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QUENCHED STRING FIELD THEORY

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A B S T R A C T

We construct a toy model of Witten's open string field theory by truncating the Virasoro group to its $SU(1,1)$ subgroup. The string field is given in terms of finite dimensional matrices, which satisfy a non-trivial BRST-cohomology. The string gauge group is found to be $SO(3)$. Finally, we perform the gauge fixing of the action, finding an explicit formula for calculating the partition function.

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In Witten's open string field theory [1], the string field $A[x^\mu(\sigma), c(\sigma)]$ is some sort of "infinite dimensional matrix", whose entries are labelled by the left- and right-handed pieces of the string. Following this analogy, one can consider the $*$ -product as a matrix multiplication and the integration $\int$ as the trace operation. In this way we encounter in Witten's theory a resemblance of the objects and operations of the ordinary matrix algebra. String fields seem to behave as matrices but they are much more complicated in their structure. The complexity of the string field is in contrast with the simplicity of the axioms it obeys.

These considerations have lead us to construct a model where the content of the string field is drastically reduced, while retaining its algebraic properties. The model can be then called a "quenched string field theory" [2], and it is far from being realistic but it may serve as a toy model of the complete theory.

Our starting point is to consider not the entire set of Virasoro operators $\{L_n, n \in \mathbb{Z}\}$, but the set $\{L_n, n=0, \pm 1\}$ which generates a $SU(1,1)$ subgroup. These operators can be represented in terms of 2×2 matrices as

$$\rho_0 = -\frac{1}{2} \sigma_3, \quad \rho_+ = \frac{1}{2} (i\sigma_1 - \sigma_2), \quad \rho_- = \frac{1}{2} (i\sigma_1 + \sigma_2) \quad (1)$$

which satisfy the "truncated Virasoro algebra":

$$\begin{aligned} [\rho_n, \rho_m] &= (n-m) \rho_{n+m} \\ n, m &= 0, \pm 1. \end{aligned} \quad (2)$$

σ_i are the usual Pauli matrices.

Following the standard construction [3] we introduce the ghost and antighost modes c_n and b_n ($n=0, +, -$), which satisfy the anticommutation relations:

$$\{b_n, c_m\} = \delta_{n+m,0} \quad (3)$$

the other ones being zero.

The ground state for the ghost sea, $|0\rangle$, is defined by the conditions

$$b_0 |0\rangle = b_+ |0\rangle = c_+ |0\rangle = 0 \quad (4)$$

The BRST-charge is then given by

$$Q = c_0 (p_0 - 1) + c_+ p_- + c_- p_+ - 2 c_- c_+ b_0 + c_0 b_- c_+ + c_0 c_- b_+ \quad (5)$$

which is normal ordered and satisfies $Q^2 = 0$.

Similarly, the ghost number operator G reads:

$$G = \frac{1}{2} [c_0, b_0] + c_- b_+ - b_- c_+ \quad (6)$$

From Eqs. (4) it follows that there are eight ghost states which can be classified according to their ghost number in four sectors, corresponding to the values $g = \pm 3/2, \pm \frac{1}{2}$.

g	$-3/2$	$-\frac{1}{2}$	$\frac{1}{2}$	$3/2$
		$ 0\rangle$	$c_0 0\rangle$	
	$b_- 0\rangle$	$c_0 b_- 0\rangle$	$c_- 0\rangle$	$c_0 c_- 0\rangle$
		$c_- b_- 0\rangle$	$c_0 c_- b_- 0\rangle$	

These are the basic ingredients of our construction. Our aim is to show how Witten's axiom for the open bosonic string field theory can be realized in this truncated string. Hence we must find out a $*$ product, a derivation Q and an integral $\int$ satisfying the following axioms:

$$(A * B) * C = A * (B * C) \quad (7a)$$

$$Q(A * B) = (QA) * B + (-)^A A * QB \quad (7b)$$

$$Q^2 = 0 \quad (7c)$$

$$\int QA = 0 \quad \forall A \quad (7d)$$

$$\int A * B = (-)^{AB} \int B * A \quad (7e)$$

where $(-)^A = +1(-1)$ if A is even (odd).

We also know that the string fields are "differential forms" of degree $g+3/2$, the $*$ product is the "exterior product" and the BRST charge is the "exterior derivative".

The Table suggests to take as zero-forms the following states

$$|\epsilon\rangle_0 = \epsilon b_- |0\rangle \quad (8)$$

where ϵ is a complex 2×2 matrix, and the subscript "0" denotes the degree of the form.

