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MATTER-COUPLED SUPERGRAVITY WITH GAUSS-BONNET INVARIANTS:
COMPONENT LAGRANGIAN AND SUPERSYMMETRY BREAKING

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ABSTRACT

We present the component Lagrangian for the general coupling of $N = 1$ matter to a higher curvature four-dimensional supergravity in which the $(\text{curvature})^2$ terms enter the Lagrangian only through the Gauss-Bonnet and Hirzebruch invariants. This is the situation suggested by the low-energy limit of heterotic strings, after compactification to four dimensions. The model obtained from the string by $SU(3)$ -invariant truncation of the toroidal compactification is discussed in detail. We give the closed-form expression for the scalar potential and discuss supersymmetry breaking via gaugino condensation. We show that, in this last case, the cosmological constant Λ remains exactly zero even when the higher curvature corrections are taken into account. We also discuss briefly the problem of auxiliary field propagation and show that the spurious states decouple (i.e. they get an infinite mass) on the physical vacuum with $\Lambda = 0$, irrespective of whether SUSY is broken or not. Some new, stringy, developments are discussed in the last section.

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1. INTRODUCTION AND DISCUSSION

It is generally thought [1] that, in any scheme of unification of all the fundamental interactions, $N = 1$ supergravity [2] should come into the game as an intermediate step.

Then, $N = 1$ SUGRA should represent a low-energy effective theory emerging from a more fundamental theory which provides a fully consistent quantum framework for (super-)gravity as well as for the other known interactions. The fundamental theory, in particular, will remove the arbitrariness of the matter coupling still present in the $N = 1$ theory [2].

The effective theory should necessarily contain also higher curvature and higher derivative terms, produced by integrating out the heavy states whose existence is implied by the good ultraviolet (UV) behaviour of the underlying fundamental theory.

In view of the above considerations, the structure of the standard $N = 1$ Lagrangians [2] (up to now confined to terms with at most two derivatives) should both be generalized (adding higher derivative terms) and specialized (using the constraints on the couplings implied by the underlying theory).

The following guiding principles have been proposed for such a program (or for part of it):

- i) extended supergravity theories [3];
- ii) "modern" Kaluza-Klein theories [4];
- iii) superstring theories [5].

The first two approaches seem to have serious problems, both from a phenomenological (e.g. chirality of fermions [6]) and from a fundamental point of view, since they do not lead to a consistent quantum theory [7] any more than ordinary supergravity does. Thus it is superstring theory that now carries the day.

Quantum consistency of a string theory seems to require at least world-sheet supersymmetry [5], and most probably space-time SUSY as well. This restricts the possible critical dimensions to be ten [5], for a superstring moving on a flat Minkowski target (however, it should be kept in mind that the dimensionality of space-time is not unambiguously defined in string theory: certain situations are better described as superstrings in a lower dimension than as compactified ten-dimensional (10D) superstrings [8]).

String theories have a finite number of massless degrees of freedom and an infinite number of massive ones. The dynamics of

the massless sector of a superstring is specified by a supersymmetric "local" field theory for the massless fields [9]. Such an effective field theory can be seen as the low-energy expansion of the effective (non-local) action for the massless modes obtained by integrating out the massive states.

The simplest of such 10D effective field theories emerging from superstrings is the $N = 1$ supergravity coupled to super-Yang-Mills [10,11], which in ten dimensions is a chiral theory. Naïvely, in the low-energy regime this effective theory reduces to just the Chapline-Manton Lagrangian [10], as is well-known.

But, as emphasized above, the full effective theory contains higher curvatures too. What it is more, the Green-Schwarz mechanism [12], to cancel the one-loop gauge and gravitational anomalies [13], implies a modification of the Chapline-Manton action, which, together with supersymmetry, requires the addition of higher curvature terms [14,15].

As is well-known from current-algebra considerations, the effective Lagrangian suitable for computing soft processes should have two fundamental properties:

- i) It must have all the symmetries which are exact symmetries of the underlying theory; (in particular, it should be supersymmetric).
- ii) It should reproduce exactly the same anomalies as the fundamental theory ('t Hooft criterion [16]).

Any supersymmetric field theory for the massless states of the superstrings which satisfies the criteria stated above (i.e. which contains the Green-Schwarz anomaly-cancelling mechanism) should contain higher curvature terms (as well as some suitable Wess-Zumino functionals [17]). Then, one is naturally led to studying higher derivative supergravities, and in particular the special ones emerging from the SUSY completion of the Green-Schwarz terms.

Partial progress towards a supersymmetric completion of these terms (in 10 dimensions) has been obtained using the component formalism [18], the group manifold [19], and the super-space approach [20].

Most recently, d'Auria and Fré [21], building upon previous work by Bonora, Pasti and Tonin [22], have succeeded in writing down the full supersymmetric completion which is (obviously) rather complicated.

Of course, we are mainly interested in the 4D effective theory resulting from the spontaneous compactification of six dimensions on some kind of 'Calabi-Yau space' [21,23], with one surviving supersymmetry.

The study of the higher curvature terms required by the Green-Schwarz modification is much more easy in 4D $N = 1$ supergravity than in ten dimensions, partly because in four dimensions we have a very powerful tensor calculus [2,24,25,26].

The structure of the new terms is quite independent of the details of the particular compactification (as long it is supersymmetric) and leads to terms which are quadratic in the gravitational [27,28] (as well as Yang-Mills) curvatures^{*}). A particularly maniable case corresponds to the SUSY completion of the trivial compactification of the 10D theory, modified by the addition of the Chern-Simons terms to the $B_{\mu\nu}$ field strength

$$H_{\mu\nu\lambda} = 3\partial_{[\mu} B_{\nu\lambda]} + \frac{1}{30} \omega_{\mu\nu\lambda}^{3YM} - \omega_{\mu\nu\lambda}^{3L} \quad (1.1)$$

as required by the Green-Schwarz mechanism [12].

Notice that the standard supergravity form [12] already contains the term - quadratic in $F_{\mu\nu}$ - implied by the Yang-Mills Chern-Simons 3-form ω^{3YM} in Eq. (1.1). The contributions coming from the supersymmetrized Lorentz Chern-Simons 3-form ω^{3L} are more delicate, and also a bit ambiguous. In fact, the terms quadratic in the Riemann tensor are predicted by (tree-level) string amplitudes up to the addition of terms containing R and $R_{\mu\nu}$ [29] which vanish for a graviton on shell. However, this ambiguity can be solved by the following perturbative argument [30].

1.1 Ghost decoupling and the Gauss-Bonnet invariant

In general, a theory containing terms quadratic in the gravitational curvatures propagates - besides the usual massless states - massive particles of spin ≤ 2 . This can be seen already at the linearized level, see Refs. [31] for the ordinary and Ref.

*) This is true, strictly speaking, only before the elimination of the auxiliary fields. Solving the auxiliary field equations introduces terms of any degree in the curvatures. However, the higher order terms so obtained differ from the ones coming from more complicated SUSY invariants. For instance, they all vanish for an on-shell graviton.

[32] for the SUSY case. {In the simplest cases a fully non-linear analysis is also available [33,34]}. It results that, whereas the new massive particles with spin $\leq 1/2$ are physically acceptable positive-norm states^{*}), the ones with spin ≥ 1 are bound to be poltergeists [31,32] (i.e. negative-norm states).

From the linearized analysis we know that, in order not to have poltergeists in the spectrum, the terms quadratic in the Riemann tensor should have the following structure

$$\alpha R^2 + \beta {}^*R^*_{abcd} R^{abcd} \quad (1.2)$$

where α and β are some coefficients ($\alpha > 0$ not to have tachyons) and the combination

$${}^*R^*_{abcd} R^{abcd} = (W_{abcd})^2 - 2 \left(R_{ab}^2 - \frac{1}{3} R^2 \right) \quad (1.3)$$

is actually a total derivative, proportional to the density of Euler characteristic (Gauss-Bonnet theorem). Since $\alpha \neq 0$ would require an additional scalar (two Wess-Zumino multiplets in the SUSY case) which is not obviously related to the string, it is natural to put $\alpha = 0$ and require, for consistency, that the (Riemann)² terms enter the Lagrangian only through the Gauss-Bonnet combination, Eq. (1.3) [30], with a field-dependent coefficient^{**}) [otherwise, Eq. (1.3) being a total derivative, the (Riemann)² terms would drop from the Lagrangian].

Is the above argument correct? Of course, the superstring should be poltergeist-free; but, since the above argument relies heavily on perturbative (that is linearized-level) considerations, it cannot be taken too seriously. Indeed, as we observed above [29], as long as tree-level S-matrix elements for massless particles are concerned, the coefficients of R^2 and R_{ab}^2 are completely arbitrary, only the coefficient of the square of the Weyl tensor, $(W_{abcd})^2$, being fixed by the on-shell amplitudes. Thus, as long as we are interested in computing just the soft

^{*}) i) Provided the quadratic terms have the right sign, otherwise they are tachyons. ii) The mixing with 'ordinary' $s \leq 1/2$ fields spoils the positivity of the norm. This happens, for example, for the case of interest here. See Section 6 for details.

^{**}) As we shall see below, this coefficient is uniquely fixed by the Green-Schwarz mechanism.

processes, we can always assume that we have the full Gauss-Bonnet combination.

Nevertheless, it makes perfect sense to ask what is the particular combination of $(\text{curvature})^2$ terms which is produced by integrating out the heavy string modes. Duff, Nilsson and Pope [35] have shown, using Kaluza-Klein techniques, that for the heterotic string, exactly the Gauss-Bonnet combination appears; this result is also confirmed by the 10D Lagrangian found by D'Auria and Fré [21]. Thus the perturbative argument of Zwanziger turns out to be correct (in the heterotic case).

However, the above arguments should - in the 4D supersymmetric case - be supplemented with an analysis of the auxiliary field sector. In a higher derivative supergravity, most of the new massive degrees of freedom are just auxiliary fields which get promoted to physical propagating ones [32]. Now, it is true (and obvious) that - if the $(\text{Riemann})^2$ terms form the poltergeist-free Gauss-Bonnet combination - also the auxiliary fields do not propagate, at the linearized level around a supersymmetric background. Of course, the auxiliary fields have some derivative coupling (SUSY partners of the derivative couplings for the gravitational fields coming from the Gauss-Bonnet term) but no kinetic term, around such a background^{*}).

However, as emphasized in Ref. [34], there is a great difference between linear-level decoupling of the unwanted states and exact, non-linear level decoupling. Indeed, in principle, we can introduce new fields for the massive degrees of freedom [34]; in the new field basis, the Lagrangian should turn out to have almost the canonical form. With this field representation, linear-level decoupling amounts to the fact that in the perturbative vacuum the new fields get an infinite mass; whereas exact decoupling would require the field to decouple from the Lagrangian {for examples see Ref. [34]}. The distinction between the two decoupling mechanisms may seem to be just a mathematical fancy, but it is not. The point is that, if other fields are present, there may be some non-perturbative vacuum, corresponding to non-vanishing v.e.v.'s for the fields introduced to represent the massive states, around which the new (and usually unwanted) fields propagate with a finite mass.

^{*}) Thus, the auxiliary field equations are no longer algebraic, as in standard supergravity, but are differential equations.

Since in higher derivative SUGRA the auxiliary field equations are no longer linear [36,37], in general there is more than one solution (even for a translationally invariant background). Then we have the problem of giving a physical interpretation (if any) to the new solutions. If the auxiliary fields turn out to be propagating degrees of freedom, these new solutions should be regarded as new physical vacua^{*)} corresponding to non-trivial values for the new fields, which, being the auxiliary components of the initial multiplets, imply broken SUSY. Thus, higher derivative SUGRA potentially contains a new 'non-perturbative' mechanism to break supersymmetry.

The above discussion is not merely academical, since in Ref. [37] it was found that, for the situation of interest here, the auxiliary field equations have two solutions in a constant background. The 'non-perturbative' one [which is not continuously connected with the standard result when we let (Riemann)² terms vanish] can be seen as a 'vacuum' in which the auxiliary fields get non-trivial v.e.v. and become propagating with a finite mass, of the order of 1 in Planck units. However, since they are poltergeists, this solution cannot be taken too seriously, but it is just a spurious effect due to the fact that our Lagrangian is not a good approximation to the real string dynamics at Planck energies. See, however, Section 10 for the discussion of some new developments on this problem which occurred after the completion of the present paper.

1.2 Purpose and plan of the paper

It is quite clear that in four dimensions, owing to the superconformal tensor calculus, one can derive the full off-shell local supersymmetric completion of any higher derivative term involving curvatures or any other interaction of the $N = 1$ theory. Moreover, 4D results give important insight into the more complicated situation of 10D anomaly-free supergravity [37].

Partially with the above motivations in mind, we have extended the construction of the general matter coupling [2] of 4D $N = 1$ supergravity to the case in which curvature squared

*) Provided they belong to the positive kinetic domain

terms are present. More precisely, we will have chiral multiplets coupled to Yang-Mills and the full Gauss-Bonnet combination.

Some of the essentials have already been presented in Refs. [14] and [37].

In view of the previous discussion, we have chosen to construct directly the super-Gauss-Bonnet combination, suggested by the unitarity requirement. Anyhow, as we have seen above, from the point of view of on-shell soft processes no generality is lost because of this assumption.

Since one of the most important issues is the dynamical realization of supersymmetry, we are primarily interested in the structure of the scalar potential. Hence, we will not present the final fermionic structure (which is, however, completely under control) but for terms involving the fermionic partners of the gauge vectors (gauginos) which may be relevant for supersymmetry-breaking through fermionic condensates [38,39], as is directly suggested by the example of the trivial compactification [40].

Notice that the auxiliary field equations - which are non-linear differential equations, coupled with the usual matter equations - reduce to algebraic ones for a translationally invariant background, which can be explicitly solved to yield the scalar potential in a closed form.

In the present paper we shall use the old minimal Poincaré tensor calculus [24]. The modifications needed to recast our results in the framework of the more powerful (but less widely known) superconformal tensor calculus [2,25,26] are described in Appendix C.

The plan of this paper is the following: in Section 2 we briefly review the main properties of the 4D $N = 1$ supergravity obtained by truncating the 10D theory compactified on a 6-torus [40]. In Section 3 we discuss the three gravitational supermultiplets [27,28] and recall the supersymmetric versions of the Gauss-Bonnet and Hirzebruch index theorems [28,41].

In Section 4, the general interaction is written down. In Section 5 we discuss the equations of motion of the auxiliary fields as well as some related topics such as the existence - beyond the linearized level - of "spurious" massive states. In Section 6 we specialize the previous results to the situation predicted by the superstring and solve the auxiliary field

equations for a symmetric background with - possibly - fermion condensates.

In Section 7 we discuss the scalar potential in the standard field basis^{*)} (the 'non-minimal scheme') and discuss the possibility of SUSY breaking via fermionic condensates, with a vanishing cosmological constant.

In Section 8 we rewrite the scalar potential in the so-called 'minimal scheme', i.e. a field basis where there are no other gravitational curvatures in the 'effective' Lagrangian besides the Einstein and Gauss-Bonnet Hirzebruch terms^{**)}. More precisely, now the mixing between the conformal part of the metric and the auxiliary field U is diagonalized by a suitable Weyl rescaling. Of course, the physics turns out to be the same as in Section 7. In Appendix A we present our conventions and some other useful formulae. In Appendix B the reader can find the bosonic components of the three fundamental chiral multiplets $R\bar{\Sigma}(\bar{R})$, $\Sigma(E_\alpha E^a)$, and $W_{\alpha\beta\gamma} W^{\alpha\beta\gamma}$. Finally, in Appendix C it is explained how these multiplets are described in the framework of the superconformal tensor calculus [2,25,26].

After the completion of the present work, we added a section (see Section 10 below) to discuss some new major developments in the field. Indeed, a study of the realization of supersymmetry at the full string level (which was, partly, motivated just by the results of the present paper) leads to the conclusion that the correct auxiliary field formulation corresponding to the heterotic string case is the new-minimal one. The main properties of the interactions via new-minimal (super-) Gauss-Bonnet invariants are discussed in some detail in Section 10.

2. THE 4D EFFECTIVE LAGRANGIAN COMING FROM THE STRING

In this section we shall briefly review the main features of the low-energy Lagrangian one obtains by an $SU(3)$ -invariant truncation [40] of the 4D effective Lagrangian resulting from a

*) In a superconformal language this basis corresponds to the usual Kugo-Uehara superconformal gauge [2,25], whereas the 'minimal scheme' may be seen as an improved gauge.

**) In the 'non-minimal scheme' the other curvatures appear after the elimination of the auxiliary fields (a particular case of integrating out 'massive' states) owing to their mixing with the metric. Anyhow, the new curvature terms vanish on-shell, and thus the difference between the two schemes is just an example of the basic ambiguity discussed before.

compactification on a 6-torus of the 10D Chapline-Manton theory [10] as modified by Green and Schwarz [12].

Here we describe only the general, model-independent properties which we need to describe the supersymmetric completion [14] of the Lorentz Chern-Simons terms in Sections 3 and 4 below. More details of this particular model, which are more specific to the special case described by Witten [40], will be presented in Section 6 and their consequences explored in Sections 7 and 8.

The bosonic terms of the relevant 4D Lagrangian are [40]:

$$\begin{aligned}
\frac{1}{\epsilon} \mathcal{L} = & -\frac{1}{2} R - 3 \partial_r \sigma \partial^r \sigma - \frac{9}{16} \frac{1}{\varphi^2} \partial_r \varphi \partial^r \varphi + \\
& -\frac{1}{2} e^{-2\sigma} \varphi^{-3/2} \left(\partial_r a - \frac{i}{\sqrt{2}} \kappa \bar{C}_x \overleftrightarrow{D}_r C^x \right)^2 + \\
& -\frac{3}{4} \varphi^{-3/2} e^{6\sigma} \hat{H}^{\mu\nu\lambda} \hat{H}_{\mu\nu\lambda} - \frac{1}{4} \varphi^{-3/4} e^{3\sigma} f F^{\mu\nu} F_{\mu\nu} + \\
& -3 e^{-\sigma} \varphi^{-3/4} D_r \bar{C}_x D^r C^x + \\
& -\frac{3}{2} \tilde{g}^2 \varphi^{-3/4} e^{-5\sigma} \left| \frac{\partial W}{\partial C} \right|^2 + \\
& + \frac{9}{2} \frac{\tilde{g}^2}{f} \varphi^{-3/4} e^{-5\sigma} \sum_i (\bar{C}_i, \lambda_i C_i)^2 + \\
& -8 \kappa^2 \tilde{g}^2 \varphi^{-3/2} e^{-6\sigma} |W|^2
\end{aligned} \tag{2.1}$$

φ is the scalar field (dilaton) of the 10D $M = 1$ SUGRA multiplet, σ the breathing mode of the internal manifold ($G_{IJ} = e^\sigma \delta_{IJ}$, $I, J = 1, \dots, 6$) and $\hat{H}_{\mu\nu\lambda}$ is the improved field strength of the 2-form $B_{\mu\nu}$

$$\hat{H} = dB + \frac{1}{30} \omega_{3YM} - \omega_{3L} \tag{2.2}$$

where

$$\begin{aligned}
\omega_{3YM} &= \text{Tr} \left(AF - \frac{1}{3} A^3 \right) \\
\omega_{3L} &= \text{Tr} \left(\omega R - \frac{1}{3} \omega^3 \right)
\end{aligned}$$

are the Yang-Mills and Lorentz Chern-Simons 3-forms, respectively.

