta^2 \end{bmatrix} \otimes \begin{bmatrix} 1 & \\ & V_g \end{bmatrix} \quad (31)$$

In terms of U_W and U_T we have

$$U_W = [1] \otimes \begin{bmatrix} i & & \\ & i & \\ & & i \end{bmatrix} \otimes \begin{bmatrix} 1 & & \\ & 0 & i \\ & i & 0 \end{bmatrix}$$

$$U_T = [1] \otimes [1] \otimes \begin{bmatrix} 1 & & \\ & i & 0 \\ & 0 & i \end{bmatrix} \quad (32)$$

with this embedding $u^c + \bar{u}^c$ components remain light from $(\underline{27} + \overline{\underline{27}})$. Note that with G'' as the symmetry group there is no way to avoid having at least one component of $(\underline{27} + \overline{\underline{27}})$ remaining light. The transformation properties

of the fields under G'' are listed in Table IV. The corresponding polynomials can easily be constructed and are given in the same table. The global symmetries of the manifold are $S_1(Z_i \rightarrow Z_{i+1})$ and $S_2(Z_i \rightarrow Z_{i+2})$. It is easy to see that S_2 is an exact symmetry of the manifold whereas S_1 is a pseudosymmetry. Since we will only be interested in the cubic couplings, the superpotential must be invariant under both S_1 and S_2 . Unfortunately, in this particular example the transformation properties of the fields under S_1 and S_2 are of limited use. The symmetry S_1 does not put any constraint on the couplings at all. To obtain transformation properties under S_2 , note that

$$S^4 W S_2 = S_2 W$$

$$S^4 = W^2 \tag{33}$$

i.e., S_2 does not commute with G'' . Simple application of the above eq. (33), shows that the fields transforming as β or β^3 under W must be parts of two-dimensional representations of S_2 . For example, $\begin{pmatrix} d_1^c \\ d_4^c \end{pmatrix}$ is a doublet under S_2 , whose two-dimensional representation is given by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Similar results apply to $\begin{pmatrix} N_4 \\ N_1 \end{pmatrix}$; $\begin{pmatrix} N_2 \\ N_3 \end{pmatrix}$; $\begin{pmatrix} \nu_1 \\ \nu_4 \end{pmatrix}$; $\begin{pmatrix} \nu_3 \\ \nu_2 \end{pmatrix}$; $\begin{pmatrix} d_2^c \\ d_3^c \end{pmatrix}$ etc. Clearly the lack of constraints imposed by S_1 and S_2 leads to disastrous phenomenology for this model. In particular, ultra rapid proton decay and large ν masses are problematic.

The lesson to be learned from the above exercise is that on this manifold the symmetry group is such that the fields transforming as β or β^3 under W always remain unconstrained. We may pursue this point by considering an alternate embedding of U_G in E_6 in such a way that no fields transform as either β or β^3 under W . One such embedding is.

$$U_G - [1] \otimes \begin{bmatrix} \alpha^2 & & \\ & \alpha^2 & \\ & & 1 \end{bmatrix} \otimes \begin{bmatrix} \beta^{-2} & \\ & \beta^V & \\ & & \beta \end{bmatrix} \quad : \quad \begin{array}{l} \beta \in W \\ \alpha \in T \\ \beta^4 - \alpha^4 = 1 \end{array} \quad (34)$$

$$U_W - [1] \otimes [1] \otimes \begin{bmatrix} 1 & & \\ & 0 & i \\ & i & 0 \end{bmatrix}$$

$$U_T - [1] \otimes [1] \otimes \begin{bmatrix} 1 & & \\ & i & 0 \\ & 0 & i \end{bmatrix} \quad (35)$$

With this embedding, the field transformations and the appropriate polynomial constructions are those given in Table V.

The transformations of fields under the global symmetries S_1 and S_2 is summarized in Table VI. Note the remarkable similarity between this Z_2 symmetry and that proposed in Section II. The detailed phenomenology of this model is presented in the next section.

V. Phenomenology of a specific 4-generation model

Of all the 2- and 4-generation models discussed in the previous sections, the most promising seems to be the 4-generation model based on the manifold M_2 with the embedding (34, 35) of the discrete group G'' (29). We will now see what it predicts for baryon decay, discussing also its related predictions for Cabibbo-Kobayashi-Maskawa mixing among quarks, and for neutrino masses. For this propose, we denote fields in the first two generations by F and those in the last two generations by L. It is then a simple matter to deduce from the transformation properties listed in Table VII(b) the patterns of allowed Yukawa superpotential terms, which are shown in Table VI.