As the zero-forms must be closed under the $*$ multiplication, we define the product

$$|\epsilon_1\rangle_0 * |\epsilon_2\rangle_0 \equiv |\epsilon_1 \epsilon_2\rangle_0 \quad (9)$$

This product is associative but non-commutative and it reduces, for zero-forms, to ordinary matrix multiplication.

The structure of the one-, two- and three-forms can be easily guessed from the Table and is chosen to be

$$\begin{aligned} |A\rangle_1 &\equiv |A_0, A_+, A_-\rangle \\ |F\rangle_2 &\equiv |F_0, F_+, F_-\rangle \\ |D\rangle_3 &\equiv |D\rangle \end{aligned} \quad (10)$$

where D , A_n and F_n ($n=0,+,-$) are complex 2×2 matrices.

The $*$ product of a zero-form with a p -form ($p=1,2,3$) is the straightforward generalization of Eq. (9), i.e.,:

$$\begin{aligned} |\epsilon\rangle_0 * |A_0, A_+, A_-\rangle_1 &\equiv |\epsilon A_0, \epsilon A_+, \epsilon A_-\rangle_1 \\ |A_0, A_+, A_-\rangle_1 * |\epsilon\rangle_0 &\equiv |A_0 \epsilon, A_+ \epsilon, A_- \epsilon\rangle_1 \end{aligned} \quad (11)$$

and similarly for $p = 2$ and 3 .

A less trivial step is to define the operator Q which must behave as a derivation of the $*$ product.

For a start, we define the action of Q on zero-forms as:

$$Q|\epsilon\rangle_0 \equiv |[\rho_0, \epsilon], [\rho_+, \epsilon], [\rho_-, \epsilon]\rangle_1 \quad (12)$$

or, in other words, Q acts on zero-forms as a Lie derivative.

The rest of the construction proceeds along the same lines: finding a $*$ product and an action of Q such that the axioms (7a) and (7b) hold.

The derivation Q is completely specified by Eq. (12) and the following definitions:

$$\begin{aligned} Q|A\rangle_1 &\equiv | \{ \rho_+, A_- \} + \{ \rho_-, A_+ \}, \{ \rho_0, A_+ \} + \{ \rho_+, A_0 \}, \{ \rho_0, A_- \} + \{ \rho_-, A_0 \} \rangle_2 \\ Q|F\rangle_2 &\equiv | [\rho_0, F_0] + [\rho_+, F_-] + [\rho_-, F_+] \rangle_3 \\ Q|D\rangle_3 &\equiv 0 \end{aligned} \quad (13)$$

The remaining non-trivial $*$ products are given by

$$\begin{aligned} |A\rangle_1 * |B\rangle_1 &\equiv | A_+ B_- + A_- B_+, A_+ B_0 + A_0 B_+, A_- B_0 + A_0 B_- \rangle_2 \\ |A\rangle_1 * |F\rangle_2 &\equiv | A_0 F_0 + A_+ F_- + A_- F_+ \rangle_3 \\ |F\rangle_2 * |A\rangle_1 &\equiv | F_0 A_0 + F_+ A_- + F_- A_+ \rangle_3 \end{aligned} \quad (14)$$

The fact that Q is a derivation follows automatically from its definition and does not require any property of the ρ matrices.

The nilpotency of Q imposes the following equations on the ρ matrices:

$$[\{ \rho_n, \rho_m \}, X] = 0 \quad (15)$$

$n, m = 0, +, -$ and $n \neq m$ for any complex 2×2 matrix X .

Equations (15) are equivalent to the conditions:

$$\{p_n, p_m\} \sim 0 \text{ or } 1 \quad (16)$$

which are indeed satisfied for the choice (1), but they do not force the matrices $\{p_n\}$ to generate the $SU(1,1)$ subgroup of the Virasoro group. We shall, however, assume this to be the case, in order to maintain some contact with the strings.

Finally, we define the integral according to:

$$\begin{aligned} \int |\psi\rangle_p &= 0 & p &= 0, 1, 2 \\ \int |D\rangle_3 &= \text{Tr } D \end{aligned} \quad (17)$$

Then Eqs. (7d) and (7e) follow from the identity:

$$\text{Tr } AB = \text{Tr } BA \quad (18)$$

This ends up our explicit realization of Witten's axioms in an extremely simplified version of the strings in terms of finite dimensional matrices. We may now proceed to study the dynamics of the quenched string.

The action for the string field $A \equiv |A\rangle_1$ is given by

$$S = \int A * QA + \frac{2}{3} g A * A * A \quad (19)$$

where we have included the gauge coupling constant g .

This action is invariant under the gauge transformation:

$$\delta A = \frac{1}{g} Q\epsilon - \epsilon * A + A * \epsilon \quad (20)$$

where $\epsilon \equiv |\epsilon\rangle_0$.