In Eq. (2.1) W is the chiral superpotential, which is a cubic polynomial in the charged fields C^i (see also Section 6). By a duality transformation [40]

$$e^{6\sigma} \varphi^{-3/2} \hat{H}_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda\gamma} \partial^\gamma D \tag{2.3}$$

we can introduce a chiral multiplet S (axion-dilaton multiplet) with spin-zero component:

$$Z = e^{3\sigma} \varphi^{-3/4} + i 3\sqrt{2} D \quad (2.4)$$

An integration by part in (2.1) and Bianchi identities give the coupling $\text{Im } z(\text{tr } F\tilde{F} - \text{tr } R\tilde{R})$. In the effective 4D SUGRA theory the axion D should have a kinetic term determined by the Kähler potential $K = -\log(S + \bar{S})$ plus terms independent of S , since K should be invariant (as well as the full theory) under the shift $S \rightarrow S + ic$ (c real). The term $\text{Im } z \text{tr } F\tilde{F}$ is already present in the Lagrangian obtained by Witten [40]. Collecting the terms quadratic in the Yang-Mills curvatures we recognize the bosonic part of the last component of the superfield expression sw_α^2 , where W_α is the Yang-Mills superfield strength. In this way we have the supersymmetric 4D completion of the contribution to the effective Lagrangian from the Yang-Mills Chern-Simons 3-form.

The term $-\text{Im } z \text{tr } R\tilde{R}$ is the interaction induced in four dimensions by the Lorentz Chern-Simons modification of the field strength H , Eq. (22), and the problem is to derive its supersymmetric completion. This has been done in Ref. [14].

In the next sections we shall give full details on these issues, in particular for the fermionic structure of the theory.

We have seen here a first reason for local supersymmetrization of a higher curvature term. Since this is quadratic in the Riemann tensor, the supersymmetrization requires the knowledge of the properties of the (super-) gravitational curvature multiplets. These multiplets are described in the next section.

3. GRAVITATIONAL SUPERMULTIPLETS

The analysis of the 4D gravitational multiplets can be carried out in the framework of the superconformal tensor calculus^{*)} [2,24,25], in the super Poincaré gauge [27,28] or of the superspace geometry [42]. In this last approach one solves the superspace Bianchi identities and torsion constraints in terms of three basic superfields [42]:

*) For details see Appendix C

Weyl superfield $W_{\alpha\beta\gamma}$ (chiral)
 Scalar curvature R (chiral)
 Ricci (or Einstein) $E_{\alpha\dot{\alpha}}$ (real vector)

They contain, respectively, as part of their bosonic components, the Weyl tensor $W_{\mu\nu\sigma\tau}$, the scalar curvature R , and the tensor $R_{\mu\nu} = (1/6) \delta_{\mu\nu} R$. The three superfields are also subjected to the following constraints:

$$\begin{aligned}\bar{\nabla}_{\dot{\alpha}} R &= \bar{\nabla}_{\dot{\alpha}} W_{\alpha\beta\gamma} = 0 \\ \nabla^{\alpha} E_{\alpha\dot{\alpha}} &= \bar{\nabla}_{\dot{\beta}} \bar{R} \\ \nabla^{\alpha} W_{\alpha\beta\gamma} &= \nabla_{\beta}^{\dot{\gamma}} E_{\gamma\dot{\gamma}} + \nabla_{\gamma}^{\dot{\delta}} E_{\beta\dot{\delta}}\end{aligned}\tag{3.1}$$

Three chiral superfields, quadratic in the supercurvatures, can be constructed out of the three superfields [42]:

$$\begin{aligned}W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} &\quad (\text{gravitational analogue of } W_{\alpha} W^{\alpha}) \\ (\bar{\nabla}^2 - 8R)E^2 & \\ (\bar{\nabla}^2 - 8R)R\bar{R} &.\end{aligned}\tag{3.2}$$

In the notation of the Poincaré tensor calculus [27,28] they are written as

$$W_{\alpha\beta\gamma} W^{\alpha\beta\gamma}, \quad \Sigma(E_a E^a), \quad R T(R)\tag{3.3}$$

where T [2,25,26] denotes the operation which converts a chiral multiplet into another chiral multiplet (kinetic multiplet).

More precisely, given the chiral multiplet $S = (z, \chi_L, h)$, then

$$\begin{aligned}T(S) &= \left(\bar{h} + \frac{1}{3} \bar{U} z, \quad \not{D}^c \chi_R, \quad \not{U}^c \bar{z} + \right. \\ &\quad \left. - \frac{2}{3} U (\bar{h} + \frac{1}{3} \bar{U} \bar{z}) \right)\end{aligned}\tag{3.4}$$

where covariant derivatives and the covariant box are defined in Appendix A, and $u = S - iP$ and A_{μ} are the auxiliary fields of the $N = 1$ supergravity (we are always in the 'old minimal' Poincaré formulation [24]).

Within this last formalism we recall here the explicit content of the components of the three gravitational multiplets.

All of them can be understood as resulting from the superconformal tensor calculus.

- Scalar curvature chiral multiplet R

$$R = \left(U^*, -\gamma \cdot Z_R, \frac{1}{2} R^{\omega\omega}(M) + \frac{1}{2} \bar{\Psi}_a Z_a + \right. \\ \left. - \frac{2}{3} U^* U - \frac{1}{3} A_a^2 - i D_a^P A_a \right) \quad (3.5)$$

where

$$D_a^P A_a = e_a^\mu \left(\partial_\mu A_a + \omega_{\mu ab} A_b - i \frac{1}{2} \bar{\Psi}_\mu \gamma_5 Z_a \right) \quad (3.6)$$

A table of covariant curvatures and Z_a can be found in Appendix A.

- Weyl chiral multiplet $W_{ab,\alpha}$

The Weyl multiplet $W_{ab,\alpha}$ is a chiral multiplet with two local Lorentz (flat) indices (a,b) and one spinor index α , which contains the Weyl tensor in its second component. Its components are:

$$W_{ab,\alpha} = \left(z R_{ab,\alpha}(Q)_L, \left[\sigma_{cd} W_{abcd}^{\omega\omega} + \right. \right. \\ \left. \left. - \frac{3}{2} i R_{cd}(A) \gamma_5 T_{cdab} \right]_{\beta d,L}, 2 Z_{ab,\alpha L} \right) \quad (3.7)$$

where

$$Z_{ab,\alpha} = \left[z R_{cd}^{\omega\omega}(S) T_{cdab} - \frac{1}{2} R_{ab}(Q) (S - i \gamma_5 P) \right]_\alpha \\ T_{abcd} = \frac{1}{6} \left(\delta_{ac} \delta_{bd} - \frac{1}{2} \gamma_5 \epsilon_{abcd} - \delta_{ac} \sigma_{bd} + \delta_{ad} \sigma_{bc} \right) - (a \leftrightarrow b) \quad (3.8)$$

and the matrix T_{abcd} is a projection operator ($T^2 = T$) (notice that $[T_{abcd} R_{cd}(Q) = R_{ab}(Q)]$. Because $R_{ab}(Q) \gamma_a = 0$, the whole multiplet is transverse in spinor space, i.e. $W_{ab,\alpha} (\gamma^a)_\beta^\alpha = 0$.

- Ricci real vector multiplet E_a

The Ricci multiplet E_a is a real vector multiplet with an external local Lorentz index, whose components [28] ($C_a, \zeta_a, H_a, K_a, B_{ab}, \lambda_a, D_a$) are the following:

$$C_a = A_a$$

$$Z_a = Z_a$$

$$H_a = D_a^P P + A_a S$$

$$K_a = D_a^P S - A_a P$$

$$B_{ab} = \frac{3}{2} [R_{ab}^{\omega\omega}(M) - \frac{1}{6} S_{ab} R^{\omega\omega}(M)] + \frac{i}{2} \tilde{F}_{ab}^{\omega\omega} + \\ - \frac{1}{6} (UU^* + A_c^2) S_{ab} + \frac{1}{3} A_a A_b + \frac{1}{2} \bar{\Psi}_a Z_b$$

$$\lambda_a = (D_a^P + \frac{i}{2} A_a \gamma_5) \gamma \cdot Z - (\not{D}^P - \frac{i}{2} A \gamma_5) Z_a + \\ + \frac{1}{2} (S + i \gamma_5 P) Z_a + \frac{i}{2} \gamma_5 A Z_a + \frac{i}{3} \gamma_5 A S_{ab} Z_b + (3.9) \\ + \text{TERMS WHICH VANISH ON MULTIPLICATION} \\ \text{BY } A_a$$

$$D_a = D_a^P (D_b^P A_b) - D_b^P (D_a^P A_a) - \frac{1}{3} (\text{Tr } B_{cd}) A_a + \\ - \frac{5}{6} B_{ab} A_b - \frac{1}{6} A_a UU^* + \frac{1}{6} B_{ba} A_b - \frac{i}{6} \epsilon_{abcd} (D_c^P A_b) A_d + \\ + \frac{i}{6} \bar{Z}_a \gamma_5 \gamma \cdot Z - \frac{i}{6} \bar{Z}_b \gamma_5 \gamma_a Z_b + \frac{i}{12} \bar{Z}_b \gamma_c Z_a \epsilon^{abcd} + \\ - i \bar{Z}_b \gamma_5 [(D_b - \frac{i}{2} A_b \gamma_5) \psi_a - (a \leftrightarrow b)] + \\ + \text{TERMS WHICH VANISH ON MULTIPLICATION} \\ \text{BY } A_a$$

where

$$F_{ab}^{\omega\omega} = \frac{3}{2} R_{ab}(A)$$

$$D_a^P S = \partial_a S + \frac{1}{2} \bar{\Psi}_a \gamma \cdot Z \quad (3.10)$$

$$D_a^P P = \partial_a P - \frac{i}{2} \bar{\Psi}_a \gamma_5 \gamma \cdot Z$$

We need quadratic forms in the previous multiplets. As a matter of fact the very same 'square' of the Weyl multiplet contains in its last component h, the Hirzebruch density, thus providing the means for the supersymmetric completion of $\text{Im } z \text{ tr } R\tilde{R}$ which we are looking for.

Unfortunately, it would also lead to a term of the form $\text{Re } z W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma}$ where $W_{\mu\nu\rho\sigma}$ is the Weyl tensor, which propagates negative metric states and also the auxiliary field A_a already at the linearized level.

To solve this problem, and in agreement with the general discussion of these topics in the Introduction, we consider the supersymmetric Gauss-Bonnet combination [41,28]. The ingredients of such a combination are the chiral multiplets $RT(R)$, $\bar{W}_{ab} W_{ab}$, and the chiral multiplet obtained by the standard conversion of the square of the Ricci real vector multiplet. The bosonic terms in the components of such quadratic forms were originally computed in Refs. [28,37]. Here we present the fermionic terms as well. For the chiral multiplet $RT(R)$, we have

$$[RT(R)]_z = U^* \hat{H}^* \quad (3.11a)$$

$$[RT(R)]_{\chi_L} = U^* \not{D}^L (-\gamma \cdot Z_L) - \hat{H}^* \gamma \cdot Z_R \quad (3.11b)$$

$$[RT(R)]_h = U^* \not{D}^L U - \frac{2}{3} U U^* \hat{H}^* + \hat{H}^* \left(\frac{1}{2} R^{\text{cov}}(M) + \frac{1}{2} \bar{\Psi}_a Z_a - \frac{2}{3} U U^* + \right. \\ \left. - \frac{1}{3} A_a^2 - i D_a^P A_a \right) + 2 (\gamma \cdot Z_R) \not{D}^L (-\gamma \cdot Z_L) \quad (3.11c)$$

where

$$\hat{H}^* = \frac{1}{2} R^{\text{cov}}(M) + \frac{1}{2} \bar{\Psi}_a Z_a - \frac{1}{3} U U^* - \frac{1}{3} A_a^2 + i D_a^P A_a \quad (3.11d)$$

$$D_a^L U = \partial_a U + \bar{\Psi}_{aR} \gamma \cdot Z_L - \frac{i}{3} A_a U \quad (3.11e)$$

The components of the square of the Weyl multiplets ($\bar{W}_{ab} W_{ab}$) are

$$[\bar{W}_{ab} W_{ab}]_z = 4 \overline{R_{\mu\nu}(Q)_L} R^{\mu\nu}(Q)_L \quad (3.12a)$$

$$[\bar{W}_{ab} W_{ab}]_{\chi_L} = (4 \delta_{ab} W_{\mu\nu ab}^{\text{cov}} - 6i R_{\mu\nu}(A)) R^{\mu\nu}(Q)_L \quad (3.12b)$$

$$\begin{aligned}
[\bar{W}_{ab} W_{ab}]_h = & 2 (W^{\text{cov}}_{\mu\nu ab})^2 - 2 W^{\text{cov}}_{\mu\nu ab} \tilde{W}^{\text{cov}}{}^{\mu\nu}{}_{ab} + \\
& - 3 R_{\mu\nu}(A) R^{\mu\nu}(A) + 3 R_{\mu\nu}(A) \tilde{R}^{\mu\nu}(A) + \\
& + 16 \overline{R_{\mu\nu}(Q)}_L R^{\mu\nu \text{cov}}(S) + \\
& - 4 U \overline{R_{\mu\nu}(Q)}_L R^{\mu\nu}(Q)_L
\end{aligned} \tag{3.12c}$$

where use has been made of the multiplication rule for multiplets with an anticommuting first component:

$$S_1 \cdot S_2 = (Z_1 Z_2, -Z_1 \chi_{2L} + Z_2 \chi_{1L}, -Z_1 h_2 + Z_2 h_1 + 2 \bar{\chi}_{1L} \chi_{2L}) \tag{3.13}$$

The square of the Ricci multiplet is a real vector multiplet with components:

$$\begin{aligned}
E_a^2 = & (A_a^2, 2 A_a Z_a, 2 A_a (D_a^P P + A_a S) - \frac{1}{2} \bar{Z}_a Z_a, \\
& 2 A_a (D_a^P S - A_a P) + \frac{i}{2} \bar{Z}_a \gamma_5 Z_a, 2 A_a B_{ab} + \\
& + \frac{i}{2} \bar{Z}_a \gamma_5 \gamma_b Z_a, \Lambda, D)
\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
\Lambda = & 2 A_a \gamma_b [(D_a^P - \frac{i}{2} A_a \gamma_5) Z_b - (a \leftrightarrow b)] + (D_a^P P + \\
& + A_a S - i \gamma_5 D_a^P S + i \gamma_5 A_a P - i B_{ab} \gamma^b \gamma_5 - \not{D}^P A_a) Z_a + \\
& + (S + i \gamma_5 P) A_a Z_a + \frac{i}{3} A_c^2 \gamma_5 \gamma_c Z - \frac{2}{3} i \not{A} \gamma_5 A_a Z_a
\end{aligned} \tag{3.15a}$$

$$\begin{aligned}
D = & (D_a^P P + A_a S)^2 + (D_a^P S - A_a P)^2 - (B_{ab})^2 - (D_a^P A_a)^2 + \\
& + 2 A_a D_a^P (D_b^P A_b) - 2 A_a D_b^P (D_b^P A_a) - \frac{2}{3} \text{Tr} (B_{ab}) A_a^2 + \\
& - \frac{4}{3} B_{ab} A_a A_b - \frac{i}{3} A_a^2 U U^* + \bar{Z}_1 (\not{D}^P - \frac{i}{2} \not{A} \gamma_5) Z_a + \\
& - 2 \bar{Z}_a (D_a^P + \frac{i}{2} A_a \gamma_5) \gamma_c Z - \bar{Z}_a (S + i \gamma_5 P) Z_a + \\
& + \frac{i}{3} \bar{Z}_1 A \gamma_5 \gamma_c Z + \frac{4}{3} i \bar{Z}_a \gamma_5 \not{A} Z_a + \\
& + \frac{i}{2} \bar{Z}_b \gamma_c Z_d A_a \epsilon^{abcd} - 2i \bar{Z}_b \gamma_5 [(D_b - \frac{i}{2} A_b \gamma_5) \psi_a + \\
& - (a \leftrightarrow b)] A_a
\end{aligned} \tag{3.15b}$$

To transform this real vector multiplet into a chiral multiplet, it is sufficient to use the following general embedding formula [Ref. [43], formula (3.20)] (in the 'old minimal' formulation): given $V = [C, \bar{Z}_L, \bar{\chi}_L, B_m, A_R, D]$

$$\begin{aligned} \Sigma(V) = & \left[-\frac{1}{2} \mathcal{H}^* + \frac{2}{3} U^* C, -i \Lambda_L + \frac{i}{3} U \bar{Z}_L - \frac{1}{3} \mathcal{H} \bar{Z}_R + \right. \\ & - i \not{D}^c \bar{Z}_R, D - \frac{2}{9} (2 U U^* + A_a^2) C + \frac{2}{3} B_a A_a + \\ & + \not{D}^c C + \frac{i}{2} (\bar{\Psi}_m \eta - 2 \bar{\Phi}_m)_R \gamma_m \bar{Z}_L + \\ & + i \not{D}_m (B_m - \frac{2}{3} C A_m) - \frac{1}{3} \mathcal{H} U^* + \\ & \left. + \frac{2}{3} U \mathcal{H}^* \right] \end{aligned} \quad (3.16)$$

All other conventions are described in Appendix A.

Supersymmetric Gauss-Bonnet and Hirzebruch theorems

According to the Gauss-Bonnet theorem the combination

$$\begin{aligned} \sqrt{g} \left[(W_{abcd})^2 - 2 ((R_{ab})^2 - \frac{1}{3} R^2) \right] = \\ = \sqrt{g} {}^* R_{abcd} {}^* R^{abcd} = \\ = \sqrt{g} \left[(R_{\mu\nu ab})^2 - 4 (R_{\mu b})^2 + R^2 \right] \end{aligned} \quad (3.17)$$

of the three possible scalar densities quadratic in the Riemann tensor is a total derivative proportional to the Euler characteristic density. Then, as an action density it is locally trivial and has only a topological content.