Denoting the vacuum expectation values of $H_e, \bar{H}$ and N fields by $v, \bar{v}$ and x respectively, we have the following quark mass matrices.

$$(d_F, d_L) \begin{bmatrix} \bar{v}_F & \bar{v}_L \\ \bar{v}_L & v_F \end{bmatrix} \begin{pmatrix} d_F^c \\ d_L^c \end{pmatrix}; (u_F, u_L) \begin{bmatrix} v_F & v_L \\ v_L & v_F \end{bmatrix} \begin{pmatrix} u_F^c \\ u_L^c \end{pmatrix}; (D_F, D_L) \begin{bmatrix} x_L & x_F \\ x_F & x_L \end{bmatrix} \begin{pmatrix} D_F \\ D_L \end{pmatrix} \quad (36a)$$

lepton mass matrices:

$$(L_F, L_L) \begin{bmatrix} \bar{v}_F & \bar{v}_L \\ \bar{v}_L & \bar{v}_F \end{bmatrix} \begin{pmatrix} e_F^c \\ e_L^c \end{pmatrix}; (\nu_F, \nu_L) \begin{bmatrix} v_L & v_F \\ v_F & v_L \end{bmatrix} \begin{pmatrix} \nu_F^c \\ \nu_L^c \end{pmatrix} \quad (36b)$$

and Higgs mixing matrices:

$$(H_F, H_L) \begin{bmatrix} x_L & x_F \\ x_F & x_L \end{bmatrix} \begin{pmatrix} \bar{H}_F \\ \bar{H}_L \end{pmatrix} \quad (36c)$$

where each entry is in independent 2×2 submatrix. It is clear from the forms of the matrices (36) that all the particle masses would be non-zero even if only one out of each pair $v_{F,L}, \bar{v}_{F,L}, x_{F,L}$ were non-zero. On the other hand, Cabibbo-Kobayashi-Maskawa mixing would be confined to two 2×2 submatrices if only one each out of $v_{F,L}, \bar{v}_{F,L}$ were non-zero. Since it is obvious that rapid proton decay would be inevitable if all the v.e.v.'s were non-zero, we will ignore this defect in mixing for the moment, and investigate the hypothesis that only one of $v_{F,L}$ and one of $\bar{v}_{F,L}$ are non-zero. The normal scenario for weak gauge symmetry breaking induced by radiative corrections holds that some Higgs v.e.v. develops because of the large Yukawa coupling of the heaviest quark, taken to be the top quark in 3-generation models. In a 4-generation model the heaviest quark might couple either to an H field or to an $\bar{H}$ field, but for our purposes it does not matter which. It was proposed previously^[11] that a large $DD^c N$ or $H\bar{H}N$ coupling would also generate some non-zero v.e.v. $\langle 0|N|0 \rangle = x$, which would induce $H\bar{H}$ mixing and hence a v.e.v. for whichever of H or $\bar{H}$ did not already have a v.e.v. induced by its own Yukawa coupling. It is evident from equation (36c) that according to this scenario there are 4 possible minimal patterns of v.e.v.'s:

$$x_L \neq 0 \rightarrow v_F, \bar{v}_F \neq 0 \text{ or } v_L, \bar{v}_L \neq 0 \quad (37a)$$

$$x_F \neq 0 \rightarrow v_F, \bar{v}_L \neq 0 \text{ or } v_L, \bar{v}_F \neq 0 \quad (37b)$$

Another ambiguity in this model is in the assignment of the u and d quarks in the proton and neutron. Once we have assigned them, e.g. as states among the u_F and d_F , then the assignments of the u^c and d^c are fixed by the choices of v.e.v.'s for H and $\bar{H}$ fields, e.g. as u_F^c and d_F^c respectively if v_F and $\bar{v}_F \neq 0$. Therefore there are a total of 4 binary ambiguities in model-building, and hence $2^4 = 16$ separate cases to study.

Only models in which $\Delta B \neq 0$ interactions of the u and d quarks involve at most one each of $L_{F,L}$, $L_{F,L}^c$ and $\nu_{F,L}^c$ are not obviously unacceptable. If only one of the charged $L_{F,L}$ and $L_{F,L}^c$ is involved, we can always postulate that it is the τ or the heavier fourth-generation charged lepton. The cases of the $\nu_{F,L}$ and $\nu_{F,L}^c$ are more delicate. In these models, we can see from (36a) that all the neutrinos have Dirac masses, and we can always postulate that the fourth-generation neutrino is heavier than the proton and neutron. This option is not available for the ν_τ which is known^[24] to weigh ≤ 70 MeV. We will ignore this problem for the time being, and not discard immediately models in which only one of $\nu_{F,L}$ and $\nu_{F,L}^c$ are involved in the $\Delta B \neq 0$ interactions of the u and d quarks.