The explicit form of the action (19) is in our model:

$$\begin{aligned} S = \text{Tr} \Big(& \frac{1}{2} p_0, \{A_+, A_-\} + \{p_+, \{A_0, A_-\}\} + \{p_-, \{A_0, A_+\}\} \\ & + g \{A_0, \{A_+, A_-\}\} \Big) \end{aligned} \quad (21)$$

and the gauge transformation reads:

$$\delta A_n = \frac{1}{g} [p_n + g A_n, \epsilon] \quad (22)$$

We must further impose a reality on the string field which guarantees the reality of the action. The A_n - matrices are taken to satisfy the same hermiticity conditions as the ρ_n matrices, namely:

$$(A_0)^\dagger = A_0, \quad (A_\pm)^\dagger = -A_\pm \quad (23)$$

Moreover, from Eqs. (22) and (23), we deduce that the gauge parameter ϵ must satisfy:

$$\epsilon^\dagger = -\epsilon \quad (24)$$

from which we conclude that the "string gauge group" is $SO(3)$.

It is easy to see that the free field equation $QA = 0$ has three independent solutions which are not pure gauge, i.e., the first cohomology group $H^1(Q)$ is isomorphic to $\mathbb{R}^3$. We get for the other cohomology groups:

$$\begin{aligned} H^0 &\approx H^3 \approx \mathbb{R} \\ H^1 &\approx H^2 \approx \mathbb{R}^3 \end{aligned} \quad (25)$$

Thus duality holds as for the entire string. It is convenient to parametrize the string in the following way:

$$\begin{aligned} A_0 &= -\frac{1}{2} A_3^\mu \sigma_\mu \\ A_\pm &= \frac{1}{2} (i A_1^\mu - A_2^\mu) \sigma_\mu \end{aligned} \quad (26)$$

where A_i^μ ($i=1,2,3$; $\mu=0,1,2,3$) are real and $\sigma_\mu = (1, \vec{\sigma})$.

In these variables, the action (21) takes the form:

$$\begin{aligned} S = 2 \bigg\{ & \vec{p}_1 \cdot (\vec{A}_1 A_3^0 + \vec{A}_3 A_1^0) + \vec{p}_2 \cdot (\vec{A}_2 A_3^0 + \vec{A}_3 A_2^0) \\ & + \vec{p}_3 \cdot (\vec{A}_1 A_1^0 + \vec{A}_2 A_2^0) \bigg\} \\ & + g \bigg\{ 2 A_1^0 \vec{A}_1 \cdot \vec{A}_3 + 2 A_2^0 \vec{A}_2 \cdot \vec{A}_3 \\ & + A_3^0 (A_1^0 A_1^0 + \vec{A}_1 \cdot \vec{A}_1 + A_2^0 A_2^0 + \vec{A}_2 \cdot \vec{A}_2) \bigg\} \end{aligned} \quad (27)$$

and the gauge transformation (22) reads

$$\begin{aligned}\delta A_i^0 &= 0 \\ \delta \vec{A}_i &= \frac{1}{g} \vec{E} \times (\vec{p}_i + g \vec{A}_i) \quad i=1,2,3\end{aligned}\quad (28)$$

where $p_i^a = \delta_i^a$ ($i,a=1,2,3$), corresponding to the choice (1), and represent in some sense the background.

We may now define a Euclidean string partition function as

$$\int dA \, e^{-S(A)} \quad (29)$$

The action $S(A)$ is gauge invariant so that we must factorize the volume of the gauge group. It is interesting to see how this works in our case. Let us first define for convenience the 3×3 matrix:

$$\phi_i^a \equiv p_i^a + g A_i^a \quad (30)$$

which is the analogue of the field $A_0 + A$ ($A_0 \equiv$ background field) of Ref. [4].

The gauge group acts on the matrix ϕ on the left:

$$\phi_i^a \rightarrow \phi_i'^a = R^a_b \phi_i^b \quad (31)$$

where R^a_b is a rotation matrix.