The supersymmetric extension is made in terms of the following chiral multiplet, which is a linear combination of the above quadratic forms:

$$\hat{W}_L^2 = R T(R) + \Sigma(E_\alpha E^\alpha) + \frac{g}{16} W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} \quad (3.18)$$

In the superspace formalism the superspace integral is

$$\int d^2\theta \, \mathcal{E}(\hat{W}_L^2)_{\text{SUP. SPACE}} \quad (3.19)$$

where E is the chiral superspace invariant measure [42] and $(\hat{W}_L^2)_{\text{sup. space}}$ is the counterpart of $\hat{W}_L^2$ in the superspace framework^{*}, i.e. the chiral superfield

$$\hat{W}_L^2 \propto W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} + \frac{1}{2} (\bar{\nabla}^2 - 8R) G_{\alpha\dot{\alpha}} G^{\alpha\dot{\alpha}} - 2(\bar{\nabla}^2 - 8R) R \bar{R} \quad (3.20)$$

is a total derivative [41,44].

In the component formalism, the super-Gauss-Bonnet theorem states that the following F-type action density [corresponding to the superspace expression in Eq. (3.19)]

$$\begin{aligned} [\hat{W}_L^2]_F d^4x = & \left\{ [\hat{W}_L^2]_h + [\hat{W}_L^2]_z \right. \\ & \left. + \bar{\Psi}_R \gamma [\hat{W}_L^2]_{\chi_L} + \bar{\Psi}_R \sigma^{\mu\nu} \psi_{\nu R} [\hat{W}_L^2]_z \right\} e d^4x \end{aligned} \quad (3.21)$$

is a total derivative [28]. More precisely, we have that

$$\begin{aligned} \frac{8}{9} \frac{1}{32\pi^2} \text{Re} [\hat{W}_L^2]_F d^4x = & \frac{1}{32\pi^2} \sqrt{g}^* R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau} + \\ & + \text{TOTAL DERIVATIVE OF A GLOBALLY DEFINED,} \\ & \text{LOCAL 3-FORM} \end{aligned} \quad (3.22)$$

So that it is a 'natural' representative of the Euler class; whereas,

$$\begin{aligned} \frac{8}{9} \frac{1}{16\pi^2} \text{Im} [\hat{W}_L^2]_F d^4x = & \frac{1}{16\pi^2} \sqrt{g}^* R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau} + \\ & + \text{TOTAL DERIVATIVE OF A GLOBALLY DEFINED,} \\ & \text{LOCAL 3-FORM} \end{aligned} \quad (3.23)$$

naturally represents the first Pontragin class (= 3 times the Hirzebruch signature class = 24 x [$\hat{A}$ -genus] class). In Appendix B this is explicitly shown for the bosonic sector.

We conclude that the supersymmetric completion of the Witten Lagrangian [40], plus the Lorentz Chern-Simons interactions is

$$\mathcal{L}_0 + \int d^2\theta \mathcal{E} S \left\{ \hat{W}_L^2 - \frac{1}{30} W_{YM}^2 \right\} + \text{h.c.} \quad (3.24)$$

^{*}) The difference in the coefficients between Eqs. (3.18) and (3.20) is due to the fact that superspace and tensor calculus people use different normalizations for the three basic, gravitational multiplets.

where $\mathcal{L}_0$ contains kinetic terms of the chiral multiplets as well as the superpotential terms, $\mathcal{L}_0$ is rather model-dependent; for the simplest case, corresponding to an $SU(3)$ -invariant compactification on a torus, $\mathcal{L}_0$ will be specified in Section 6 below.

Equation (3.24), on the other hand, is essentially model-independent following from duality and gauge invariance alone; in Eq. (3.24) S is the axion-dilaton chiral multiplet [40].

At this point, having gained supersymmetric control over the quadratic gravitational curvature terms, we are in a position to write down any general coupling of, say, the super-Gauss-Bonnet combination with matter, (be it) represented by chiral as well as by vector multiplets. Keeping in mind the original motivation we will consider a general coupling to $N = 1$ matter.

4. CONSTRUCTION OF THE ACTION

The general form of the supergravity Lagrangian which includes a coupling of chiral multiplets Z_i to the supersymmetric Gauss-Bonnet combination reads in superspace notation [2]:

$$\begin{aligned} \mathcal{L} = & \int d^4\theta E \Phi(Z_i, (e^{2V} Z)_i) + \text{Re} \int d^2\theta \mathcal{E} f_{AB}(Z_i) W^A W^B + \\ & + \text{Re} \int d^2\theta \mathcal{E} g(Z_i) + \text{Re} \int d^2\theta \mathcal{E} f(Z_i) \hat{W}_L^2 \end{aligned} \quad (4.1)$$

where E is the superspace determinant and the chiral measure $\int d^2\theta E$ is equal to the full-superspace measure $\int d^4\theta E R^{-1}$, R being the chiral scalar curvature superfield. The W^A superfield denotes the Yang-Mills field strength multiplet, the chiral function $g(Z_i)$ is the superpotential, and the real function $\mathcal{E}$ is related to the Kähler potential J via the relation [2] $J = -3 \log(-\mathcal{E}/3)$. The holomorphic functions f_{AB} and f are the couplings of chiral superfields to the Yang-Mills and gravitational curvature squared terms [2]. Notice however that we can always define a new chiral multiplet S to be equal to $f(Z_i)$, so we do not lose any generality by assuming $f(Z_i) = S$.

Here we will restrict ourselves to the bosonic terms coming from the component expression of (4.1), and to the terms containing the gauge fermions which will be discussed anyway in Sections 7 and 8. We notice that the largest modification brought about by the fourth term of (4.1) concerns terms containing the gravitino and the fermions from the chiral multiplets. At this

stage even string considerations do not lead to direct interference between Yang-Mills and gravitational curvatures. The full component Lagrangian for the first three terms in (4.7) is well known, see Ref. [2], Eqs. (3.14), (3.15), and (3.16). Here we shall discuss only the last term, which is a chiral density.

In the 'old minimal' Poincaré component formalism [24] the density expression for a generic chiral multiplet reads

$$\frac{1}{e} \mathcal{L} = h + v z + \bar{\Psi}_R \cdot \gamma \chi_L + \bar{\Psi}_{rR} \sigma^{r\nu} \Psi_{vR} z \quad (4.2)$$

In the present discussion we consider only the first two bosonic terms. In our case the component of the chiral multiplet $\hat{S}\hat{W}_L^2$ can be computed through the standard multiplication formula

$$\begin{aligned} f(Z_i) \hat{\tilde{Z}} = & \left[f(z) [\hat{\tilde{Z}}]_z, f(z) [\hat{\tilde{Z}}]_{\chi_L} + f'^i(z) \chi_{Li} \cdot \right. \\ & \cdot [\hat{\tilde{Z}}]_z, f(z) [\hat{\tilde{Z}}]_h + (f'^i(z) h_i - f''^{ij}(z) \bar{\chi}_{Li} \chi_{Lj}) \cdot \\ & \left. \cdot [\hat{\tilde{Z}}]_z - 2 \overline{(f''^{ij}(z) \chi_{Li})} [\hat{\tilde{Z}}]_{\chi_L} \right] \end{aligned} \quad (4.3)$$

where

$$Z = [z, \chi_L, h], \quad \hat{\tilde{Z}} = [[\hat{\tilde{Z}}]_z, [\hat{\tilde{Z}}]_{\chi_L}, [\hat{\tilde{Z}}]_h]$$

The bosonic components of the super-Gauss-Bonnet combination can be easily computed from the results of the previous section and are given in Appendix B.

We can then write, for the density:

$$\frac{1}{e} \mathcal{L}_{GB}^{GB} [f(Z) \hat{\tilde{Z}}] = \alpha \left\{ f(z) [\hat{\tilde{Z}}]_h + v [\hat{\tilde{Z}}]_z + f'^i(z) h_i [\hat{\tilde{Z}}]_z + h.c. \right\} \quad (4.4)$$

We introduce a parameter α in front of the Gauss-Bonnet terms, although α is fixed from the anomaly free $D = 10$ theory ($\alpha = 1/9$ with our conventions). We can then split the bosonic sector of the Lagrangian into $\mathcal{L}_0 + \mathcal{L}_\alpha$ where $\mathcal{L}_0$ refers to the standard matter-supergravity interaction, linear in the gravitational curvatures.

Also, according to the usual procedure, we separate the auxiliary field terms in a similar way, so we write:

$$\begin{aligned} \mathcal{L} &= \hat{\mathcal{L}} + \mathcal{L}^{AUX} \\ \hat{\mathcal{L}} &= \hat{\mathcal{L}}_0 + \hat{\mathcal{L}}_\alpha & \mathcal{L}^{AUX} &= \mathcal{L}_0^{AUX} + \mathcal{L}_\alpha^{AUX} \end{aligned} \quad (4.5)$$

The explicit expression for the $\mathcal{L}_0$ part is given in Ref. [2]:

$$\begin{aligned} \frac{1}{e} \hat{\mathcal{L}}_0 = & \frac{1}{12} \phi R + \frac{1}{2} f_{AB} \left(-\frac{1}{4} F_{\mu\nu}^A F^{\mu\nu B} + \frac{1}{4} F_{\mu\nu}^A \tilde{F}^{\mu\nu B} \right) + \\ & - \frac{1}{2} \phi''^i{}_j D_\mu z_i D^\mu z^{*j} + \text{h.c.} \end{aligned} \quad (4.6a)$$

$$\begin{aligned} \frac{1}{e} \mathcal{L}_0^{\text{Aux}} = & \frac{1}{18} \phi (U U^* - A_a^2) + \frac{1}{2} \phi''^i{}_j h_i h^{*j} + \\ & + \frac{1}{2} g^{ii} h_i + \frac{1}{3} U^* \left(\phi'^i h_i + \frac{3}{2} g^* \right) + \\ & + \frac{i}{3} A_a \phi'^i D_a z_i + \frac{1}{2} K \phi'^i T_i^{A j} z_j D^A + \\ & + \frac{1}{4} f_{AB} D^A D^B + \text{h.c.} \end{aligned} \quad (4.6b)$$

where k is the gauge coupling constant, D^a is the auxiliary field of the gauge multiplet, and T_i^{AJ} are the matrix generators of the gauge group in the matter representation.

The explicit expressions for the α -terms are

$$\frac{1}{e} \hat{\mathcal{L}}_\alpha = d \frac{g}{8} f(z) \left({}^* R^*_{abcd} R^{abcd} - R^*_{abcd} R^{abcd} \right) + \text{h.c.} \quad (4.7a)$$

$$\begin{aligned} \frac{1}{e} \mathcal{L}_\alpha^{\text{Aux}} = & d \left\{ f(z) D_a L_a + f''^i h_i \left[\frac{1}{2} R U^* - \frac{1}{3} U^* (U U^* + \right. \right. \\ & \left. \left. + 5 A_a^2) + i U^* D_a A_a + 2 i A_a \partial_a U^* \right] \right\} + \text{h.c.} \end{aligned} \quad (4.7b)$$

where

$$\begin{aligned} L_a = & U^* D_a^c U + D_b (A_a A_b) + 2 i A_b B_{ba} - \frac{2}{3} i A_a A_b^2 + \\ & + \frac{3}{2} \epsilon^{abcd} A_b D_c A_d \end{aligned} \quad (4.8)$$

$$D_a^c U = \left(\partial_a - \frac{i}{3} A_a \right) U$$

Obviously for $S = \text{const}$ the $\mathcal{L}_\alpha$ contribution becomes a total derivative.

The presence of square gravitational curvature terms has drastically changed the structure of the auxiliary field terms: they appear with higher non-linearity and with derivatives. Their equations of motions will no longer be a linear algebraic system but a coupled system of non-linear differential equations.

In the next sections it will be argued that, nevertheless, all new degrees of freedom propagated by the auxiliary fields, decouple in perturbation theory around Minkowski space. Thus the new terms in the auxiliary field equations amount to just new derivative couplings for the standard physical fields.

Similarly, no ghost state appears perturbatively in the gravitational sector, when expanding around flat space, since the quadratic piece of the $(\text{Riemann})^2$ terms just cancels, because of the Gauss-Bonnet theorem. However, as already indicated in the Introduction, for the more general situation (i.e. for field configurations far from Minkowski space) a more careful analysis is required; see the sections below.

5. AUXILIARY FIELD EQUATIONS

The auxiliary fields in the theory are:

- i) D^A from the Yang-Mills multiplets,
- ii) h_i from the matter chiral multiplets,
- iii) u, A_a from the gravitational multiplet.

For the general model specified by the Lagrangian in Eq. (4.1) the equations for the fields D^A are the same as in the standard case [2]

$$- \text{Re } f_{AB}(z) D^B = \kappa \phi^{ii} T_{Ai}{}^j z_j \quad (5.1)$$

The equations for the h_i fields are now modified, but they can still be solved in general. Their solutions have the form:

$$h_i = - (\phi^{ii})^{-1} \left[\frac{1}{2} g^{*i} + \frac{1}{3} u \phi^i + \alpha f^{*i} \Pi \right] \quad (5.2)$$

where

$$\Pi = \frac{1}{2} R u - \frac{1}{3} u u^* u - \frac{5}{3} u A_a^2 - i u D_a A_a - 2i A_a \partial_a u \quad (5.3)$$

Upon the elimination of the matter auxiliary fields the Lagrangian $\mathcal{L}^{\text{aux}}$ takes the form:

$$\mathcal{L}^{\text{aux}} = \mathcal{L}_1 + \mathcal{L}_2 \quad (5.4)$$

$$\begin{aligned} \frac{1}{e} \mathcal{L}_1 = & -\frac{1}{4} \text{Re} f_{AB}^{-1} \left[\kappa \phi^{i\bar{j}} T_i^{A\bar{j}} z_j \right] \left[\kappa \phi^{l\bar{k}} T_{\bar{k}}^{B\bar{l}} z_{\bar{l}} \right] + \\ & -\frac{1}{8} (\phi^{\prime\prime-1})^{\bar{j}}_i g^{*1}_{\bar{j}} g^{1i} + \text{h.c.} \end{aligned} \quad (5.5a)$$

$$\begin{aligned} \frac{1}{e} \mathcal{L}_2 = & \frac{1}{18} \phi (v v^* - A_a^2) + \frac{i}{3} A_a \phi^{i\bar{l}} D_a z_i + \frac{1}{2} v^* g^* + \\ & - \frac{1}{6} v^* (\phi^{\prime\prime-1})^{\bar{j}}_i g^{*1}_{\bar{j}} \phi^{i\bar{l}} - \frac{1}{18} (\phi^{\prime\prime-1})^{\bar{j}}_i \phi^{i\bar{l}} \phi'_{\bar{j}} v v^* + \\ & - d f^{i\bar{l}} D_a z_i \left[v^* D_a^c v + 3i A_b (R_{ab} - \frac{1}{6} \delta_{ab} R) + \right. \\ & \left. - 4 A_b \tilde{F}_{ba} - \frac{i}{3} A_a v v^* - \frac{i}{3} A_a A_b^2 \right] - d f^{i\bar{l}} D_a z_i \cdot \\ & \cdot D_b (A_a A_b) - d \left[\frac{1}{2} (\phi^{\prime\prime-1})^{\bar{j}}_i f^{i\bar{l}} g^{*1}_{\bar{j}} + \frac{1}{3} v (\phi^{\prime\prime-1})^{\bar{j}}_i \cdot \right. \\ & \left. \cdot f^{i\bar{l}} \phi'_{\bar{j}} \right] \cdot \left[\frac{1}{2} R v^* - \frac{1}{3} v v^* v - \frac{5}{3} v^* A_a^2 + \right. \\ & \left. + i v^* D_a A_a + 2i A_a \partial_a v^* \right] - \frac{d^2}{2} (\phi^{\prime\prime-1})^{\bar{j}}_i f^{i\bar{l}} f^{*1}_{\bar{j}} \left| \frac{1}{2} R v + \right. \\ & \left. - \frac{1}{3} v v^* v - \frac{5}{3} v A_a^2 - i v D_a A_a - 2i A_a \partial_a v \right|^2 + \text{h.c.} \end{aligned} \quad (5.5b)$$

where, as through the whole paper, D_a is the Lorentz and Yang-Mills covariant derivative, and an integration by part has been performed in $\mathcal{L}_2$, and the explicit expression (3.9) for B_{ab} has been used.

From $\mathcal{L}_2$ the exact equations of motion for u and A_a can be derived. As already mentioned, they form a non-linear system of partial differential equations which, in general, is not solvable in a closed form.

Do these auxiliary fields propagate some unwanted degree of freedom, besides the usual ones corresponding to the 'physical' components of the superfields? As we have discussed in the Introduction, this is a non-trivial question in a non-supersymmetric background. (In a supersymmetric vacuum these particles - if present at all - are expected to get infinite masses, by simple superfield considerations).

It is clear that the transverse component of A_μ does not get promoted to be a propagating spin-1 particle, since there is no other vector field to mix with in order to get a kinetic term.

The situation for the u field is less clear. For a generic combination of (curvature)² terms, there are four new propagating scalars [32], i.e. $\text{Re } u$, $\text{Im } u$, $D^a A_a$, and the conformal factor of the metric $g_{\mu\nu}$. They build up, together with their fermionic partners, two Wess-Zumino multiplets, which typically, have positive metric. The same argument as for (A_μ) transverse suggests^{*}) that these four states cannot all be propagating.

Nevertheless, it may happen that still one unwanted Wess-Zumino multiplet propagates, via non-trivial mixing of the above four modes. It may even be that the presence or absence of such states is controlled by the geometry of the Kähler manifold underlying the scalars' σ -model. In fact, it is a general feature of higher derivative supergravity that the σ -model geometry controls the 'non-perturbative' (i.e. fully non-linear) dynamics [34].

In the present paper we are primarily interested in the supergravity model which is predicted by the low-energy limit of the heterotic string. In this case (see next section for further details) the scalars' σ -model corresponds to a product manifold [40]

$$\left[\frac{SU(1,1)}{U(1)} \right] \otimes \left[\frac{SU(N,1)}{SU(N) \otimes U(1)} \right] \quad (5.6)$$

with coupling described by the Kähler-invariant function [40]

$$\begin{aligned} G = & -3 \ln \left(-\frac{\Phi}{3} \right) + \ln |g|^2 = \\ = & -\ln(S+\bar{S}) - 3 \ln(T+\bar{T} - 2 \sum_k \bar{C}^k C_k) + \\ & + \ln |g|^2 \end{aligned} \quad (5.7)$$

^{*}) At least if we limit ourselves to maximally symmetric backgrounds.

Even if this geometry has some remarkable features, unfortunately it does not seem to prevent the appearance of the unwanted states beyond perturbation theory.

Remarkably enough, in the case of Eqs. (5.6) and (5.7), which is the one of more physical interest, the above equations for the auxiliary fields can also be solved in a closed form, provided we limit ourselves to a constant bosonic background.

This will allow for a complete analysis of the scalar potential and of the issue of spontaneous breaking of local supersymmetry (at the classical level). These points are elaborated in full detail in Sections 6, 7, and 8 below.

Here we limit ourselves to a simple superconformal tensor calculus argument which explains how the peculiar geometry of Eqs. (5.6) and (5.7) produces solvable equations for the auxiliary fields and how to a certain extent, it controls the full non-linear dynamics.