Among all the 16 cases mentioned above, we find just two which do not have too many $\Delta B \neq 0$ operators, which are

$$\begin{array}{llll} \text{Case (A):} & x_L \neq 0 & v_F, \bar{v}_F \neq 0 & u \in u_F \\ & x_F = 0 & v_L, \bar{v}_L = 0 & d \in d_F \end{array} \quad (38a)$$

$$\begin{array}{llll} \text{Case (B):} & x_L \neq 0 & v_F, \bar{v}_F \neq 0 & u \in u_L \\ & x_F = 0 & v_L, \bar{v}_L = 0 & d \in d_L \end{array} \quad (38b)$$

In both cases the non-zero mass terms are

$$u_F u_F^c, u_L u_L^c, d_F d_F^c, d_L d_L^c, e_F e_F^c, e_L e_L^c; \nu_F \nu_L^c, \nu_L \nu_F^c; D_F D_F^c, D_L D_L^c \quad (39)$$

In case (A) the available $\Delta B \neq 0$ interactions mediated by D and D^c exchange are

$$\begin{aligned} & (u_F^c u_F^c) (L_L q_F)^*, (u_F^c d_F^c) (L_L^c u_F^c), (u_F^c d_F^c) (\nu_F^c d_F^c), \\ & (q_F q_F) (L_L^c u_F^c)^*, (q_F q_F) (\nu_F^c d_F^c)^*, (q_F q_F) (q_F L_L) \end{aligned} \quad (40a)$$

where the $u_F, d_F, u_F^c, d_F^c, L_F, L_F^c, \nu_L^c$ are assumed to be light and the other fields heavy, while in case (B) they are

$$\begin{aligned} & (u_L^c u_L^c) (L_F q_L)^*, (u_L^c d_L^c) (L_F^c u_L^c), (u_L^c d_L^c) (\nu_L^c d_L^c), \\ & (q_L q_L) (L_F^c u_L^c)^*, (q_L q_L) (\nu_L^c d_L^c)^*, (q_L q_L) (q_L L_F) \end{aligned} \quad (40b)$$

where the $u_L, d_L, u_L^c, d_L^c, L_L, L_L^c, \nu_F^c$ are assumed to be light and the other fields heavy. Note that because generalized Cabibbo mixing only occurs inside 2×2 submatrices of the full mass matrices (36a), it cannot induce via electroweak radiative corrections $\Delta B \neq 0$ interactions among light fields alone.

The conclusion of this analysis of $\Delta B \neq 0$ interactions in a specific 4-generation model is: good try, but no banana. Rapid proton (and neutron)

decay is avoided but at the price of an unrealistic pattern of Cabibbo-Kobayashi-Maskawa mixing--the b quark would be stable in this model--and an unacceptable mass for the ν_τ --it would have to be larger than $m_{p,n}$ in this model. On the other hand, no other known 4-generation comes even close.

VI. Conclusions

In this paper we have explored the structure of several superstring models based on compactification on Calabi-Yau manifolds with non-Abelian fundamental groups, of the type that could yield a rank-5 gauge group in four dimensions. There is no model known with 3 generations, so we have focussed on 4-generation models. None of them avoid completely the well-known phenomenological problems with baryon decay and neutrino masses, although one (based on the M_2 manifold with discrete group G'') comes much closer than the others. However, it only avoids proton decay at the expense of an unrealistic pattern of Cabibbo-Kobayashi-Maskawa mixing and a heavy ν_τ .

The following directions remain for future research along the lines pursued here. Foremost is the quest for a 3-generation model with a non-Abelian discrete group. Clearly it is not easy to find one, and one can only hope that if one is found, its symmetries will be as easy to analyze as those of the 4-generation models studied in detail in this paper. Another possibility is that some particular algebraic relation between the parameters of the polynomials defining the manifold (see e.g. (4), (24), (26), (27)) may lead to interesting additional symmetries. This was not the case in the models studied here, but one may have better luck next time. Even if a rank-5 gauge group of the type studied here cannot be obtained from a Calabi-Yau manifold without being accompanied by catastrophic proton decay and neutrino masses, such a rank-5 gauge group might be obtainable from orbifold compactifications or formulations of superstrings directly in 4 dimensions.

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Table I

Transformation properties under $SU(3) \times SU(2) \times U_Y(1) \times U_E(1)$ of the chiral matter superfields contained in $\underline{27}$ of E_6 .

Chiral Superfield	$SU(3) \times SU(2) \times U(1) \times U(1)$ quantum numbers
Q	(3, 2, 1/6, 1/3)
u^c	($\bar{3}$, 1, -2/3, 1/3)
d^c	($\bar{3}$, 1, 1/3, -1/6)
L	(1, 2, -1/2, -1/6)
e^c	(1, 1, 1, 1/3)
ν^c	(1, 1, 0, 5/6)
H	(1, 2, 1/2, -2/3)
$\bar{H}$	(1, 2, -1/2, -1/6)
N	(1, 1, 0, 5/6)
D	(3, 1, -1/3, -2/3)
D^c	($\bar{3}$, 1, 1/3, -1/6).