This means that we can gauge away the orthogonal part of ϕ , going to the symmetric gauge where ϕ is a positive definite symmetric matrix, i.e.:

$$\phi \in \frac{GL(3, \mathbb{R})}{O(3)} \quad (32)$$

The gauge fixing conditions are therefore:

$$\begin{aligned}f_1 &= -\phi_3^2 + \phi_2^3 \\ f_2 &= \phi_3^1 - \phi_1^3 \\ f_3 &= -\phi_2^1 + \phi_1^2\end{aligned}\quad (33)$$

and the Faddeev-Popov determinant reads:

$$\det \frac{\delta f_A}{\delta \epsilon_a} = \det \begin{pmatrix} \phi_2^2 + \phi_3^3 & -\phi_2^1 & -\phi_3^1 \\ -\phi_1^2 & \phi_1^1 + \phi_3^3 & -\phi_3^2 \\ -\phi_1^3 & -\phi_2^3 & \phi_1^1 + \phi_2^2 \end{pmatrix}$$

$$= \det [(\text{tr} \phi) \mathbb{1} - \phi] = 8 \det \left(\mathbb{1} + \frac{g}{2} (\text{tr} A - A) \right) \quad (34)$$

In this gauge, the matrix A_1^a is symmetric and will be denoted by a new symbol ψ_{ij} . The action (27) takes a simple form in the symmetric gauge if we define the vector:

$$\vec{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \equiv \begin{pmatrix} -2A_1^0 \\ -2A_2^0 \\ A_3^0 \end{pmatrix} \quad (35)$$

reading

$$\mathcal{S}(\psi) = 2 (\text{tr} \psi - \psi)_{ij} \psi_i \psi_j$$

$$+ g \left[\frac{1}{4} v_i (\delta_{jk} - v_j v_k) \psi_i \psi_j \psi_k + (\text{tr}(\psi^2) - \psi^2)_{ij} v_i \psi_j \right] \quad (36)$$

where we have introduced the unit vector $\vec{v} = (0,0,1)$.

It is worthwhile noticing that the quadratic part in Eq. (36) can be absorbed into the cubic part by means of the field redefinition:

$$\delta_{ij} + g \psi_{ij} = g \chi_{ij} \quad (37)$$

as it is done in Ref. [4], but in our case one gets a linear term proportional to $1/g \psi_i v_i$. The free classical solutions that follow from Eq. (39) (setting $g=0$) take the form:

$$\psi_i = 0, \quad \psi_{ij} = \begin{pmatrix} \psi_{11} & \psi_{12} & 0 \\ \psi_{12} & -\psi_{11} & 0 \\ 0 & 0 & \psi_{33} \end{pmatrix} \quad (38)$$

which justify the claim that $H^1 \approx \mathbb{R}^3$.

The partition function of our model is therefore:

$$Z = \int_D d\psi \det \left(1 + \frac{g}{2} (\tau\psi - \psi) \right) e^{-S(\psi)} \quad (39)$$

where

$$d\psi = \prod_{i=1}^3 d\psi_i \prod_{i \leq j} d\psi_{ij} \quad (40)$$

and D is the integration region.

D is not simply $\mathbb{R}^9$ since we already know that the matrix $1 + g\psi$ belongs to the coset space (32). Moreover, the convergence of the integral (39) imposes the variable ψ_3 to be positive, while ψ_1 and ψ_2 can be either positive or negative. We thus find that:

$$D = \mathbb{R}_+ \times \mathbb{R}^2 \times \frac{GL(3, \mathbb{R})}{O(3)} \quad (41)$$

where $\mathbb{R}_+$ is the positive real axis.

The Faddeev-Popov determinant in (39) takes a simple form if we define the matrix χ_{ij} as

$$\chi = 1 + \frac{g}{2} (\tau\psi - \psi) \quad (42)$$

which is also positive definite and symmetric.

The integral (39) then reads:

$$Z = \frac{1}{g^6} \int_{\mathbb{R}^2} d\psi_1 d\psi_2 \int_{\mathbb{R}_+} d\psi_3 \int_{GL(3, \mathbb{R})/O(3)} d\chi \det \chi e^{-S(\psi, \chi)} \quad (43)$$

where the action S is in the new variables:

$$\begin{aligned} S = & -\frac{8}{g} \psi_i \psi_i + \frac{g}{4} \psi_3 (\psi_1^2 + \psi_2^2) \\ & + \frac{4}{g} \left(\tau\chi^2 - \chi^2 + \chi \tau\chi - \frac{1}{2} (\tau\chi)^2 \right)_{ij} \psi_i \psi_j \end{aligned} \quad (44)$$

Equation (43) can be taken as the starting point for constructing "string amplitudes".

Let us now summarize our results. In this article we have constructed a toy model of Witten's open string field theory by considering the $SU(1,1)$ subgroup of the Virasoro group. The string field is given in terms of finite dimensional

matrices which satisfy Witten's axioms. We have chosen a representation of these matrices which leads to a non-trivial cohomology. In this representation, the string gauge group is $SO(3)$. One should further investigate the dependence of this cohomology on the representation of these finite dimensional matrices. We have performed the gauge fixing of the action, finding a simple formula for the partition function. It is not clear to us whether this toy model has any physical significance but it may still capture some information about the real string.

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