In superconformal notation^{*}), the Lagrangian of Eq. (4.1) reads (specializing to the case of interest)

$$\begin{aligned} \mathcal{L} = & -\frac{3}{2} [S_c \bar{S}_c \exp(-\frac{1}{3}G)]_D + \left([S_c^3 + S(W_{YM}^2 - \hat{W}_L^2)]_F + \right. \\ & + \text{h.c.} \Big) = -\frac{3}{2} [S_0 \bar{S}_0 (S + \bar{S})^{1/3} (T + \bar{T} - 2\hat{Z}_\kappa \bar{C}^\kappa C_\kappa)]_D + \\ & + \left([S_c^3 + S(W_{YM}^2 - \hat{W}_L^2)]_F + \text{h.c.} \right) \end{aligned} \quad (5.8)$$

Now, using the same tricks as in Ref. [34], we have

$$\begin{aligned} & -\frac{3}{2} [S_0 \bar{S}_0 (S + \bar{S})^{1/3} (T + \bar{T})]_D = \\ & = -\frac{3}{2} \left([S_c T \Sigma \{ \bar{S}_c (S + \bar{S})^{1/3} \}]_F + \text{h.c.} \right) + \text{surface term.} \end{aligned} \quad (5.9)$$

Since all the other terms in Eq. (5.8) do not contain T , we conclude from Eq. (5.9) that T is just a Lagrange multiplier chiral multiplet enforcing the constraint

$$\Sigma \{ \bar{S}_c (S + \bar{S})^{1/3} \} = 0 \quad (5.10a)$$

that is

$$S_0 \bar{S}_0 (S + \bar{S})^{1/3} = \text{Real linear multiplet } (w = 2, n = 0). \quad (5.10b)$$

^{*}) S_c is the compensating chiral multiplet ($w = n = 1$), see Refs. [2, 24, 26].

Most importantly, the constraint - Eq. (5.10a) - can be solved for the scalar curvature multiplet R , which in the superconformal framework reads

$$R \propto S_0^{-1} \sum (\bar{S}_0) \quad (W=M=1) \quad (5.11)$$

(see Appendix C), in terms of the components of the chiral multiplet S and of their superconformal derivatives.

The solution to Eq. (5.10) is known in closed form, but the techniques and tricks (such as duality transformations) necessary for a useful application would take us outside the plan of the present pages. So, here, it is convenient to solve for the com-
ponents of R in terms of those of S (and derivatives thereof). For instance, the first component of the constraint reads

$$\begin{aligned} [\bar{\Sigma} \{ \bar{S}_0 (S + \bar{S})^{1/3} \}]_z &= -\frac{1}{2} [\bar{S}_0 (S + \bar{S})^{1/3}]_{\#} + \\ + \frac{i}{2} [\bar{S}_0 (S + \bar{S})^{1/3}]_k &= \frac{1}{3} (S + \bar{S})^{1/3} \{ (S + \bar{S})^{-1} \bar{S}_0 h_s^* + \bar{U} \} \quad (5.12) \\ &+ \text{FERMIONS} \\ \text{i.e.} \end{aligned}$$

$$\begin{aligned} [R]_z &= S_0^{-1} \bar{U} = - \left(\frac{\bar{S}_0}{S_0} \right) (S + \bar{S})^{-1} h_s^* + \\ &+ \text{FERMIONS} \end{aligned} \quad (5.13)$$

($\bar{S}_0/S_0$ can always be chosen to be 1, as an R-gauge condition).

It is exactly the occurrence of equations such as (5.13) that makes the computations easier for the Witten compactification. Even more, these equations control the 'non-perturbative' dynamics. In fact, the fields U and $D^a A_a$, which are those which may get independent propagation, are components of the multiplet R , and then are not independent but are constrained by Eqs. (5.10)*). This will be shown more explicitly in Section 6.3.

6. SPECIALIZING TO THE COMPACTIFIED HETEROTIC STRING

6.1 The low-energy effective supergravity

Following Witten [40], we divide the matter multiplets Z_i into two multiplets S , T , and N charged multiplets C_I . These are

*) Unfortunately, contrary to the examples of Ref. [34], these constraints are too weak to prevent unwanted states from propagating at all.

in a suitable representation of the gauge group G surviving after dimensional reduction. They are related to the $SU(3)$ -invariant harmonics of the given compactification. The scalar components of the S and T singlets are related to combinations of the 10D dilaton Φ and of the internal conformal factor σ . The pseudoscalar component of T , $\text{Im } T$, is related to the harmonic part of the 2-form B proportional to the Kähler form of the internal manifold. The pseudoscalar $\text{Im } S$, as already anticipated in Section 2, is obtained from the 4D part of B , $B_{\mu\nu}$, by a duality transformation. We will use the notation:

$$\begin{aligned} Z_i &= (z_i, 0, h_i) \quad i = 1, \dots, n \\ Z_1 &= (S, 0, h_S) \\ Z_2 &= (T, 0, h_T) \\ Z_I &= C_I = (z_I, 0, h_I) \quad I = 3, \dots, n \end{aligned} \tag{6.1}$$

The dimensional reduction leads to the following special forms for the Φ , f_{AB} , and g functions:

$$\begin{aligned} \Phi &= -3(S + \bar{S})^{1/3} (T + \bar{T} - 2 C_I \bar{C}^I) \\ g &= 8\sqrt{2} \kappa d_{IJK} Z_I Z_J Z_K + c \\ f_{AB} &= \delta_{AB} S \end{aligned} \tag{6.2}$$

where κ is the gauge coupling constant of $E_6 \subset G$.

The 4D supergravity of Ref. [40] differs from our Lagrangian obtained by the Poincaré calculus by a suitable Weyl rescaling and field redefinition for the fermions.

The presence of an arbitrary complex constant c in the superpotential Eq. (6.2) can be understood geometrically (see Ref. [39]). Indeed, from the Bianchi identity obeyed by the 3-form H [45],

$$dH = \frac{1}{30} \text{Tr } F \wedge F - \text{Tr } R \wedge R \tag{6.3}$$

we see that, once the gauge and gravitational field configurations are given, H is determined up to a close form α , i.e. the general solution to Eq. (6.3) is

$$\begin{aligned}
H &= H_0 + d \\
d\alpha &= 0
\end{aligned}
\tag{6.4}$$

where H_0 is any particular solution. α is also restricted by the equation of motion of the B field

$$S(\phi^{-3/2} H) = 0
\tag{6.5}$$

In our case we are interested only in the SU(3)-invariant components of H ; on the compact space K_6 there are only two invariant forms, the (3,0) form [11]

$$\epsilon = dz^1 \wedge dz^2 \wedge dz^3$$

and its complex conjugate $\bar{\epsilon}$. If $i : K_6 \rightarrow M_4 \times K_6$ is the natural inclusion, our truncation implies [39]

$$i^* H = \frac{1}{2\sqrt{3}} [g\epsilon + \bar{g}\bar{\epsilon}]
\tag{6.6}$$

where the function g must be identified with the 4D superpotential and is independent of the internal coordinates. Comparing Eqs. (6.4) and (6.6) we see that the superpotential is determined up to an additive constant C ; moreover, Eq. (6.5) is satisfied for all values of C . An important point is that C is a constant of motion only in the massless truncation of the theory [46]. There are physical processes involving massive Kaluza-Klein modes which make the value of C change, then allowing, at least in principle, its value to relax to the one consistent with the vacuum Einstein equations. Unfortunately, the above discussion is correct only in the 10D field theory limit. Indeed, at the string level we have another consistency requirement [46]: α in Eq. (6.4) must belong to an integral cohomology class. This fact makes a non-vanishing C less attractive from a phenomenological point of view because it would imply (see next section) supersymmetry breaking at a very large scale, unless some more subtle mechanism [46] is at work.

6.2 Auxiliary field equations

We now have the following simplifications and relations

$$\begin{aligned}
 \Phi''^2_i = \Phi''^i_2 = 0 \quad i \neq 1, \quad \Phi''^I_K = 0 \quad I \neq K, \quad (\Phi''^{-1})^I_4 = 0, \\
 (\Phi''^{-1})^I_I = (\Phi''^{-1})^I_4 = (\Phi''^{-1})^I_K = 0 \quad I \neq K, \quad (\Phi''^{-1})^I_i f'' f'^*_{ij} = 0, \\
 (\Phi''^{-1})^I_i f'' g'^*_{ij} = 0, \quad \phi - (\Phi''^{-1})^I_i \phi'_i \phi'^i = -3\phi, \quad (6.7) \\
 (\Phi''^{-1})^I_i g'^*_{ij} \phi'^i = (\Phi''^{-1})^K_i \phi'^i = 0, \\
 (\Phi''^{-1})^I_i f'' \phi'_i = 3(s + \bar{s}), \quad (\Phi''^{-1})^I_i g'^*_{ij} g'^i = \frac{g'^*_{IK} g'^K}{6(s + \bar{s})^{1/3}}
 \end{aligned}$$

As we want to explore the possibility of a spontaneous breaking of supersymmetry induced by fermion condensates, it is convenient to reintroduce in the Lagrangian the fermionic terms which are relevant in the case of condensation of the gauge fermions. As already mentioned, such terms originate in the standard Lagrangian and, owing to the previous relations (6.2) and (6.7), they amount to the simple term:

$$-\frac{1}{4} f''_{AB} h_i \bar{\lambda}^A_L \lambda^B_L + h.c. = -\frac{i}{4} h_s \bar{\lambda}^A_L \lambda^A_L - \frac{1}{4} h_s^* \bar{\lambda}^A_R \lambda^A_R \quad (6.8)$$

The equations of motion for the matter auxiliary fields h_i now read:

$$\frac{\delta \mathcal{L}}{\delta h_I^*} = \Phi''^I_I h_I + \Phi''^I_J h_J + \frac{1}{2} g'^*_{I1} + \frac{1}{3} v \phi'_I = 0 \quad (6.9a)$$

$$\frac{\delta \mathcal{L}}{\delta h_1^*} = \Phi''^i_s h_i + \frac{1}{3} v \phi'_s - \frac{i}{4} \bar{\lambda}^A_R \lambda^A_R + \alpha \Pi = 0 \quad (6.9b)$$

$$\frac{\delta \mathcal{L}}{\delta h_2^*} = \Phi''^1_2 h_1 + \frac{1}{3} v \phi'_2 = 0 \quad (6.9c)$$

[compare Eq. (6.9c) with Eq. (5.13)].

After the elimination of the h_i fields, L^{aux} reads

$$\mathcal{L}^{\text{aux}} = \mathcal{L}_1 + \mathcal{L}_2 \quad (6.10a)$$

$$\begin{aligned} \frac{1}{e} \mathcal{L}_1 = & -\frac{1}{4} \frac{2}{s+\bar{s}} \left[\kappa \phi^{I\bar{I}} T_I^{A\bar{J}} z_{\bar{J}} \right] \left[\kappa \phi^{J\bar{K}} T_{\bar{K}}^{A\bar{L}} z_L \right] + \\ & -\frac{1}{48} \frac{g^{*I} g^{I\bar{I}}}{(s+\bar{s})^{1/3}} + \text{h.c.} \end{aligned} \quad (6.10b)$$

$$\begin{aligned} \frac{1}{e} \mathcal{L}_2 = & -\frac{1}{6} \phi U U^* - \frac{1}{18} \phi A_a^2 + \frac{i}{3} A_a \phi^{i\bar{i}} D_a z_i + \frac{1}{2} U^* g^* + \\ & -2(\partial_a s) \left[U^* D_a^c U + 3i A_b (R_{ab} - \frac{1}{6} \delta_{ab} R) + \right. \\ & -4 A_b \tilde{F}_{ba} - \frac{i}{3} A_a U U^* - \frac{i}{3} A_a A_b^2 \left. \right] + \\ & -2(\partial_a s) D_b (A_a A_b) - 2(s+\bar{s}) \left[\frac{1}{2} R U U^* + \right. \\ & -\frac{1}{3} (U U^*)^2 - \frac{5}{3} U U^* A_a^2 + i U U^* D_a A_a + \\ & \left. + 2i A_a U \partial_a U^* \right] + \frac{1}{4} U^* \bar{\lambda}^A_R \lambda^A_R (s+\bar{s}) \end{aligned} \quad (6.10c)$$

Varying Eqs. (6.10) with respect to U and A_a we get the following equations:

$$\begin{aligned} & -\frac{1}{3} \phi U + \frac{1}{2} g^* - 2 \partial_a (s-\bar{s}) D_a^c U + i \frac{7}{3} \partial_a (s+\bar{s}) A_a U + \\ & -2 \partial (s+\bar{s}) \left[\frac{1}{2} R U - \frac{2}{3} U U^* U - \frac{5}{3} U A_a^2 \right] + 4i \partial (s+\bar{s}) A_a \partial_a U \\ & + 2i \partial (s+\bar{s}) D_a A_a U + 2 U D_a (D^a \bar{s}) = 0 \end{aligned} \quad (6.11)$$

and

$$\begin{aligned} & -\frac{2}{9} \phi A_a + \frac{i}{3} \left[\phi^{i\bar{i}} D_a z_i - \phi^{i\bar{i}} D_a z^{*i} \right] + \frac{20}{3} \partial (s+\bar{s}) U^* U A_a + \\ & -i \partial \partial_b (s-\bar{s}) \left[3(R_{ba} - \frac{1}{6} \delta_{ab} R) + 2 A_a A_b \right] + i \partial \partial_a (s-\bar{s}) \left[\frac{1}{3} U U^* - A_a^2 \right] + \\ & + 8 \partial \partial_b (s-\bar{s}) \tilde{F}_{ab} + 2i \partial (s+\bar{s}) (U \partial_a U^* - U^* \partial_a U) + \\ & + 2 A_a D_a (\partial_a (s+\bar{s})) + 2 A_c D_a (\partial_c (s+\bar{s})) = 0 \end{aligned} \quad (6.12)$$

The equations are still rather difficult to solve. The most interesting aspect, from the physical point of view, concerns the scalar potential which controls the mechanism of spontaneous breaking of supersymmetry. We shall see that on a Poincaré invariant (or anti-de-Sitter) background a complete analysis is perfectly manageable.

6.3. Propagation of the auxiliary fields?

After the elimination of the D^A and h_i auxiliary fields, the overall Lagrangian reads

$$\mathcal{L} = \hat{\mathcal{L}} + \mathcal{L}^{AUX} \quad (6.13)$$

where

$$\begin{aligned} \hat{\mathcal{L}} = & \frac{1}{6} \phi R - \phi''^i D_r z_i D^r z^{*j} - \frac{1}{4} (\text{Re } f) F_{\mu\nu}^A F^{A\mu\nu} + \\ & + \frac{i}{4} (\text{Im } f) F_{\mu\nu}^A \tilde{F}^{A\mu\nu} + \frac{g}{4} d (\text{Re } f)^* R_{abcd}^* R^{abcd} + \end{aligned} \quad (6.14)$$

$$- \frac{g}{4} i d (\text{Im } f) R_{abcd}^* R^{abcd} \quad (6.15)$$

$$\mathcal{L}^{AUX} = \mathcal{L}^{AUX} \quad (6.10)$$

and

$$\mathcal{T} = -3 \ln \left(-\frac{\phi}{3} \right) \quad (6.16a)$$

$$h_s = \mathcal{T}_s^{-1} \cup \quad \mathcal{T}_s = -\frac{1}{s+\bar{s}} \quad (6.16b)$$

As is well known [2], there are three sources of kinetic terms for the scalar particles:

- i) the explicit terms with two derivatives in Eqs. (6.13) to (6.15).
 - ii) the terms which are generated by the Weyl rescaling needed in order to put the Einstein term in Eqs. (6.13) and (6.14) into the canonical form;
 - iii) terms quadratic in the derivatives obtained by eliminating the auxiliary fields A_μ .
- (ii) and (iii) correspond to the terms coming from the diagonalization of the mixings with the conformal factor and the longitudinal part of A_μ , as discussed in Section 5.

From a direct inspection of Eqs. (6.13) to (6.15) it is clear that there are no terms of type (i) involving the auxiliary field h_s except the one due to mixing with the field S . As we shall see below, these are exactly the terms responsible for all the trouble.

For terms of type (ii) things are a bit more tricky. The new terms, quadratic in the derivatives, produced by the Weyl rescaling, are [2]

$$- \frac{3}{4} e \left[\partial_r \ell_n \left(\phi + 6\alpha (s+\bar{s})^{-1} |h_s|^2 \right) \right]^2 \quad (6.17)$$

Let us perform the following non-holomorphic field redefinition:

$$T \rightarrow T + 3\alpha (s+\bar{s})^{-4/3} |h_s|^2 = T' \quad (6.18)$$

then, Eq. (6.17) becomes (neglecting the C's)

$$- \frac{3}{4} e \left\{ \partial_r \ell_n \left[(s+\bar{s})^{1/3} (T' + \bar{T}') \right] \right\} \quad (6.19)$$

and the Weyl factor assumes everywhere the same form as in the corresponding standard case. Does the field redefinition (6.18) cancel the derivatives of $|h_s|^2$, from everywhere in the Lagrangian (6.13)? To see this let us rewrite all the terms in Eq. (6.13) which contain $\partial_\mu (T + \bar{T})$ or $\partial_\mu |h_s|^2$, other than those that we have already discussed in Eqs. (6.17) to (6.19)*). They are

$$\begin{aligned} & \frac{1}{3} (s+\bar{s})^{-2/3} \partial_r \left(\frac{s+\bar{s}}{2} \right) \partial_r \left(\frac{T+\bar{T}}{2} \right) - \partial_r \left(\frac{s+\bar{s}}{2} \right) \partial_r \left[\alpha (s+\bar{s})^{-2} |h_s|^2 \right] = \\ & = \frac{4}{3} \alpha |h_s|^2 (s+\bar{s})^{-3} \partial_r \left(\frac{s+\bar{s}}{2} \right) \partial_r \left(\frac{s+\bar{s}}{2} \right) + \\ & + \frac{1}{6} (s+\bar{s})^{-2/3} \partial_r \left(\frac{s+\bar{s}}{2} \right) \partial_r \left[(T+\bar{T}) - 6\alpha (s+\bar{s})^{-4/3} |h_s|^2 \right] \end{aligned} \quad (6.20)$$

Notice that, were it not for the minus sign inside the square bracket, the last term in Eq. (6.20) would be just proportional to $(\partial_\mu \text{Re } S)(\partial_\mu \text{Re } T')$, by Eq. (6.18), and hence $|h_s|^2$ would not propagate at all. Unfortunately, the sign in Eq. (6.20) is the other way around, and - after the field redefinition (6.18) - we have still a derivative acting on $|h_s|^2$, i.e.