Table II
Polynomial Representations and Transformation Properties in the Two-
Generation Model

Field	Transformation under G	Polynomial representation	Transformation under		
			A	B	C
u_1^c	β^2	$\begin{bmatrix} Z_0 Z_2 X(p^1) \\ Z_1 Z_3 X(p^2) \\ 0 \\ 0 \end{bmatrix}$	+	+	+
u_2^c	β^2	$\begin{bmatrix} 0 \\ 0 \\ W_0 W_2 X(p^3) \\ W_1 W_3 X(p^4) \end{bmatrix}$	+	+	+
d_1^c	$(\beta V_g)^{-1}$	$\begin{bmatrix} (Z_0 W_3 - Z_2 W_1) X(p^1) \\ 0 \\ 0 \\ 0 \end{bmatrix}$	+	*	D_2^c
D_1^c		$\begin{bmatrix} (\alpha^{-1} Z_1 W_0 + \alpha^{-3} Z_3 W_3) X(p^2) \\ 0 \\ 0 \\ 0 \end{bmatrix}$	+	*	d_2^c
d_2^c	$(\beta V_g)^{-1}$	$\begin{bmatrix} 0 \\ 0 \\ (W_0 Z_3 - Z_1 W_2) X(p^3) \\ 0 \end{bmatrix}$	-	*	D_1^c
D_2^c		$\begin{bmatrix} 0 \\ 0 \\ 0 \\ (\alpha^{-1} W_1 Z_0 + \alpha^{-3} W_3 Z_2) X(p^4) \end{bmatrix}$	+	*	d_1^c

Q_1	α	$\begin{bmatrix} (Z_0^2 - Z_2^2) \chi(p^1) \\ (\alpha^{-1} Z_1^2 + \alpha Z_3^2) \chi(p^2) \\ 0 \\ 0 \end{bmatrix}$	+	*	-
Q_2	α	$\begin{bmatrix} 0 \\ 0 \\ (W_0^2 - W_2^2) \chi(p^3) \\ (\alpha^{-1} W_1^2 + \alpha W_3^2) \chi(p^4) \end{bmatrix}$	+	+	+
D_1	α^2	$\begin{bmatrix} Z_1 Z_3 \chi(p^1) \\ -Z_0 Z_2 \chi(p^2) \\ 0 \\ 0 \end{bmatrix}$	+	-	+
D_2	α^2	$\begin{bmatrix} 0 \\ 0 \\ W_1 W_3 \chi(p^3) \\ -W_0 W_2 \chi(p^4) \end{bmatrix}$	+	+	+
e_1^c	$\alpha^2 \beta^2$	$\begin{bmatrix} Z_0 Z_2 \chi(p^1) \\ -Z_1 Z_3 \chi(p^2) \\ 0 \\ 0 \end{bmatrix}$	+	-	+
e_2^c	$\alpha^2 \beta^2$	$\begin{bmatrix} 0 \\ 0 \\ W_0 W_2 \chi(p^1) \\ -W_1 W_3 \chi(p^2) \end{bmatrix}$	+	+	+

$$\begin{array}{lcl}
\left. \begin{array}{l} L_1 \\ \bar{H}_1 \end{array} \right\} & (\alpha^{-1} \beta v_g) \begin{bmatrix} (Z_0 W_3 + Z_2 W_1) \chi(p^1) \\ 0 \\ 0 \\ 0 \end{bmatrix} & + * \bar{H}_2 \\
& \begin{bmatrix} 0 \\ (Z_1 W_1 + Z_3 W_2) \chi(p^2) \\ 0 \\ 0 \end{bmatrix} & - * L_2 \\
\left. \begin{array}{l} L_2 \\ \bar{H}_2 \end{array} \right\} & (\alpha^{-1} \beta v_g) \begin{bmatrix} 0 \\ 0 \\ (W_0 Z_3 + Z_1 W_2) \chi(p^3) \\ 0 \end{bmatrix} & - * \bar{H}_1 \\
& \begin{bmatrix} 0 \\ 0 \\ 0 \\ (W_1 Z_0 + W_3 Z_2) \chi(p^4) \end{bmatrix} & + * \bar{L}_1 \\
\left. \begin{array}{l} N_1 \\ \nu_1^c \end{array} \right\} & (\alpha^2 \beta v_g) \begin{bmatrix} (Z_0 W_3 + \alpha^2 Z_2 W_1) \chi(p^1) \\ 0 \\ 0 \\ 0 \end{bmatrix} & + * \nu_2^c \\
& \begin{bmatrix} 0 \\ (\alpha Z_1 W_0 + \alpha^{-1} Z_3 W_2) \chi(p^2) \\ 0 \\ 0 \end{bmatrix} & - * N_2
\end{array}$$