*) Here we are also exploiting the peculiar geometry of the σ -model, Eqs. (5.6) to (5.13).

$$- \alpha (S + \bar{S})^{-2} \partial_\mu (S + \bar{S}) \partial_\mu |h_S|^2 \quad (6.21)$$

So (neglecting the fields C's as well as the pseudoscalars) the kinetic terms are

$$A (\partial_\mu \text{Re} S)^2 + B (\partial_\mu \text{Im} T')^2 + 2 C \partial_\mu (\text{Re} S) \partial_\mu |h_S|^2 \quad (6.22)$$

where A, B, C, are some functions of the scalar fields (which are rather similar to those of the "standard" Kählerian case [2]).

Since the matrix

$$\begin{bmatrix} A & C & 0 \\ C & 0 & 0 \\ 0 & 0 & B \end{bmatrix} \quad (6.23)$$

has (at least) one negative eigenvalue, we see that in general there should be a scalar poltergeist. By supersymmetry then there should be a whole Wess-Zumino poltergeist multiplet. In particular, the pseudoscalar component may be identified with the phase of h_S . Then, we can repeat almost word by word the above argument in the case of $\arg(h_S)$, with the mixing with the longitudinal component of A_μ instead of the mixing with the spin-0 part of the metric, $\text{Im} T$ instead of $\text{Re} T$, and $\text{Im} S \rightarrow \text{Re} S$.

That these unwanted states get an infinite mass when $h_S = 0$ [which, by Eqs. (5.13), is much the same as saying that the vacuum is Minkowski space^{*)}] is most easily seen if we parametrize these states in terms of the fields $\text{Re} h_S$ and $\text{Im} h_S$, rather than $|h_S|^2$ and $\arg(h_S)$. Indeed, in this field basis, the negative eigenvalue of the kinetic term matrix reads

$$- \frac{C^2 |h_S|^2}{A} \quad (6.24)$$

where, for simplicity, we have assumed $|h_S|$ to be small and $\text{Im} h_S = 0$ (in order not to have scalar-pseudoscalar mixing).

*) We stress that this condition is weaker than unbroken SUSY (in which case the decoupling would follow by a simpler argument) and is very mild from a phenomenological point of view.

From Eqs. (6.13) and (6.24) we learn that the poltergeists have a mass of order

$$m^2 = O\left(1/|h_s|^2\right) \quad (6.25)$$

and then they decouple as $h_s \rightarrow 0$, that is in a Minkowski background ($h_s = J_s^{-1} u$). This completes our argument that no new states are present in the theory, when we perturbate around a flat vacuum.

7. SCALAR POTENTIAL AND SPONTANEOUS BREAKDOWN OF SUPERSYMMETRY

Assuming a maximally symmetric background (in particular $R = \text{const.}$) and allowing for a non-vanishing scalar condensate of the gauge fermions λ^A , the Lagrangian (6.10) reduces to

$$\begin{aligned} \frac{1}{2} \mathcal{L} = & \frac{1}{6} \phi R - \frac{1}{s+\bar{s}} \left(\kappa \phi^{1\bar{i}} T_{\bar{i}}^{A\bar{j}} C_j \right)^2 - \frac{1}{24} \frac{|g_i|^2}{(s+\bar{s})^{1/3}} + \\ & - \frac{1}{3} \phi |U|^2 + \frac{1}{2} U^* \hat{g}^* + \frac{1}{2} U \hat{g} - \alpha R (s+\bar{s}) |U|^2 + \\ & + \frac{2}{3} \alpha (s+\bar{s}) |U|^4 \end{aligned} \quad (7.1)$$

where

$$\hat{g} = g(c) + \frac{1}{2} (s+\bar{s}) \bar{\lambda}_L^A \lambda_L^A \quad (7.2)$$

In order to make the geometric content of Eq. (7.1) more transparent, let us perform the following Weyl rescaling [2]

$$e_r^a \rightarrow e^\sigma e_r^a \quad (7.3)$$

$$e^{2\sigma} = -\frac{3}{\phi}$$

as well as a field redefinition $U \rightarrow h$ with

$$U = -h \hat{g}^* e^{J/3} \quad (7.4)$$

where J is the Kähler potential of the scalars' σ -model (corresponding to the standard, $\alpha = 0$, theory [2])

$$J = -3 \ln \left(-\frac{\Phi}{3} \right) = -\ln(s + \bar{s}) + \\ -3 \ln(T + \bar{T} - 2 \bar{C}^i C_i) \quad (7.5)$$

As usual, see Ref. [2], we can define a Kähler-invariant function*) G

$$G = \ln \|\hat{g}\|_J^2 = J + \ln |\hat{g}|^2 \quad (7.6)$$

Since the chiral σ -model corresponds to a product manifold, i.e. $B_{n+1} \times B_1$ (B_n denotes the open unit ball in C^n) and g may be thought of as a section of a bundle over the first factor only, we have another geometrically natural invariant function besides G , i.e. [48]

$$\tilde{G} = \ln \|\hat{g}\|_{B_{n+1}}^2 = -3 \ln(T + \bar{T} - 2 \bar{C}^i C_i) + \\ + \ln |\hat{g}|^2 \quad (7.7)$$

where $\|\cdot\|_{B_{n+1}}^2$ is (up to a factor) the Bergman metric for the canonical bundle of B_{n+1} .

Using Eqs. (6.3) to (6.7), the Lagrangian becomes [37]

$$\frac{1}{e} \mathcal{L} = -\frac{1}{2} [1 + 2\alpha e^{\tilde{G}} |h|^2] R + \\ + e^G [|h|^2 - \frac{1}{2} h - \frac{1}{2} h^* + \frac{2}{3} \alpha e^{\tilde{G}} |h|^4] - \beta^2 \quad (7.8)$$

where

*) $\|g\|_J^2$ is the (norm)² of the holomorphic section g with respect to the fibre metric induced by the Kähler potential J , see Refs. [47,48].

$$\beta^2 = \frac{1}{(S+\bar{S})} (\kappa G^{i\bar{j}} T_i A_{\bar{j}} C_j)^2 + \frac{g}{24} \frac{|g_i|^2}{(S+\bar{S})^{1/3} \phi^2} \quad (7.9)$$

Now we can proceed in two different ways. We can firstly make the Weyl transformation which puts the Einstein term in Eq. (7.8) in the canonical form $-(1/2)R$ and then eliminate the auxiliary field h through its equation of motion (which is just algebraic in this class of backgrounds). This is the procedure adopted in Ref. [37], which will be discussed in full detail in the next section.

Alternatively, one can firstly eliminate the field U from Eq. (7.8) and then perform the Weyl redefinition needed to restore the canonical form of the Einstein Lagrangian. Obviously, the two procedures give equivalent physics but different Lagrangians. In fact, the difference between the two Lagrangians, so obtained, is an example of the general ambiguity due to redefinitions of the metric, as discussed in the Introduction.

The Lagrangian obtained from the second method has additional R^2 and higher curvature terms; however, the new R^2 terms are $O(\alpha)$ relative to the ones explicit in Eq. (5.6), and so do not spoil the Gauss-Bonnet combination.

The fact that the elimination of the 'auxiliary' fields introduces such rather complicated terms is connected with the propagating nature of such fields at the 'non-perturbative' level. Indeed, were we not just interested in the low-energy phenomena in flat space, we would treat the 'auxiliary' fields as 'physical' ones (and then the procedure of eliminating them - to obtain the 'component' form of the Lagrangian - would not be legitimate); the new complications however do not affect the S-matrix for light states [29].

The first approach, leading to a Lagrangian without any spurious higher curvature term (at least around symmetric background), corresponds to the so-called 'minimal form' of the theory, to be discussed in Section 8.

In the present section we work out the potential and discuss the issue of supersymmetry breaking using the second strategy. In this formulation the mathematical analysis of the potential is somehow easier.

As we have already anticipated, the minimal formulation of Section 8 leads to two different solutions to the equation

$$\frac{\partial V}{\partial h_s} = 0 \quad (7.10)$$

since now V is no longer quadratic in h_s . They are both 'physical' in a formal sense, but only the one continuously connected with the result of the standard case as $\alpha \rightarrow 0$ is acceptable as being ghost-free, at least perturbatively.

On the contrary in the second approach, Eq. (7.10) has just one solution (corresponding to the physical one). This is obviously not a discrepancy in the physics described by the two Lagrangians, but it is a simple consequence of the fact that in the last case we have not diagonalized the mixing between $|h_s|^2$ and the conformal factor of the metric, so that the other solution will enter through the gravitational equations.

In the non-minimal scheme the equation of motion for the auxiliary field h is

$$2\lambda |h|^2 h + (1 - \frac{3}{2} \lambda e^{-G} R) h - \frac{1}{2} = 0 \quad (7.11)$$

where

$$\lambda = \frac{2}{3} \alpha e^{\tilde{G}} \quad (7.12)$$

From Eq. (7.11) it follows that h is real and positive so, identifying $h = |h|$, Eq. (7.11) reads

$$h^3 + p h + q = 0 \quad (7.13)$$

with

$$p = \frac{1}{2\lambda} (1 - \frac{3}{2} \lambda e^{-G} R) \quad (7.14a)$$

$$q = - \frac{1}{4\lambda} \quad (7.14b)$$

Since $q < 0$ and $p \geq 0$ [in anti-de-Sitter space or anyhow if R is small in Planck units*)] Eq. (7.13) admits only one real (positive) solution

$$h = \left(\frac{2}{3\lambda} \right)^{1/2} \left[1 - \frac{3}{2} \lambda e^{-G} R \right]^{1/2} \text{Sh } \hat{\rho} \quad (7.15)$$

where

$$\hat{\rho} = \frac{1}{3} \text{Sh}^{-1} \left\{ \frac{3\sqrt{6\lambda}}{4} \left(1 - \frac{3}{2} \lambda e^{-G} R \right)^{-3/2} \right\} \quad (7.16)$$

Then the Lagrangian, Eq. (7.8), equals

$$\begin{aligned} \frac{1}{e} \mathcal{L} = & -\frac{1}{2} R - \beta^2 + \frac{1}{3\lambda} e^G \left(1 - \frac{3}{2} \lambda e^{-G} R \right)^2 \text{Sh } \hat{\rho} \cdot \\ & \cdot [\text{Sh } \hat{\rho} - \text{Sh } 3\hat{\rho}] \end{aligned} \quad (7.17)$$

where the scalar curvature R is also implicitly present through $\hat{\rho}$, Eq. (7.16).

Now we make a second Weyl rescaling, to put the term in Eq. (7.17) which is linear in R into its canonical form.

First of all, we have to isolate the coefficient of R in Eq. (7.17), which is hidden in the definition of $\hat{\rho}$. Obviously,

$$\begin{aligned} (\text{coefficient of } R \text{ in the r.h.s. of Eq. (7.17)}) &= \frac{\partial \left(\frac{1}{e} \mathcal{L} \right)}{\partial R} \Big|_{R=0} = \\ &= \frac{\partial (e^{-1} \mathcal{L}(h, R))}{\partial R} \Big|_{\substack{h=(7.15) \\ R=0}} + \frac{\partial (e^{-1} \mathcal{L}(h, R))}{\partial h} \frac{\partial h}{\partial R} \Big|_{\substack{h=(7.15) \\ R=0}} = \\ &= -\frac{1}{2} (1 + 3\lambda |h|^2) \Big|_{\substack{h=(7.15) \\ R=0}} \end{aligned} \quad (7.18)$$

where $L(h, R)$ is given by Eq. (7.8) and we used the equation

$$\frac{\partial (e^{-1} \mathcal{L}(h, R))}{\partial h} \Big|_{h=(7.15)} = 0$$

*) i) Otherwise the massive Kaluza-Klein and string states would not effectively decouple from the 4D massless ones. ii) This is also the condition under which we can neglect the mixing with the conformal factor, i.e. the other solution of Section 8 is ruled out.

After the Weyl rescaling

$$e_\mu^a \rightarrow \Omega e_\mu^a$$

$$\Omega = [1 + 2 (\text{sh} \mu)^2]^{-1/2} = [\text{ch}(2\mu)]^{-1/2} \quad (7.19)$$

$$\mu = \hat{\mu}|_{R=0} = \frac{1}{3} \text{sh}^{-1} \left(\frac{3\sqrt{6}\lambda}{4} \right)$$

The Lagrangian (7.17) takes the form

$$\begin{aligned} \frac{1}{2} \mathcal{L} = & -\frac{1}{2} R - [1 + 2 (\text{sh} \mu)^2]^{-2} [\beta^2 + \\ & + \frac{e^G}{3\lambda} \text{sh} \mu (\text{sh} 3\mu - \text{sh} \mu)] + O(R^2) \end{aligned} \quad (7.20)$$

The scalar potential of the 'non-minimal form' is

$$V = \frac{\text{sh} \mu (\text{sh} 3\mu - \text{sh} \mu) + 2\alpha (S + \bar{S}) \beta^2}{2\alpha (S + \bar{S}) [1 + 2 (\text{sh} \mu)^2]^2} \quad (7.21)$$

with μ as in Eq. (7.19) and β^2 as in Eq. (7.9).

Since by Eq. (7.19), $\mu \geq 0$

$$\text{sh} 3\mu \geq \text{sh} \mu \geq 0$$

so the numerator in the r.h.s. of Eq. (7.21) is the sum of two positive terms, simply related to the corresponding positive terms of the standard $\alpha = 0$ case.

The cosmological constant will vanish only if

i) $\beta^2 = 0$ which from Eq. (7.9) requires

$$g_i = 0$$

$$G^i T_i^{\text{A}^\dagger} C_I = 0 \quad (7.22)$$

which exactly as in the standard case, requires $\langle c_j \rangle = 0$.

ii) $\mu = 0$ which implies $\lambda = 0$. Since by Eqs. (7.7) and (7.12)

$$\lambda = 18\alpha \frac{|\hat{g}|^2}{(\tau + \bar{\tau} - 2\bar{c}^I c_I)^3} \quad (7.23)$$

we have that $\langle \hat{g} \rangle = 0$ and $\langle u \rangle = 0$.

We now have two possibilities:

- i) No gauge fermion condensate:

$$\langle \lambda_L^A \lambda_L^A \rangle = 0.$$

This implies, in the expression of the superpotential, $C = 0$ and supersymmetry cannot be broken, as already seen in Ref. [39].

- ii) $\langle \lambda_L^A \lambda_L^A \rangle \neq 0$

In this case $C = (1/2) \langle \lambda_L^A \lambda_L^A \rangle \langle S + \bar{S} \rangle$ and supersymmetry is broken. In fact, from Eqs. (5.2), (6.7), and (6.9), it results that

$$\begin{aligned} \langle h_I \rangle &= \langle h_S \rangle = 0 \\ \langle h_T \rangle &= - \frac{\langle S + \bar{S} \rangle^{2/3}}{4} \langle \bar{\lambda}_R^A \lambda_R^A \rangle \end{aligned} \quad (7.24)$$

and, from the transformation rules of the fermions of the chiral multiplet Z_i

$$\delta \chi_{L,i} = \frac{1}{2} \not{D} Z_i \epsilon_R + \frac{1}{2} h_i \epsilon_R \quad (7.25)$$

we have, for $\chi_{L,T}$:

$$\langle \delta \chi_{L,T} \rangle = - \frac{1}{8} \langle (S + \bar{S})^{2/3} \rangle \langle \bar{\lambda}_R^A \lambda_R^A \rangle \epsilon_L \quad (7.26)$$

This identifies the Goldstone fermion with χ_T . We see then that in the supergravity theory induced by the dimensional reduction of the 10D theory the spontaneous breaking of supersymmetry can only be caused by a fermionic condensate. Two comments are in order:

- i) As discussed in Ref. [39], the dynamics of the interactions responsible for $\langle \bar{\lambda} \lambda \rangle$ being non-vanishing may be simulated by a chiral superpotential

$$g(C, S) = g(C) + G + a e^{bS} \quad (7.27)$$

The resulting Lagrangian is

$$(\mathcal{L} \text{ in Eq. (7.1)}) - \frac{1}{2} [g_S(S + \bar{S}) v + \text{h.c.}] \quad (7.28)$$

After the same manipulations as before, it becomes

$$(\mathcal{L} \text{ as in Eq. (7.8)}) + \frac{1}{2} e^{\tilde{G}} \left(\frac{g_s}{g} h + \text{h.c.} \right) \quad (7.29)$$

Then the equation for h is given by Eq. (7.13), where q is now

$$q = - \frac{1}{4\lambda} \left| 1 - \frac{g_s}{g} (s + \bar{s}) \right| \quad (7.30)$$

so the potential is still given by Eq. (7.21), but μ is now given by

$$\mu = \frac{1}{3} S h^{-1} \left\{ \frac{3\sqrt{6\lambda}}{4} \left| 1 - \frac{g_s}{g} (s + \bar{s}) \right| \right\} \quad (7.31)$$

and λ is given by Eq. (7.23) with $\hat{g} \rightarrow g(C_1, S)$.

Then if g has a non-trivial dependence on S , the only field configuration with vanishing cosmological constant is given by $\langle C_1 \rangle = 0$ and $\mu = 0$, that is

$$-g + (s + \bar{s}) \frac{\partial g}{\partial S} = 0 \quad (7.32)$$

which is the condition found in Ref. [39] for $\alpha = 0$.

- ii) A priori it is not obvious that, once supersymmetry is broken by $\langle \bar{\lambda}\lambda \rangle \neq 0$, with $\Lambda = 0$ at the level of standard supergravity, Λ would remain zero when higher orders in α' are taken into account. Here we have shown in an explicit example, corresponding to the first non-trivial correction in α' , that Λ remains zero. More general arguments, valid to all orders, may be found in Ref. [36].

8. THE SCALAR POTENTIAL IN THE 'MINIMAL' FORMULATION

In this section we discuss the scalar potential in the minimal formulation described in Section 7.

We start from Eq. (7.8) and, before eliminating the auxiliary field h , we put the Einstein term into the canonical form by the Weyl rescaling

$$e_\nu^a \rightarrow e^\sigma e_\nu^a$$

$$e^{2\sigma} = [1 + 3\lambda |h|^2]^{-1} \quad (8.1)$$

Then the potential V is

$$V = - e^G \frac{[|h|^2 - \frac{1}{2}h - \frac{1}{2}h^* + \lambda |h|^4 - e^{-G} \beta^2]}{(1 + 3\lambda |h|^2)^2} \quad (8.2)$$

Now we eliminate h using its equation of motion $\partial V / \partial h = 0$. Again h is real positive. Then the equation for h reads

$$h^3 - \frac{9}{2} h^2 - (6e^{-G} \beta^2 + \lambda^{-1}) h + \frac{\lambda^{-1}}{2} = 0 \quad (8.3)$$

By Descartes theorem, Eq. (8.3) has two real positive solutions for all positive values of λ . As discussed in the previous section, only the one with the correct behaviour as $\lambda \rightarrow 0$ (i.e. $h \rightarrow 1/2$) is physical. This solution is^{*})

$$h = \frac{3}{2} - \frac{2}{\sqrt{27\lambda}} \left[1 + \frac{27}{4}\lambda + 4\alpha (S + \bar{S}) \beta^2 \right]^{1/2} \sin \theta \quad (8.4)$$

where

$$\theta = \frac{1}{3} \arcsin \left\{ \frac{\sqrt{27\lambda} \left[1 + \frac{27}{4}\lambda + 6\alpha (S + \bar{S}) \beta^2 \right]}{2 \left[1 + \frac{27}{4}\lambda + 4\alpha (S + \bar{S}) \beta^2 \right]^{3/2}} \right\} \quad (8.5)$$

with

$$0 \leq \theta \leq \frac{\pi}{6} \quad (8.6)$$

in particular

$$0 < h < \frac{3}{2} \quad (8.7)$$

^{*}) The other solution is obtained with the substitution $\theta \rightarrow \theta - 2\pi/3$ in Eq. (8.4).

Using Eq. (8.3) V can be rewritten as

$$V = \frac{\beta^2 + \frac{1}{2} e^G h \left(\frac{3}{2} - h \right)}{1 + 3\lambda h^2} \quad (8.8)$$

Equations (8.4) to (8.8) give the explicit form of the scalar potential in the minimal formulation.

Notice that V , as given in Eq. (8.8), is manifestly positive in the interval (8.7), so the potential is positive (semi-) definite.

The only configuration with vanishing Λ is

$$\begin{aligned} \beta^2 &= 0 \\ e^G = e^T |\hat{g}|^2 &= 0 \Rightarrow \hat{g} = 0 \end{aligned} \quad (8.9)$$

in agreement with the results of Section 7.

The rest of the analysis is exactly the same as in Section 7.

9. CONCLUSIONS

In this paper we have written down the component form of the most general matter-coupled 4D $N = 1$ supergravity with (curvature)² terms of the form which is uniquely selected by the two principles:

- i) They provide the 4D SUSY completion of the Lorentz Chern-Simons terms added to $H_{\mu\mu p}$ by Green and Schwarz [12] in order to cancel the anomalies of the 10D theory^{*)}.
- ii) The perturbative-level decoupling of the unwanted states, in particular, of the poltergeists [30,35].

These requirements allow only the Gauss-Bonnet ${}^*R_{abcd} {}^*R^{abcd}$ and Hirzebruch ${}^*R_{abcd} R^{abcd}$ invariants to appear. We remark in passing that the 4D SUSY completion of the Green-Schwarz term can also be - equivalently - studied in the 'dual' formulation^{**)} with a linear multiplet. Full details can be found in Refs. [51,52]. This dual language is closer in spirit to the geometry

*) There are also string theories formulated directly in four dimensions, where the anomalies are cancelled by the Green-Schwarz mechanism, see, for example, Ref. [49].

**) For a recent discussion of duality in $N = 1$ supergravity, see Ref. [50].

of the anomaly-cancelling mechanism, since it is formulated in terms of a 2-form (and SUSY partners thereof) rather than the dual scalar as we have done here (see Sections 2 and 3 above).

We have also analysed in detail how the new higher curvature terms modify the structure of the scalar potential^{*)} as well as the physical properties of the vacuum of the $\alpha \neq 0$ theory.

Our results show that the higher order gravitational corrections do not change the vacuum configuration; in particular, the cosmological constant remains exactly zero. Classically, SUSY is not broken but it can be broken by the same quantum effects as suggested in the framework of the standard supergravity limit (i.e. $\alpha = 0$), that is by gluino condensation [38,39]. In this last case too, the vacuum energy vanishes even when the higher derivative corrections are taken into account.

The fact that the vacuum is not 'renormalized' by the addition of the Gauss-Bonnet interactions (though the potential and the auxiliary field equations are dramatically modified) is just an explicit example of the general situation, which is discussed in Ref. [36]. The arguments of that paper imply that the 'non-renormalization' of the vacuum will persist even if higher curvature corrections are considered, at least for the case with unbroken SUSY. [Notice that these arguments are of a rather different nature from the usual non-renormalization theorems. These latter ones assume the standard supergravity form of the scalar potential, which is incorrect for a higher derivative theory, and, on top of that, there is the problem of the potential propagation of the auxiliary fields].

Finally, we have seen explicitly how in higher derivative supergravity the auxiliary field equations of motion are no longer algebraic, let alone linear. (For the particular case, we are dealing with, this is connected with the peculiar equation for the torsion in the anomaly-free 10D theory, see Refs. [15,21,22] for a discussion). As a consequence, in general, the auxiliary fields are expected to become propagating, a phenomenon whose consequences (at the full non-linear level) we do not completely understand, at present. (However there is some work in

*) For a general discussion of the structure of the scalar potential in an $N = 1$ 4D supergravity with higher curvature terms see Ref. [36].

progress that will be reported elsewhere.) It may happen - as is the case for the models considered here - that these 'new' particles decouple in the true vacuum (or - equivalently - their masses diverge). The non-linearity of the equation also allows the presence of more than just one solution. Each of these solutions - if physically acceptable - when inserted back into the original Lagrangian gives a physically different low-energy effective theory for the light fields^{*}).

In this last case, the new 'exotic' solutions would break SUSY simply because the 'exotic' vacua are characterized by non-vanishing values of the auxiliary fields. Clearly if this could happen it would produce a new non-perturbative mechanism to break SUSY, different from all the known ones.

Unfortunately, we are not aware of a model in which the exotic solutions have simultaneously all the properties required to be physically acceptable - in particular, absence of both negative-metric and tachyonic states. In the present case, for instance, it seems that the exotic solutions lead to poltergeists. However, this trouble might be just a consequence of the fact that our Lagrangian is a very inadequate description of the superstring for large values of the curvature: the additional interactions present in the string may stabilize the exotic solution (or add different, better behaving exotics), so that it cannot be excluded a priori that a non-perturbative mechanism of this kind is effective in the real string case to break supersymmetry. Unfortunately, to work it out would require - most probably - a complete understanding of the coupling to supergravity of $N = 1$ multiplets of arbitrary high spin - a formidable task.

10. GAUSS-BONNET INVARIANTS IN THE NEW-MINIMAL FORMULATION AND THE HETEROTIC STRING (note added as a postscript)

After the completion of the present paper, there were some new developments in the field, which put the above results into a rather different perspective.

^{*}) The auxiliary fields - if they become propagating at all - are very heavy ($\approx$ Planck mass) and so they should be integrated over to get the massless effective theory, i.e. they should be eliminated using their equation of motion.

As we have already discussed above, all the previous calculations were done using the so-called 'old-minimal formulation' [24] of the supergravity auxiliary fields. Other formulations also exist [53,54], but they are known to lead to theories equivalent to those described by the old-minimal formulation [55]. In fact, all these different formulations are just related by simple Legendre transformations [55].

However, recently it has been shown {see Ref. [56]} that the equivalence between different auxiliary field formulations breaks down when higher derivative terms are added to the Lagrangian (this is, indeed, a general property of variational formulations related by Legendre transformations).

In particular, each formulation contains its own specific super-Gauss-Bonnet theorem, which, when interacting with some matter field (as is the case in the present paper) leads to physically inequivalent theories.

Obviously, the most interesting situations correspond to the minimal formulations, i.e. those with the smallest set of auxiliary fields. The old-minimal case is described in full detail in Sections 2 to 8 above and in Appendix C below. The new-minimal situation [54] has been analysed (in the linearized approximation) in Ref. [56]. Here we discuss just the conclusions of that paper which are relevant for the Gauss-Bonnet invariant and interactions.

Let us consider the following component of the superspace torsion:

$$T_{ab}^{\alpha}.$$

From the first Bianchi identity {and the usual constraints on the components of the torsion [57]} we have

$$\hat{D}_a \tilde{T}^{ab\alpha} = 0 \quad (10.1)$$

where

$$\tilde{T}^{ab\alpha} = \frac{1}{2} \epsilon^{abcd} T_{cd}^{\alpha} \quad (10.2)$$

and

$$(\hat{D}_a)^{\alpha}_{\beta} = D_a \delta_{\beta}^{\alpha} + T_{a\beta}^{\alpha} \quad (10.3)$$

The explicit expression of T_{ab}^α is [57]

$$T_{\gamma\dot{\gamma}\beta\dot{\beta}\alpha} = \epsilon_{\gamma\dot{\gamma}} T_{\gamma\beta\alpha} + \epsilon_{\gamma\beta} T_{\gamma\dot{\gamma}\dot{\beta}\alpha} \quad (10.4)$$

with

$$T_{\gamma\beta\alpha} = -2 W_{\gamma\beta\alpha} + \frac{1}{2} \epsilon_{\gamma\alpha} W_{\beta} + \frac{1}{3} \epsilon_{\beta\alpha} W_{\gamma} \quad (10.5a)$$

$$T_{\gamma\dot{\gamma}\beta\dot{\beta}\alpha} = \frac{1}{2} \bar{D}_{\gamma} G_{\alpha\dot{\beta}} + \frac{1}{2} \bar{D}_{\dot{\beta}} G_{\alpha\dot{\gamma}} \quad (10.5b)$$

Here $W_{\gamma\beta\alpha}$, $G_{\alpha\dot{\alpha}}$, and W_{α} are the three basic superspace gravitational curvatures of the new minimal formulation, namely:

- $W_{\gamma\beta\alpha}$: the Weyl multiplet (it is basically the same multiplet that we discussed above in the context of the old-minimal formulation).
- $G_{\alpha\dot{\alpha}}$: the Einstein multiplet (which should not be confused with the Ricci multiplet $E_{\alpha\dot{\alpha}}$ of the old-minimal geometry).
- W_{α} : the trace multiplet (a chiral-spinor multiplet, which contains in its components the scalar curvature).

These multiplets obey the following Bianchi identities [56, 57]:

$$D^{\dot{\alpha}} G_{\alpha\dot{\alpha}} = -2 \bar{W}_{\dot{\alpha}} \quad (10.6a)$$

$$\bar{D}_{\dot{\alpha}} W_{\alpha} = 0 \quad (10.6b)$$

$$\bar{D}_{\dot{\alpha}} W_{\alpha\beta\gamma} = 0 \quad (10.6c)$$

$$D^{\dot{\alpha}} W_{\alpha} = \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \quad (10.6d)$$

$$D^{\gamma} W_{\alpha\beta\gamma} = \frac{i}{2} D_{\alpha}^{\dot{\gamma}} G_{\beta\dot{\gamma}} - \frac{1}{3} D_{\alpha} W_{\beta} + \frac{i}{2} D_{\beta}^{\dot{\gamma}} G_{\alpha\dot{\gamma}} - \frac{1}{3} D_{\beta} W_{\alpha} \quad (10.6e)$$

From Eqs. (10.6a) and (10.6b) we have

$$\begin{aligned} \bar{D}^{\dot{\beta}} \bar{D}^{\dot{\alpha}} G_{\alpha\dot{\alpha}} = 0 & \Rightarrow \\ \bar{D}^{\dot{\beta}} \bar{D}_{\dot{\beta}} G_{\alpha\dot{\alpha}} = D_{\beta} D^{\beta} G_{\alpha\dot{\alpha}} = 0 \end{aligned} \quad (10.7)$$

and from Eqs. (10.6a) and (10.6d) we get

$$D^{\alpha\dot{\alpha}} G_{\alpha\dot{\alpha}} = 0 \quad (10.8)$$

Moreover, from Eqs. (10.4) to (10.6) we see that T_{ab}^{α} satisfies the following two identities

$$\bar{D}_{\dot{\alpha}} T_{ab}^{\alpha} = 0 \quad (10.9a)$$

$$D_{\alpha} T_{ab}^{\alpha} = \bar{D}^{\dot{\alpha}} \bar{T}_{ab\dot{\alpha}} \quad (10.9b)$$

Notice that Eqs. (10.9) have exactly the same form as the Bianchi identities satisfied by the field strength supermultiplet of an Abelian vector.

Hence, in the linearized case, the general solution to Eqs. (10.9) is [56]

$$T_{ab}^{\alpha} = \bar{D}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}} D^{\alpha} V_{ab} \quad (10.10)$$

for some real vector V_{ab} . This vector is further constrained by Eq. (10.1) which has the usual structure of a Bianchi identity for a gauge field strength. Then, at the linear level, we have [56]

$$V_{ab} = -\frac{1}{8} (\partial_a \phi_b - \partial_b \phi_a) \quad (10.11)$$

for some real vector multiplet Φ_a .

Next, let us consider the combination $T_{ab}^{\alpha} \tilde{T}^{ab}_{\alpha}$. At the linear level we have

$$\begin{aligned} T_{ab}^{\alpha} \tilde{T}^{ab}_{\alpha} &= \frac{1}{64} \epsilon^{abcd} [\bar{D}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}} D^{\alpha} \partial_a \phi_b] \cdot \\ &\cdot [\bar{D}^{\dot{\beta}} \bar{D}_{\dot{\beta}} D_{\alpha} \partial_c \phi_d] = \partial_a \left\{ \frac{1}{64} \epsilon^{abcd} \bar{D}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}} \cdot \right. \\ &\cdot \left. [(D^{\alpha} \phi_b) \bar{D}^{\dot{\beta}} \bar{D}_{\dot{\beta}} D_{\alpha} \partial_c \phi_d] \right\} \end{aligned} \quad (10.12)$$

In this approximation, it is manifest, from Eq. (10.12), that $T_{ab}^\alpha \tilde{T}^{ab}_\alpha$ is a total derivative.

Moreover, from Eq. (10.9a) we learn that $T_{ab}^\alpha \tilde{T}^{ab}_\alpha$ is a chiral multiplet^{*}), so it makes sense to consider the Lagrangian density

$$\left[T_{ab}{}^d \tilde{T}^{ab}{}_d \right]_F \quad (10.13)$$

which is a total derivative, using Eq. (10.12). Thus Eq. (10.13) is a topological invariant for the new-minimal formulation. Indeed, its explicit component form is given in Ref. [56] (at the linear level). We have

$$\begin{aligned} \text{Re} \left[T_{ab}{}^d \tilde{T}^{ab}{}_d \right]_F &= \frac{1}{8} R_{\mu\nu\sigma\lambda} {}^*R^{\mu\nu\sigma\lambda} + \\ &+ \epsilon_{\mu\nu\sigma\lambda} \epsilon_{\mu'\nu'\sigma'\lambda'} \partial_\mu F_{\nu\mu'\nu'} \partial_\sigma F_{\lambda\sigma'\lambda'} \end{aligned} \quad (10.14a)$$

and

$$\begin{aligned} \text{Im} \left[T_{ab}{}^d \tilde{T}^{ab}{}_d \right]_F &= \frac{1}{8} R_{\mu\nu\sigma\lambda} {}^*R^{\mu\nu\sigma\lambda} + \\ &+ \frac{i}{2} \psi_{\mu\nu} \sigma \cdot \partial \tilde{\psi}^{\mu\nu} - \frac{1}{4} F_{\mu\nu} {}^*F^{\mu\nu} + \\ &+ \frac{1}{2} {}^*R_{\mu\nu\sigma\lambda} \partial_\mu F_{\nu\sigma\lambda} + \frac{1}{4} \epsilon_{\mu\nu\sigma\lambda} \partial_\mu F_{\nu\sigma\lambda} \partial_\sigma F_{\lambda\sigma\sigma} \end{aligned} \quad (10.14b)$$

Thus the real part of Eq. (10.13) gives the new-minimal form of the super-Gauss-Bonnet theorem, whereas the topological invariant corresponding to its imaginary part is a linear combination of the Hirzebruch signature and of the Pontragnyn class for the Abelian connection A_μ , corresponding to the R-symmetry.

The new-minimal super-Gauss-Bonnet theorem, however, differs in a crucial way from its old-minimal counterpart [see Eqs. (3.17) to (3.24)]. In that case it was just the last component of the superfield $\hat{W}_L^2$, which was (locally) a total derivative. Here instead, we see that the whole multiplet $T_{ab}^\alpha \tilde{T}^{ab}_\alpha$ is a total derivative, see Eq. (10.12).

This fact - that all the components of $T_{ab}^\alpha \tilde{T}^{ab}_\alpha$ are separately total derivatives - becomes crucial when the 'new-minimal'

^{*}) From a superconformal point of view it has weights $w = n = 3$, see Ref. [56].

Gauss-Bonnet invariant is coupled to matter, in the way we discuss in Section 4. This geometrical difference is the main reason why the new-minimal Gauss-Bonnet combination no longer propagates the negative-norm Wess-Zumino multiplet, as we found in Section 6.

Using the same manipulations as in Ref. [52] (see Section 3 of that paper) for the Chern-Simons multiplet of an Abelian multiplet we have (always at the linearized level)

$$[S T^a{}_{ab} \tilde{T}^{ab}{}_d]_F + \text{h.c.} = - [\Omega^a(\phi) \partial_a (S + \bar{S})]_D \quad (10.15)$$

where S is any chiral multiplet and the (linearized) Gauss-Bonnet Chern-Simons multiplet $\Omega^a(\phi)$ is given by

$$\begin{aligned} 2 \Omega^a(\phi) = \frac{1}{64} \epsilon^{abcd} \{ & (D^a \phi_b) T_{cd} + \\ & + (\bar{D}_a \phi_b) \bar{T}_{cd} + \frac{1}{2} \phi_b (D^a T_{cd} + \\ & + \bar{D}_a \bar{T}_{cd}) \} \end{aligned} \quad (10.16)$$

From these explicit formulae, it follows that (to this order, at least).

- i) The Gauss-Bonnet interaction in the l.h.s of Eq. (10.15) does not contribute to the scalar potential since it vanishes for all constant backgrounds.

This, of course, is a major difference with respect to our findings in Sections 7 and 8 above.

- ii) There is no new propagating scalar multiplet in the new formulation, since there is no scalar multiplet for S to mix with.

Thus, whereas in the old-minimal case the unwanted scalars did decouple only at the linearized level (expanding around flat space), here it seems that the decoupling is more natural and - so to say - geometrical.

It is tempting to conjecture that, since the heterotic string is (believed to be) consistent, the auxiliary field

formulation chosen by the God of the string is the new-minimal formulation. This formulation has also the further virtue that all its auxiliary fields (in the gravitational sector) are, in fact, gauge fields.

Luckily for us, the question, 'What is the correct stringy auxiliary field formulation?' makes perfectly good sense, and can be answered with available techniques. In fact, for the matter multiplets the (on-shell) vertices corresponding to the auxiliary fields have recently been constructed (and applied to the problem of radiative supersymmetry breaking in the string) in Refs. [58].

Partly motivated by the above state of affairs and conjectures, in Ref. [59] we have addressed the issue of the explicit construction (from stringy first principles) of the vertices for the auxiliary fields of the gravitational sector of any 4D $N = 1$ heterotic string.

The above conjecture turns out to be true. The 'stringy' formulation is indeed the new-minimal one. The reason why God (or Whoever) has made this choice is deeply rooted in the superconformal algebra of the underlying 2D theory.