$$\begin{array}{lcl}
\left. \begin{array}{l} N_2 \\ \nu_2^c \end{array} \right\} & (\alpha^2 \beta V_g) \begin{bmatrix} 0 \\ 0 \\ (W_0 Z_3 - Z_1 W_2) x(p^2) \\ 0 \end{bmatrix} & - * \nu_1^c \\
& \begin{bmatrix} 0 \\ 0 \\ 0 \\ (\alpha W_1 Z_0 + \alpha^{-1} W_3 Z_2) x(p^4) \end{bmatrix} & + * N_1 \\
H_1 & \alpha^{-1} \beta^2 \begin{bmatrix} (Z_1^2 + \alpha^2 Z_3^2) x(p^1) \\ (\alpha Z_2^2 + \alpha^{-1} Z_0^2) x(p^2) \\ 0 \\ 0 \end{bmatrix} & + - - \\
H_2 & \alpha^{-1} \beta^2 \begin{bmatrix} 0 \\ 0 \\ (W_1^2 + \alpha^2 W_3^2) x(p^3) \\ (\alpha W_2^2 + \alpha^{-1} W_0^2) x(p^4) \end{bmatrix} & + + +
\end{array}$$

*means that the corresponding field does not transform in any definite way under that symmetry.

Table III

The character table for the eight-element group G generated by V and W .

	1	(W, W^3)	(XW, WX)	(X, X^3)	X^2W^2
1	1	1	1	1	1
(W, W^3)	1	1	-1	-1	1
(XW, WX)	1	-1	-1	1	1
(X, X^3)	1	-1	1	-1	1
(X^2W^2)	2	0	0	0	-2

Table IV

Polynomial Representations and Transformation Properties in the Four-Generation Model with an embedding given by Eq. 31.

<u>Field</u>	<u>Transformation under G</u>	<u>Polynomial representation</u>	<u>Transformation Under S_2</u>
u_1^c	1	$\begin{bmatrix} Z_1 Z_7 + Z_3 Z_5 \\ Z_8 Z_2 + Z_4 Z_6 \\ 0 \\ 0 \end{bmatrix}$	u_3^c
u_2^c	1	$\begin{bmatrix} Z_1 Z_3 + Z_5 Z_7 \\ Z_2 Z_4 + Z_6 Z_8 \\ 0 \\ 0 \end{bmatrix}$	u_4^c
u_3^c	1	$\begin{bmatrix} 0 \\ 0 \\ Z_1 Z_3 + Z_5 Z_7 \\ Z_2 Z_4 + Z_6 Z_8 \end{bmatrix}$	u_1^c
u_4^c	1	$\begin{bmatrix} 0 \\ 0 \\ Z_1 Z_7 + Z_3 Z_5 \\ Z_2 Z_4 + Z_6 Z_8 \end{bmatrix}$	u_2^c
u_5^c	1	$\begin{bmatrix} Z_2 Z_6 \\ Z_3 Z_7 \\ Z_8 Z_4 \\ Z_5 Z_1 \end{bmatrix}$	u_5^c

D_1, e_1^c	β^2	$\begin{bmatrix} z_1 z_7 + z_3 z_5 \\ -z_8 z_2 - z_4 z_6 \\ 0 \\ 0 \end{bmatrix}$	D_3
D_2, e_2^c	β^2	$\begin{bmatrix} z_1 z_3 + z_5 z_7 \\ -z_2 z_4 - z_6 z_8 \\ 0 \\ 0 \end{bmatrix}$	D_4
D_3, e_3^c	β^2	$\begin{bmatrix} 0 \\ 0 \\ z_1 z_3 + z_5 z_7 \\ -z_2 z_4 - z_6 z_8 \end{bmatrix}$	D_1
D_4, e_4^c	β^2	$\begin{bmatrix} 0 \\ 0 \\ z_1 z_7 + z_3 z_5 \\ -z_8 z_2 - z_4 z_6 \end{bmatrix}$	D_2
Q_1	β	$\begin{bmatrix} z_1 z_7 - z_3 z_5 \\ -iz_4 z_6 + iz_8 z_2 \\ 0 \\ 0 \end{bmatrix}$	*
Q_2	β	$\begin{bmatrix} z_1 z_3 - z_5 z_7 \\ -iz_4 z_2 + iz_6 z_8 \\ 0 \\ 0 \end{bmatrix}$	*
Q_3	β	$\begin{bmatrix} 0 \\ 0 \\ z_1 z_7 - z_3 z_5 \\ iz_4 z_6 + iz_8 z_2 \end{bmatrix}$	*

Q_4	β	$\begin{bmatrix} 0 \\ 0 \\ z_1 z_3 - z_5 z_7 \\ i z_4 z_2 + i z_6 z_8 \end{bmatrix}$	$*$	
d_1^c	$\left. \begin{array}{c} \\ \\ \end{array} \right\}$	V_g^{-1}	$\begin{bmatrix} z_1 z_2 - z_5 z_6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	d_4^c
D_1^c				$-D_4^c$
d_2^c	$\left. \begin{array}{c} \\ \\ \end{array} \right\}$	V_g^{-1}	$\begin{bmatrix} z_3 z_4 - z_7 z_8 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$-d_3^c$
D_2^c				D_3^c
d_3^c	$\left. \begin{array}{c} \\ \\ \end{array} \right\}$	V_g^{-1}	$\begin{bmatrix} 0 \\ 0 \\ z_1 z_2 - z_5 z_6 \\ 0 \end{bmatrix}$	d_2
D_3^c				$-D_2^c$
			$\begin{bmatrix} 0 \\ 0 \\ 0 \\ i z_4 z_7 - i z_8 z_3 \end{bmatrix}$	

d_4^c	}	v_g^{-1}	$\begin{bmatrix} 0 \\ 0 \\ z_3 z_4 - z_7 z_8 \\ 0 \end{bmatrix}$	$-d_1$
D_4^c				D_1^c
L_1	}	$v_g \beta^{-1}$	$\begin{bmatrix} z_1 z_4 + z_5 z_8 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	L_3
$\bar{H}_1$				$\bar{H}_3$
L_2	}	$v_g \beta^{-1}$	$\begin{bmatrix} z_3 z_6 + z_7 z_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	L_4
$\bar{H}_2$				$\bar{H}_4$
			$\begin{bmatrix} 0 \\ z_2 z_3 + z_6 z_7 \\ 0 \\ 0 \end{bmatrix}$	

L_3	}	$v_g \beta^{-1}$	$\begin{bmatrix} 0 \\ 0 \\ Z_1 Z_4 + Z_5 Z_8 \\ 0 \end{bmatrix}$	L_1
$\tilde{H}_3$				$\tilde{H}_1$
L_4	}	$v_g \beta^{-1}$	$\begin{bmatrix} 0 \\ 0 \\ Z_3 Z_6 + Z_7 Z_2 \\ 0 \end{bmatrix}$	L_2
$\tilde{H}_4$				$\tilde{H}_2$
H_1		β^{-1}	$\begin{bmatrix} Z_1 Z_7 - Z_3 Z_5 \\ iZ_4 Z_6 - iZ_8 Z_2 \\ 0 \\ 0 \end{bmatrix}$	*
H_2		β^{-1}	$\begin{bmatrix} Z_1 Z_3 - Z_5 Z_7 \\ iZ_4 Z_2 - iZ_6 Z_8 \\ 0 \\ 0 \end{bmatrix}$	*
H_3		β^{-1}	$\begin{bmatrix} 0 \\ 0 \\ Z_1 Z_7 - Z_3 Z_5 \\ iZ_4 Z_6 - iZ_8 Z_2 \end{bmatrix}$	*

H_4		β^{-1}	$\begin{bmatrix} 0 \\ 0 \\ Z_1 Z_3 - Z_5 Z_7 \\ iZ_4 Z_2 - iZ_6 Z_8 \end{bmatrix}$	*
N_1	}	$\beta^2 v_g$	$\begin{bmatrix} Z_1 Z_4 - Z_5 Z_8 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	N_4
ν_1^c			$\begin{bmatrix} 0 \\ iZ_4 Z_5 - iZ_8 Z_1 \\ 0 \\ 0 \end{bmatrix}$	$-\nu_4^c$
N_c	}	$\beta^2 v_g$	$\begin{bmatrix} Z_3 Z_6 - Z_7 Z_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$-N_3$
ν_2^c			$\begin{bmatrix} 0 \\ iZ_2 Z_3 - iZ_6 Z_7 \\ 0 \\ 0 \end{bmatrix}$	ν_3^c
N_3	}	$\beta^2 v_g$	$\begin{bmatrix} 0 \\ 0 \\ Z_1 Z_4 - Z_5 Z_8 \\ 0 \end{bmatrix}$	N_2
ν_3^c			$\begin{bmatrix} 0 \\ 0 \\ 0 \\ iZ_4 Z_5 - iZ_8 Z_1 \end{bmatrix}$	$-\nu_2^c$

$$\left. \begin{matrix} N_4 \\ \nu_4^c \end{matrix} \right\} \beta^2_{V_g} \begin{bmatrix} 0 \\ 0 \\ Z_3 Z_6 - Z_7 Z_2 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ iZ_2 Z_3 - iZ_6 Z_7 \end{bmatrix} \begin{matrix} -N_1 \\ -\nu_1^c \end{matrix}$$

Table V

Polynomial Representations and Transformation Properties in the Four-Generation Model with an Embedding Given by Eq. 35.