Indeed, as is shown in Ref. [60], space-time $N = 1$ supersymmetry requires, at the world-sheet level, an $N = 2$ superconformal invariance [61]. In particular, this means that there exists a conformal current $J(z)$ which, acting on the various operators of the 2D theory, generates the $U(1)$ symmetry of the $N = 2$ superconformal algebra. The current $J(z)$ acts also on the vertices corresponding to physical particles. From the point of view of space-time, this action on the physical states is interpreted as the R-symmetry. The R-symmetry (in the heterotic case) should be considered as a local symmetry, in the sense that there exists a vertex operator

$$\frac{2}{3} J(z) \bar{\partial} \chi^r(\bar{z}) e^{i \kappa \cdot X(z, \bar{z})} \quad (10.17)$$

which can be consistently interpreted as the emission vertex for the (non-propagating) connection A_μ of such a symmetry. [The vertex in Eq. (10.17) is not BRS-invariant, but this is appropriate for a vertex which corresponds to a non-propagating field [58,59]].

As is well-known [2,25], the new-minimal formulation is the only one with a local R-symmetry gauged by an auxiliary field A_μ , since in all other formulations the superconformal U(1) symmetry gets broken by the value of the first component of the compensating multiplet.

The above symmetry is already a major hint in favour of the new-minimal formulation. Other physical motivations naturally leading to this formulation can be found by analysing the relationships between the gravitational and dilaton multiplet vertices.

Anyhow, the most direct and convincing way to see that the new-minimal formulation is indeed the stringy one, is to look for the particular realization of the superspace geometry given by the heterotic string vertices. In practice, this amounts to showing that the vertices of the gravitational sector can be organized into full superfields, and that these superfields obey all the Bianchi identities characterizing the new-minimal superspace geometry.

This is exactly the method that was used in Ref. [59] to show that - as a matter of fact - the heterotic string uses the new-minimal formulation.

Here we limit ourselves to giving some arguments in favour of the plausibility of the above result. More specific and technical arguments are given in Ref. [59].

The most elementary way to see that the new formulation is the correct one is to realize that the gravitational vertices should belong to a real linear multiplet [56]. For brevity, we shall consider here only the right-moving part of the string, since - as far as supersymmetry is concerned - the left-movers are just spectators. Then, let us consider the (right-moving part of a) bosonic vertex for a massless state. Since it has a vector index, it is natural to identify it as the vector component of some general supermultiplet.

The vector component of the general supermultiplet transforms as

$$\delta B^r = \gamma^r Z + \gamma^r \bar{Z} \quad (10.18)$$

Now, for a real linear multiplet, Λ is constrained to be equal to

$$\Lambda = - \not{\partial} Z \quad (10.19)$$

which substituted back in Eq. (10.18) gives us

$$\delta B^\mu = \gamma^{\nu\mu} \partial_\nu Z \quad (10.20)$$

Now, Eq. (10.20) should be compared with the SUSY transformation of a general (massless) vertex given by Friedan, Martinec and Shenker [62],

$$\begin{aligned} [Q_\alpha, V_B(z, \kappa, Z)] &= \\ &= V_F \left(U^\beta = i \kappa^\mu Z^\nu (\gamma_{\mu\nu})^\beta_\alpha, \kappa, Z \right) \end{aligned} \quad (10.21)$$

Equations (10.20) and (10.21) coincide, since the fermionic vertex V_F is just the Z -component of the supervertex {see Ref. [59] for details}.

Another elementary way to see that the above interpretation of the vector vertex as the vector component of a real linear supermultiplet is indeed correct is to realize that the standard constraint on the vector component of a linear multiplet [26]

$$\partial_\mu B^\mu = 0 \quad (10.22)$$

boils down to the usual gravitational and gauge Ward identities for the (on-shell) physical amplitudes. Again, a more precise discussion can be found in Ref. [59].

Thus, even if the relevance of perturbative unitarity in the context of effective Lagrangian may be a priori questioned [63], nevertheless all available information suggests that the heterotic string (but, most probably, not the other superstrings) manages not to have propagating ghosts even in this approximation. For the purely bosonic sector this was shown to be the case in Ref. [35]. Here we have generalized that result to the full (new-minimal) SUSY completion.

However, it remains to understand why. A lot of work is still to be done, but we think that the bridge between the underlying 2D superconformal theory and the 4D effective Lagrangian provided by the full understanding of the SUSY properties of the string vertices [59], will make the task ahead much easier.

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APPENDIX A: CONVENTIONS AND USEFUL FORMULAE

We list some of our notations:

$$\begin{aligned}\psi_L &= \frac{1+\gamma_5}{2} \psi & \psi_R &= \frac{1-\gamma_5}{2} \psi \\ \bar{\psi}_L &= \bar{\psi} \frac{1+\gamma_5}{2} & \bar{\psi}_R &= \bar{\psi} \frac{1-\gamma_5}{2} & \tilde{F}_{\mu\nu} &= \frac{1}{2} e \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \quad (\text{A.1})\end{aligned}$$

$$\text{Chiral multiplet} \quad S = (z, \chi_L, h)$$

$$\text{Vector multiplet} \quad V = (c, \beta_L, \mathcal{H}, B_\omega, \lambda_R, D) \quad \mathcal{H} = H + iK$$

$$S_1 \cdot S_2 = (z_1 z_2, z_1 \chi_{2L} + z_2 \chi_{1L}, z_1 h_2 + z_2 h_1 - 2 \bar{\chi}_{1L} \chi_{2L})$$

$$\frac{1}{e} \mathcal{L}_S = h + v z + \bar{\psi}_{aR} \gamma^a \chi_L + \bar{\psi}_{aR} \sigma^{ab} \psi_{bR} z + \text{h.c.}$$

D_μ is the Lorentz and Yang-Mills covariant derivative.

The following is the table of the superconformal curvatures in the 'old minimal' Poincaré gauge [2,25]:

$$\overline{R_{\mu\nu,d}}(Q) = (D_\nu \bar{\psi}_\mu + \bar{\psi}_\mu \gamma_\nu + i/2 A_\nu \bar{\psi}_\mu \gamma_5)_d - (\mu \leftrightarrow \nu)$$

$$R_{\mu\nu ab}^{\text{cov}}(M) = R_{\mu\nu ab}(M) - \frac{1}{2} R_{ab}(Q) (\gamma_\mu \psi_\nu - \gamma_\nu \psi_\mu)$$

$$R_{\mu\nu ab}(M) = R_{\mu\nu ab}(\omega) - \bar{\psi}_\mu \sigma_{ab} \phi_\nu + \bar{\psi}_\nu \sigma_{ab} \phi_\mu$$

$$R_{\mu\nu}(A) = \gamma_3 \partial_\mu A_\nu - i \bar{\psi}_\mu \gamma_3 \phi_\nu - (\mu \leftrightarrow \nu)$$

$$\overline{R_{\mu\nu,d}^{\text{cov}}}(S) = \overline{R_{\mu\nu,d}}(S) - \frac{i}{4} \left\{ \bar{\psi}_\nu \gamma^a [\gamma_5 R_{\lambda\mu}(A) - \tilde{R}_{\lambda\mu}(A)] - (\mu \leftrightarrow \nu) \right\}_d$$

$$\begin{aligned}\overline{R_{\mu\nu,d}}(S) &= (D_\nu \bar{\phi}_\mu - \bar{\psi}_\mu \gamma^a f_{a\nu} - i/2 A_\nu \bar{\phi}_\mu \gamma_5)_d + \\ &\quad - (\mu \leftrightarrow \nu)\end{aligned} \quad (\text{A.2})$$

$$R_{\mu\nu}^a(P) = D_\nu e_\mu^a + \frac{1}{4} \bar{\psi}_\mu \gamma^a \psi_\nu - (\mu \leftrightarrow \nu)$$

$$R_{\mu\nu}^a(K) = D_\nu f_\mu^a - \frac{1}{4} \bar{\phi}_\mu \gamma^a \phi_\nu - (\mu \leftrightarrow \nu)$$

$R_{\mu\nu ab}(M)$ is a modified curvature the exact superconformal curvature is:

$$\hat{R}_{\mu\nu ab}(M) = R_{\mu\nu ab}(M) - 2 [e_\mu^a f_\nu^b - e_\nu^b f_\mu^a - (\mu \leftrightarrow \nu)] \quad (\text{A.3})$$

In the whole of the paper $R_{\mu\nu ab}(M)$ curvature is used; notice that the f_v^a term cancels in the Weyl tensor and

$$\begin{aligned}\hat{W}_{\mu\nu ab}^{\text{cov}} &= W_{\mu\nu ab}^{\text{cov}} = R_{\mu\nu ab}^{\text{cov}}(M) - \frac{1}{2} (e_{\mu a} R_{\nu b}^{\text{cov}}(M) + \\ &\quad - e_{\mu b} R_{\nu a}^{\text{cov}}(M) - e_{\nu a} R_{\mu b}^{\text{cov}}(M) + e_{\nu b} R_{\mu a}^{\text{cov}}(M)) + \\ &\quad + \frac{R^{\text{cov}}(M)}{6} (e_{\mu a} e_{\nu b} - e_{\mu b} e_{\nu a})\end{aligned}$$

The ω_μ^{ab} , f_μ^a , Φ_μ fields are eliminated with the superconformal constraints:

$$R_{\mu\nu}^a(P) = 0$$

$$R_{\mu\nu}(Q)\gamma^\nu = 0$$

(A.4)

$$\hat{R}_{\mu\nu}^{\text{cov}}(M) + i/2 \tilde{R}(A) = 0$$

which give

$$\begin{aligned}\omega_r^{ab} &= \omega_r^{ab}(e, \psi) = \left\{ \frac{1}{2} [e^{\nu a} (\partial_r e_\nu^b - \partial_\nu e_r^b) + \right. \\ &\quad + e^{a\lambda} e^{b\sigma} \partial_\sigma e_{c\lambda} e_r^c] - (a \leftrightarrow b) \left. \right\} + \frac{1}{4} (\bar{\psi}_r \gamma^a \psi^b + \\ &\quad - \bar{\psi}_r \gamma^b \psi^a + \bar{\psi}^a \gamma_r \psi^b)\end{aligned}$$

(A.5)

$$\phi_a = \frac{1}{3} Z_a + \frac{1}{2} \mathcal{M} \psi_a$$

$$e_r^a f_{av} = -\frac{1}{4} (R_{rv}^{\text{cov}}(M) - \frac{1}{6} g_{rv} R^{\text{cov}}(M)) - \frac{i}{8} \tilde{R}_{rv}(A)$$

and

$$Z_a = \frac{3}{2} (R_a^{\text{cov}} - \frac{1}{3} \gamma_a \gamma_b R_b^{\text{cov}})$$

$$R^{\mu\nu, \text{cov}} = e^\mu e^{\nu\sigma} \gamma_3 \gamma_\nu D_\sigma^{\text{cov}} \psi_\sigma$$

$$D_\sigma^{\text{cov}} \psi_\sigma = (D_\sigma + i/2 A_\sigma \gamma_5 - i/2 \gamma_\sigma \mathcal{M}) \psi_\sigma$$

(A.6)

$$\mathcal{M} = -1/3 (S - i \gamma_5 P - i \not{A} \gamma_5)$$

$$D_\sigma \psi_\sigma = (\partial_\sigma + \frac{1}{2} \omega_\sigma^{ab} \sigma_{ab}) \psi_\sigma$$

Usually one can achieve some simplification if one uses Lorentz flat indices (latin letters) instead of world indices (greek letters). To do this the vierbein is used

$$R_{ab}(\gamma) = e_a{}^\mu e_b{}^\nu R_{\mu\nu}(\gamma)$$

$$\psi_a = e_a{}^\mu \psi_\mu$$

$$\partial_\mu A_\nu = e_\mu{}^a e_\nu{}^b D_a A_b - e_\nu{}^b \omega_{\mu, bc}(\psi) A_c \quad (A.7)$$

$$\frac{1}{e} \partial_\nu (g^{\mu\nu} e \partial_\mu z) = D_a (\partial_a z) + \frac{1}{2} \bar{\psi}_c \gamma \cdot \psi \partial_c z$$

$$\text{where} \quad \omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e, \psi) = \omega_\mu{}^{ab}(e) + \omega_\mu{}^{ab}(\psi)$$

$$\text{and} \quad D_a A_b = \partial_a A_b + e_a{}^\mu \omega_{\mu}{}^{bc} A_c$$

We list now some super-Poincaré and conformal (in the 'old minimal' formulation) covariant derivatives: chiral multiplets $S = (z, X_L, h)$

$$D_\mu^P z^* = \partial_\mu z^* - \bar{\psi}_{\mu R} \chi_R$$

$$D_\mu^P \chi_R = D_\mu \chi_R - \frac{1}{2} \not{D}^\mu z^* \psi_{\mu L} - \frac{1}{2} h^* \psi_{\mu R}$$

$$D_\mu^C z^* = \partial_\mu z^* - \bar{\psi}_{\mu R} \chi_R - \frac{i}{3} A_\mu z^* = D_\mu^P z^* - \frac{i}{3} A_\mu z^* \quad (A.8)$$

$$D_\mu^C \chi_R = D_\mu \chi_R + \frac{i}{6} A_\mu \chi_R - \frac{1}{2} \not{D}^\mu z^* \psi_{\mu L} - \frac{1}{2} (h^* + \frac{1}{3} U^* z^*) \psi_{\mu R} + z^* \phi_{\mu R}$$

$$\begin{aligned} \square^C z^* &= D_a (D_a^C z^*) + \frac{1}{6} R^{\text{cov}}(H) z^* + \bar{\phi}_L \cdot \gamma \chi_R + \\ &- \bar{\psi}_{\mu R} D_\mu^C \chi_R - \frac{i}{3} A_a D_a^C z^* \end{aligned}$$

real vector multiplets $V = (c, \beta_L, \mathcal{H}, \beta_{\mu}, \lambda_R, D)$

$$D_\mu^C c = \partial_\mu c - \frac{i}{2} \bar{\psi}_{\mu L} \beta_L$$

$$\begin{aligned} D_\mu^C \beta_L &= D_\mu \beta_L - \frac{i}{2} \mathcal{H} \psi_{\mu L} + \frac{1}{2} \not{D} \psi_{\mu R} + \frac{i}{2} \not{D}^C c \psi_{\mu R} + \\ &+ \frac{i}{6} c \psi_{\mu L} + \frac{1}{6} c \not{D} \psi_{\mu R} \end{aligned} \quad (A.9)$$

APPENDIX B. BOSONIC COMPONENT OF $(\text{CURVATURE})^2$ MULTIPLETS

The bosonic sectors of the $\Sigma[\text{RT}(R)]$, $\Sigma(W^2)$ and $\Sigma(E^2)$ multiplets are [see (2.7), (2.8), and (2.9)]:

$$[\Sigma(\text{RT}(R))]_z = \frac{1}{2} R U^* - \frac{1}{3} U^* (U U^* + A_a^2) + i U^* D_a A_a$$

$$\begin{aligned} [\Sigma(\text{RT}(R))]_h = & U^* D_a (D_a U) + \frac{1}{4} R^2 - \frac{1}{3} R (2 U U^* + A_a^2) + \\ & + \frac{4}{9} U U^* (A_a^2 + U U^*) + \frac{1}{9} (A_a^2)^2 + (D_a A_a)^2 + \\ & - \frac{i}{3} U^* A_a \partial_a U - i U U^* D_a A_a \end{aligned}$$

$$[\Sigma(W^2)]_z = 0$$

$$\begin{aligned} [\Sigma(W^2)]_h = & 2 (W_{abcd})^2 - \frac{4}{3} (D_a A_b - D_b A_a)^2 + \\ & - 2 W_{abcd} \tilde{W}^{abcd} + \frac{4}{3} \frac{1}{2} \epsilon_{abcd} (D_a A_b - D_b A_a) \cdot \\ & \cdot (D_c A_d - D_d A_c) \end{aligned}$$

(B.1)

$$[\Sigma(E^2)]_z = -\frac{4}{3} A_a^2 U^* + 2i A_a \partial_a U^*$$

$$\begin{aligned} [\Sigma(E^2)]_h = & \partial_a U^* \partial_a U + \frac{1}{6} R (U U^* + 2 A_a^2) - i A_a (U \partial_a U^* + \\ & - U^* \partial_a U) - \frac{9}{4} (R_{ab}^2 - \frac{2}{9} R^2) - R_{ab} A_a A_b + \\ & + \frac{1}{4} \tilde{F}_{ab} \tilde{F}^{ab} - \frac{1}{9} U U^* (U U^* + A_a^2) - \frac{1}{9} (A_a^2)^2 + \\ & - (D_a A_b)^2 + 2 A_a (D_a D_b A_b - D_b D_b A_a) + \\ & + \frac{4}{3} A_a^2 U U^* + D_a (\partial_a A_b^2) - \frac{i}{3} A_a (U \partial_a U^* + \\ & - U^* \partial_a U) + 2i D_a (A_b B_{ba}) - \frac{2}{3} i D_a (A_a A_b^2) + \\ & - i A_a (U \partial_a U^* + U^* \partial_a U) \end{aligned}$$

where [see (2.6)]

$$B_{ab} = \frac{3}{2} (R_{ab} - \frac{1}{6} S_{ab} R) + i \frac{1}{2} \tilde{F}_{ab} - \frac{1}{6} (UU^* + A_c^2) S_{ab} + \frac{1}{3} A_a A_b$$

$$F_{ab} = D_a A_b - D_b A_a$$

$$D_a^c U = \partial_a U - i \frac{1}{3} A_a U$$

and the Ricci multiplet was transformed into a chiral one with the embedding formula (2.10):

$$\begin{aligned} \Sigma(V)|_{\text{Bcs}} = & \left[-\gamma\gamma^* + \frac{2}{3} U^* C, 0, D + D_a (\partial_a C) + \right. \\ & + \frac{2}{3} C \left(\frac{1}{2} R - \frac{2}{3} UU^* - \frac{1}{3} A_a^2 \right) + \frac{2}{3} A_a B_a + \\ & \left. - \frac{1}{3} \gamma\gamma U^* + \frac{2}{3} U \gamma\gamma^* + i D_a \left(B_w - \frac{2}{3} C A_w \right) \right]_{(B.2)} \end{aligned}$$

APPENDIX C: SUPERCONFORMAL CURVATURE MULTIPLETS

In this appendix, we discuss the three gravitational multiplets $W_{\alpha\beta\gamma}$, E_m , and R from the point of view of the superconformal tensor calculus [2,25,26], which is the most general and powerful approach to 4D $N = 1$ supergravity.

Our first problem is to construct three multiplets which are covariant under local superconformal transformations and which, under the appropriate gauge conditions, reproduce $W_{\alpha\beta\gamma}$, E_m , and R [28].

The superconformal multiplets carry a Weyl weight w and a chiral weight n (as well as a Lorentz representation). The Weyl and chiral weights can be adjusted to the required values multiplying the given multiplet by a suitable power of the compensator field S_0 and of its conjugate $\bar{S}_0$. We choose S_0 to be a chiral superconformal multiplet with $w = n = 1$. Then the results of the Poincaré tensor calculus are recovered by fixing the dilation, chiral and S -supersymmetry gauges by the conditions [24].

$$S_0 = (1, 0, \frac{1}{3} u) \quad \text{POINCARÉ GAUGE} \quad (C.1)$$

where u is to be identified with the auxiliary field of Poincaré supergravity.