<u>Field</u>	<u>Transformation</u> <u>Under G</u>	<u>Polynomial</u> <u>Representation</u>	<u>S₂</u>
D ₁	1	$\begin{bmatrix} Z_1 Z_7 + Z_3 Z_5 \\ Z_8 Z_2 + Z_4 Z_6 \\ -Z_1 Z_3 - Z_5 Z_7 \\ -Z_2 Z_4 - Z_6 Z_8 \end{bmatrix}$	-1
D ₂	1	$\begin{bmatrix} Z_1 Z_3 + Z_5 Z_7 \\ Z_2 Z_4 + Z_6 Z_8 \\ -Z_1 Z_7 - Z_3 Z_5 \\ -Z_8 Z_2 - Z_4 Z_6 \end{bmatrix}$	-1
D ₃	1	$\begin{bmatrix} Z_1 Z_7 + Z_3 Z_5 \\ Z_8 Z_2 + Z_4 Z_6 \\ -Z_1 Z_3 - Z_5 Z_7 \\ -Z_2 Z_4 - Z_6 Z_8 \end{bmatrix}$	1
D ₄	1	$\begin{bmatrix} Z_1 Z_3 + Z_5 Z_7 \\ Z_2 Z_4 + Z_6 Z_8 \\ Z_1 Z_7 + Z_3 Z_5 \\ Z_8 Z_2 + Z_4 Z_6 \end{bmatrix}$	1
D ₅	1	$\begin{bmatrix} Z_2 Z_6 \\ Z_3 Z_7 \\ Z_8 Z_4 \\ Z_5 Z_1 \end{bmatrix}$	1
H ₁	$\alpha^2 \beta^2$	$\begin{bmatrix} Z_2 Z_8 + Z_4 Z_6 \\ -Z_1 Z_7 + Z_3 Z_5 \\ Z_2 Z_4 + Z_6 Z_8 \end{bmatrix}$	1

H_2	$\alpha^2 \beta^2$	$\begin{bmatrix} Z_2 Z_4 + Z_6 Z_8 \\ -Z_1 Z_3 - Z_5 Z_7 \\ Z_2 Z_8 + Z_4 Z_6 \\ -Z_1 Z_3 - Z_5 Z_7 \end{bmatrix}$	1
H_3	$\alpha^2 \beta^2$	$\begin{bmatrix} Z_2 Z_8 + Z_4 Z_6 \\ -Z_1 Z_7 + Z_3 Z_5 \\ Z_2 Z_4 + Z_6 Z_8 \\ -Z_1 Z_3 - Z_5 Z_7 \end{bmatrix}$	-1
H_4	$\alpha^2 \beta^2$	$\begin{bmatrix} Z_2 Z_4 + Z_6 Z_8 \\ -Z_1 Z_3 - Z_5 Z_7 \\ -Z_2 Z_8 - Z_4 Z_6 \\ +Z_1 Z_3 - Z_5 Z_7 \end{bmatrix}$	-1
Q_1	α^2	$\begin{bmatrix} Z_2 Z_8 + Z_4 Z_6 \\ Z_1 Z_7 + Z_3 Z_5 \\ -Z_2 Z_4 - Z_6 Z_8 \\ -Z_1 Z_3 - Z_5 Z_7 \end{bmatrix}$	-1
Q_2	α^2	$\begin{bmatrix} Z_2 Z_4 + Z_6 Z_8 \\ -Z_1 Z_3 - Z_5 Z_7 \\ Z_2 Z_8 + Z_4 Z_6 \\ -Z_1 Z_7 - Z_3 Z_5 \end{bmatrix}$	-1
Q_3	α^2	$\begin{bmatrix} Z_2 Z_8 + Z_4 Z_6 \\ Z_1 Z_7 + Z_3 Z_5 \\ Z_2 Z_4 + Z_6 Z_8 \\ Z_1 Z_3 + Z_5 Z_7 \end{bmatrix}$	1

Q_4		α^2	$\begin{bmatrix} z_2 z_4 + z_6 z_8 \\ z_1 z_3 + z_5 z_7 \\ z_2 z_8 + z_4 z_6 \\ z_1 z_7 + z_3 z_5 \end{bmatrix}$	1
N_1	}	β^V_g	$\begin{bmatrix} z_1 z_4 + z_5 z_8 \\ 0 \\ -z_3 z_6 - z_7 z_2 \\ 0 \end{bmatrix}$	-1
ν_1^c			$\begin{bmatrix} 0 \\ -z_4 z_5 - z_8 z_1 \\ 0 \\ z_2 z_3 + z_6 z_7 \\ 0 \end{bmatrix}$	
N_2	}	β^V_g	$\begin{bmatrix} z_3 z_6 + z_7 z_2 \\ 0 \\ -z_1 z_4 - z_5 z_8 \\ 0 \end{bmatrix}$	-1
ν_2^c			$\begin{bmatrix} 0 \\ -z_2 z_3 - z_6 z_7 \\ 0 \\ z_4 z_5 + z_8 z_1 \end{bmatrix}$	
N_3	}	β^V_g	$\begin{bmatrix} z_1 z_4 + z_5 z_8 \\ 0 \\ z_3 z_6 - z_7 z_2 \\ 0 \end{bmatrix}$	-1
ν_3^c			$\begin{bmatrix} 0 \\ -z_4 z_5 - z_8 z_1 \\ 0 \\ z_2 z_3 + z_6 z_7 \end{bmatrix}$	

N_4	}	βV_g	$\begin{bmatrix} z_3 z_6 + z_7 z_2 \\ 0 \\ z_1 z_4 + z_5 z_8 \\ 0 \end{bmatrix}$	1
d_1^c, L_3	}	$(\beta V_g)^{-1}$	$\begin{bmatrix} 0 \\ -z_2 z_3 - z_6 z_7 \\ 0 \\ -z_4 z_5 + z_8 z_1 \end{bmatrix}$	-1
d_2^c, L_4	}	$(\beta V_g)^{-1}$	$\begin{bmatrix} 0 \\ -z_4 z_7 - z_8 z_3 \\ 0 \\ z_2 z_5 + z_6 z_1 \end{bmatrix}$	-1
$\begin{bmatrix} z_3 z_4 + z_7 z_8 \\ 0 \\ -z_1 z_2 - z_5 z_6 \\ 0 \end{bmatrix}$	-1			
		$\begin{bmatrix} 0 \\ -z_2 z_5 - z_6 z_1 \\ 0 \\ +z_4 z_7 + z_8 z_3 \end{bmatrix}$	-1	

$$\begin{array}{c}
 \left. \begin{array}{l} d_3^c, L_1 \\ \\ d_3^c, \bar{H}_1 \end{array} \right\} \\
 \\
 \left. \begin{array}{l} d_4^c, L_2 \\ \\ d_4^c, \bar{H}_2 \end{array} \right\}
 \end{array}
 \begin{array}{c}
 (\beta v_g)^{-1} \\
 \\
 (\beta v_g)^{-1}
 \end{array}
 \begin{array}{c}
 \begin{bmatrix} z_1 z_2 + z_5 z_6 \\ 0 \\ z_3 z_4 + z_7 z_8 \\ 0 \end{bmatrix} \\
 \\
 \begin{bmatrix} 0 \\ -z_4 z_7 - z_8 z_3 \\ 0 \\ -z_2 z_5 - z_6 z_1 \end{bmatrix} \\
 \\
 \begin{bmatrix} z_3 z_4 + z_7 z_8 \\ 0 \\ z_1 z_2 + z_5 z_6 \\ 0 \end{bmatrix} \\
 \\
 \begin{bmatrix} 0 \\ -z_2 z_5 - z_6 z_1 \\ 0 \\ -z_4 z_7 - z_8 z_3 \end{bmatrix}
 \end{array}
 \begin{array}{c}
 1 \\
 \\
 1 \\
 \\
 1 \\
 \\
 1
 \end{array}$$

D_1, e_3^c	β^2	$\begin{bmatrix} z_1 z_7^+ & z_3 z_5 \\ -z_8 z_2^- & z_4 z_6 \\ -z_1 z_3^- & z_7 z_5 \\ -z_4 z_2^+ & z_8 z_6 \end{bmatrix}$	-1
D_2, e_4^c	β^2	$\begin{bmatrix} z_1 z_3^+ & z_5 z_7 \\ -z_2 z_4^- & z_6 z_8 \\ -z_1 z_7^- & z_3 z_5 \\ -z_8 z_2^+ & z_4 z_6 \end{bmatrix}$	-1
D_3, e_1^c	β^2	$\begin{bmatrix} z_1 z_7^+ & z_3 z_5 \\ -z_8 z_2^- & z_4 z_6 \\ z_1 z_3^+ & z_5 z_7 \\ -z_2 z_4^- & z_6 z_8 \end{bmatrix}$	1
D_4, e_2^c	β^2	$\begin{bmatrix} z_1 z_3^+ & z_5 z_7 \\ -z_2 z_4^- & z_6 z_8 \\ z_1 z_7^+ & z_3 z_5 \\ -z_8 z_2^- & z_4 z_6 \end{bmatrix}$	1

Table VI

Transformation properties of fields in the 27 under the global symmetry S_2 .

$[3, \bar{3}]SU(3)$	$\rightarrow$	$(-1)[3, \bar{3}]SU(3)$	For the first two generations.
(ν^c, N)	$\rightarrow$	$-1(\nu^c, N)$	
(L, H, e^c)	$\rightarrow$	(L, H, e^c)	
$[3, \bar{3}]SU(3)$	$\rightarrow$	$[3, \bar{3}]SU(3)$	For the last two generations.
(ν^c, N)	$\rightarrow$	(ν^c, N)	
(L, H, e^c)	$\rightarrow$	$-(L, H, e^c)$	