C.1 Superconformal gravitational curvature multiplets and actions

We have to construct superconformal multiplets which in the gauge (C.1) reduce to the results of Section 3. Moreover, we have to construct Lagrangians which are invariant under the full local superconformal group and such that in the above gauge they reproduce the results in the text.

C.2 Weyl multiplet

The Weyl multiplet is superconformal by itself, since it is the multiplet containing the conformal invariant part of the (super-) curvatures. It is chiral and has $W = 3/2$. Then $w_{\alpha\beta\gamma} W^{\alpha\beta\gamma}$ is a chiral multiplet with Weyl weight 3, which can be inserted into the superconformal density to obtain the invariant Lagrangian

$$(W^2)_n + \bar{\Psi}_R \cdot \gamma (W^2)_\chi + (W^2)_z \bar{\Psi}_r \sigma^{\mu\nu} \Psi_{\nu R} \quad (C.2)$$

which is just the expression of $\int d^2\theta EW^2$ in components.

Notice that the $(Weyl)^2$ sector of the Lagrangian does not contain the compensator field at all.

C.3 Scalar curvature multiplet

The superconformal counterpart of the scalar curvature multiplet can be identified with the following chiral superconformal multiplet

$$R = 3 S_o^{-1} T(S_o) = -3 S_o^{-1} \Sigma(S_o) \quad (C.3)$$

which has $w = n = 1$. In Eq. (C.3), T is the superconformal kinetic operation, which acting on a chiral multiplet with $w = n = 1$ gives a chiral multiplet with $w = n = 2$, (here D_a^C and $\square^C$ are the fully covariant derivative and d'Alembertian, respectively); Σ is the conformal operation converting a general multiplet with $w - n = 2$ into a chiral multiplet with $w' = n' = w + 1$ [2,25]

$$\begin{aligned} \Sigma(C, Z, H, K, B_\mu, \Lambda, D) = \\ = [(H - iK), i(\not{D}^C \chi_R + A_L), \\ - (D + \not{D}^C C + i D_\mu^C B^\mu)] \end{aligned} \quad (C.4)$$

More specifically, the multiplets of the Poincaré tensor calculus must be identified with superconformal multiplets having $w = n = 0$. Then the correct identification is

$$R = S_o^{-1} \mathcal{R} \Big|_{\text{POINCARÉ GAUGE}} \quad (C.6)$$

To prove Eqs. (C.3) and (C.6) we need the following Lemma.

Lemma [2,25,26]

- i) A superconformal multiplet with given w , n , and Lorentz representation is uniquely identified by its first component.

- ii) The same holds for Poincaré multiplets.
- iii) A superconformal multiplet with $w = n = 0$, restricted by the gauge condition (C.1), is a Poincaré multiplet with respect to modified supersymmetry.

Now by construction, the r.h.s. of Eq. (C.6) is a chiral superconformal multiplet with $w = n = 0$, and hence, after (C.1) is imposed, a Poincaré multiplet. Then by the Lemma, the two sides of Eq. (C.6) are equal if their first components are; and they are both equal to U^* .

Then $R\bar{R}$ is a vector superconformal multiplet with $w = 2$ which can be inserted into the superconformal invariant density for a $w = 2$ vector $(C, H, K, B_m, \lambda, D)$ [2,25]

$$e^1 \mathcal{L} \Big|_{w=2} = D - \frac{1}{2} \bar{\Psi} \cdot \gamma \cdot \gamma_5 \lambda - \bar{\Psi} \cdot \gamma \cdot \gamma_5 Z + \\ + \frac{1}{2} C (R + e^1 \bar{\Psi}_\mu R^\mu) + \frac{i}{4} \epsilon^{\mu\nu\kappa\rho} \bar{\Psi}_\mu \gamma_\nu \Psi_\kappa (B_e + \\ - A_e C - \frac{1}{2} \bar{\Psi}_e Z) \quad (C.7)$$

to obtain the superconformal version of $(1/2) \int d^2\theta \text{ ERT}(R) + \text{h.c.}$

C.4 Ricci multiplet

The Ricci multiplet E_m is the most difficult to discuss since it has an external Lorentz index m . Even in the Poincaré framework this multiplet is subtle and its components are not completely worked out. Naïvely one would expect

$$E_m = (S_e \bar{S}_e)^{-1/2} \mathcal{E}_m \quad (C.8)$$

$$\mathcal{E}_m = -\frac{3}{2} i \left[S_e^{-1} D_m^e S_e - \bar{S}_e^{-1} D_m^e \bar{S}_e \right] \quad (C.9)$$

But Eq. (C.9) does not define a superconformal multiplet since its 'first component' is not invariant under S-supersymmetry and conformal boosts, as it should be for the first component of any superconformal multiplet.

The above problem was solved in Ref. [26] by the introduction of the so-called u-associated covariant derivatives $D_m^{(U)}$, $D_\alpha^{(U)}$, $\bar{D}_{\dot{\alpha}}^{(U)}$; here a superconformal multiplet U is used to invariantize the derivatives with respect to S- and K-

transformations. The action of these new derivatives is defined on any superconformal multiplet and obeys the (graded) chain rule. Here we use as invariantizer U the same chiral multiplet S_0 which compensates the weights.

$D_m^{(S_0)}$ acting on any superconformal multiplet with given weights w and n , gives a new superconformal multiplet with $w' = w + 1$, $n' = n$, and one more Lorentz index.

We have the identities [26]

$$D_m^{(S_0)} (S_0 \bar{S}_0) = 0 \quad (C.10)$$

$$S_0^{-1} D_m^{(S_0)} S_0 = - \bar{S}_0 D_m^{(S_0)} \bar{S}_0 \quad (C.11)$$

Then the correct definition of E_m ($w = 1$, $n = 0$) is

$$\xi_m = -3i S_0^{-1} D_m^{(S_0)} S_0 = 3i \bar{S}_0^{-1} D_m^{(S_0)} \bar{S}_0 \quad (C.12)$$

and its first component is

$$-\frac{3}{2}i \left(z_0^{-1} D_m z_0 - z_0^{*-1} D_m z_0^* \right) - \frac{3}{8}i \left\{ \bar{\zeta}_L \gamma_m \right\}_R \quad (C.13)$$

where (in a generic gauge)

$$S_0 = (z_0, \zeta_L, h_0 = \frac{1}{3}U) \quad (C.14)$$

Equations (C.8) and (C.12) are again a consequence of the Lemma and the fact that the first component of E_m (in the Poincaré gauge) is A_m .

$E_m E^m$ is a superconformal multiplet with $w = 2$, which inserted in Eq. (C.7) gives the superconformal counterpart of $\int d^4\theta E(E_m E^m)$.

S_0 -associated derivatives are also useful to give a better formulation of the scalar curvature multiplet, which we shall exploit in the following.

One can define an S_0 -associated chiral projector $\Sigma^{(S_0)}$, which acting on any superconformal multiplet, will give an S_0 -chiral multiplet, which is a multiplet V such that

$$\bar{D}_2^{(S_0)} V = 0 \quad (C.15)$$

The explicit form of $\Sigma^{(S_0)}$ is [26]

$$\Sigma^{(S_0)} = -\frac{1}{4} \overline{\mathcal{D}}_2^{(S_0)} \overline{\mathcal{D}}^{(S_0)2} - \overline{S}_0^{-1} T(S_0) \quad (C.16)$$

From the identities [26]

$$\overline{\mathcal{D}}_2^{(S_0)} \overline{S}_0 = \mathcal{D}_2^{(S_0)} S_0 = 0 \quad (C.17)$$

we get

$$T^{(S_0)}(S_0) \stackrel{\text{def}}{=} -\Sigma^{(S_0)}(\overline{S}_0) = T(S_0) \quad (C.18)$$

and so by Eq. (C.3)

$$R = 3 S_0^{-1} T^{(S_0)}(S_0) \quad (C.19)$$

C.5 The superconformal Gauss-Bonnet theorem
The expression*)

$$\mathcal{L}_{GB} = 2 [\mathcal{W}_{\alpha\beta\gamma} \mathcal{W}^{\alpha\beta\gamma}]_h + \frac{32}{9} [\mathcal{E}_m \mathcal{E}^m]_D + \frac{32}{9} [R \overline{R}]_D \quad (C.20)$$

is a total derivative and

$$\int_M d^4x \mathcal{L}_{GB} = 32 \pi^2 \chi(M) - 16 \pi^2 i P(M) \quad (C.21)$$

where $\chi(M)$ is the Euler characteristic of M and $P(M) = 3\tau(M)$ is the Pontryagin number [equal to 3 times the Hirzebruch signature $\tau(M)$].

This theorem shows that the ordinary index theorems of geometry not only extend to topological invariants of 'superspace geometry' but also to invariants of a somehow larger space-parametrizing superconformal invariance.

Proof. By construction both Eqs. (C.20) and (C.21) are invariant for local superconformal transformation. But in the Poincaré gauge, Eqs. (C.20) and (C.21) give the usual supersymmetric Gauss-Bonnet theorem [28]. QED

*) Here $[-]_h$ is the chiral density of Eq. (C.2) and $[-]_D$ is the vector superconformal action density of Eq. (C.7).

C.6 Superconformal transformations of gravitational multiplets

Suppose we work in the Poincaré gauge (or, equivalently, superspace geometry) framework. We ask ourselves how the three multiplets $W_{\alpha\beta\gamma}$, R , and E_m would change when we make a super Weyl transformation, which, on the determinant of the supervierbein E , acts as

$$E \rightarrow E e^{-(\Lambda + \bar{\Lambda})} \quad (C.22)$$

where Λ is a chiral superfield, $\bar{D}_{\dot{\alpha}}\Lambda = 0$.

The infinitesimal transformations have been worked out by Howe and Tucker in Ref. [44]. Here we show the power of the above methods by constructing explicitly the highly non-trivial finite transformations, with less work than required for the infinitesimal case with standard methods.

We need the (obvious) following Lemma.

Lemma. The transformation (C.22) for a Poincaré multiplet V amounts to

$$V(S_0, \bar{S}_0) \Big|_{\text{POINCARÉ GAUGE}} \rightarrow V(e^{-\Lambda} S_0, e^{-\bar{\Lambda}} \bar{S}_0) \Big|_{\text{POINCARÉ GAUGE}} \quad (C.23)$$

where the superconformal multiplet $V(S_0, \bar{S}_0)$ has $w = n = 0$ and it is such that

$$V(S_0, \bar{S}_0) \Big|_{\text{POINCARÉ GAUGE}} = V$$

(This Lemma says that - by definition - a multiplet with zero weights is invariant under scale and chiral transformations if we vary both the gravitational fields (the Weyl multiplet) and the compensator S_0).

i) Since $S_0^{-3/2} W_{\alpha\beta\gamma}$ has $w = n = 0$ we have

$$W_{\alpha\beta\gamma} \rightarrow e^{3/2 \Lambda} W_{\alpha\beta\gamma} \quad (C.24)$$

ii) R has already $w = n = 0$. Then

$$R = 3 S_0^{-2} T^{(S_0)}(S_0) \rightarrow -3 e^{2\Lambda} S_0^{-2} \sum (e^{-\Lambda} S_0) (e^{-\bar{\Lambda}} \bar{S}_0) \quad (C.25)$$

For any scalar chiral supermultiplet A , with $w = n = 1$ we have the identity

$$\bar{\Sigma}^{(e^{-\Lambda} S_0)}(\bar{A}) = \bar{\Sigma}(\bar{A}) = \bar{\Sigma}^{(S_0)}(\bar{A}) \quad (C.26)$$

Putting $A = e^{-\Lambda} S_0$ we have

$$\begin{aligned} \bar{\Sigma}^{(e^{-\Lambda} S_0)}(e^{-\bar{\Lambda}} \bar{S}_0) &= \bar{\Sigma}^{(S_0)}(e^{-\bar{\Lambda}} \bar{S}_0) = \\ &= -\frac{1}{4} \bar{\mathcal{D}}_2^{(S_0)} \bar{\mathcal{D}}^{(S_0)\dot{2}}(e^{-\bar{\Lambda}} \bar{S}_0) - e^{-\bar{\Lambda}} T(S_0) = \\ &= -\frac{1}{4} \bar{S}_0 \bar{\mathcal{D}}_2^{(S_0)} \bar{\mathcal{D}}^{(S_0)\dot{2}} e^{-\bar{\Lambda}} - e^{-\bar{\Lambda}} T(S_0) \end{aligned} \quad (C.27)$$

where we used the identities (C.16) and (C.17).

Inserting Eq. (C.27) in Eq. (C.25) we get

$$\frac{1}{3} R \rightarrow \frac{1}{3} R e^{2\Lambda - \bar{\Lambda}} + \frac{1}{4} (\bar{S}_0 S_0^{-2}) e^{2\Lambda} \bar{\mathcal{D}}_2^{(S_0)} \bar{\mathcal{D}}^{(S_0)\dot{2}} e^{-\bar{\Lambda}} \quad (C.28)$$

iii) For the Ricci multiplet E_m , we have

$$\begin{aligned} (e^{-\Lambda} S_0)^{-1} D_{\mu\nu}^{(e^{-\Lambda} S_0)}(e^{-\Lambda} S_0) &= S_0^{-1} D_{\mu\nu}^{(S_0)} S_0 + \\ &+ \frac{1}{2} [e^{\Lambda} D_{\mu\nu}^{(S_0)} e^{-\Lambda} - e^{\bar{\Lambda}} D_{\mu\nu}^{(S_0)} e^{-\bar{\Lambda}}] + \frac{1}{8} [(\bar{\chi}_L \gamma_{\mu\nu} \chi_R)] \end{aligned} \quad (C.29)$$

where χ is the fermionic component of the chiral field Λ and [26] [(a)] denotes the superconformal multiplet whose first component is a . It is easy to check that $[(\bar{\chi}_L \gamma_{\mu\nu} \chi_R)]$ is a superconformal multiplet with $w = n = 0$.

Then, from Eqs. (C.8), (C.12), and (C.29),

$$\begin{aligned} \frac{1}{3} E_{\mu\nu} \rightarrow e^{\frac{1}{2}(\Lambda + \bar{\Lambda})} \left\{ \frac{1}{3} E_{\mu\nu} - \frac{i}{2} (S_0 \bar{S}_0)^{-1/2} D_{\mu\nu}^{(S_0)} (\Lambda - \bar{\Lambda}) + \right. \\ \left. - \frac{i}{8} (S_0 \bar{S}_0)^{-1/2} [(\bar{\chi}_L \gamma_{\mu\nu} \chi_R)] \right\} \end{aligned} \quad (C.30)$$

As for infinitesimal transformations we have

$$\begin{aligned} \delta W_{\alpha\beta\gamma} &= \frac{3}{2} \Lambda W_{\alpha\beta\gamma} \\ \delta \left(\frac{2}{3} R \right) &= (2\Lambda - \bar{\Lambda}) \left(\frac{2}{3} R \right) - \frac{1}{4} (S_0 \bar{S}_0^{-2}) \bar{\mathcal{D}}_2^{(S_0)} \bar{\mathcal{D}}^{(S_0)\dot{2}} \bar{\Lambda} \\ \delta \left(\frac{1}{3} E_{\mu\nu} \right) &= \frac{1}{2} (\Lambda + \bar{\Lambda}) \left(\frac{1}{3} E_{\mu\nu} \right) - \frac{i}{2} (S_0 \bar{S}_0)^{-1/2} D_{\mu\nu}^{(S_0)} (\Lambda - \bar{\Lambda}) \end{aligned} \quad (C.31)$$

up to terms of higher order in Λ .

To recover the results of superspace geometry, we must exploit the relation between S_0 -associated derivatives and operators and their curved superspace counterparts. They are [26]

$$\mathcal{D}_\alpha = S_0^{1/2} \bar{S}_0^{-1} \mathcal{D}_\alpha^{(S_0)} \Big|_{\text{POINCARÉ GAUGE}} \quad (\text{C.32a})$$

$$D_\mu = (S_0 \bar{S}_0)^{-1/2} D_\mu^{(S_0)} \Big|_{\text{POINCARÉ GAUGE}} \quad (\text{C.32b})$$

$$\Sigma^P = -\frac{1}{4} (\bar{\mathcal{D}}_\alpha \bar{\mathcal{D}}^{\dot{\alpha}} - 8R) = S_0^{-2} \bar{S}_0 \bar{Z}^{(S_0)} \Big|_{\text{POINCARÉ GAUGE}} \quad (\text{C.32c})$$

In Eq. (C.32) we used Eqs. (C.16) and (C.17) as well as the fact that any chiral multiplet is also S_0 -chiral, i.e.

$$\bar{\mathcal{D}}_\alpha^{(S_0)} S_0 = \mathcal{D}_\alpha^{(S_0)} \bar{S}_0 = 0 \quad (\text{C.33})$$

From Eq. (C.32c) [or - equivalently - Eqs. (C.32a), (C.33), and (C.17)] we have

$$\bar{\mathcal{D}}_\alpha \bar{\mathcal{D}}^{\dot{\alpha}} = (S_0^{-2} \bar{S}_0) \bar{\mathcal{D}}_\alpha^{(S_0)} \bar{\mathcal{D}}^{\dot{\alpha}(S_0)} \Big|_{\text{POINCARÉ GAUGE}} \quad (\text{C.34})$$

so in the Poincaré gauge Eqs. (C.31) become

$$\begin{aligned} \delta W_{\alpha\beta\gamma} &= \frac{3}{2} \Lambda W_{\alpha\beta\gamma} \\ \delta \left(\frac{2}{3} R \right) &= (2\Lambda - \bar{\Lambda}) \left(\frac{2}{3} R \right) - \frac{1}{4} \bar{\mathcal{D}}_\alpha \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\Lambda} \\ \delta \left(\frac{1}{3} E_\mu \right) &= \frac{1}{2} (\Lambda + \bar{\Lambda}) \left(\frac{1}{3} E_\mu \right) - \frac{i}{2} D_\mu (\Lambda - \bar{\Lambda}) \end{aligned} \quad (\text{C.31b})$$

which agree with Eq. (9) of Ref. [44] after the identifications^{*)}

*) The relative coefficients in the normalization of the multiplets can be checked by comparing our form and their form of the super Gauss-Bonnet theorem.

$$\Lambda \longleftrightarrow -2\bar{\Sigma}$$

$$W_{\alpha\beta\gamma} \longleftrightarrow W_{\alpha\beta\gamma}$$

$$\frac{1}{3} E_m \longleftrightarrow G_{\alpha\beta}$$

$$\frac{2}{3} R|_{\text{OURS}} \longleftrightarrow R|_{\text{THEIRS}}$$

To get the corresponding finite transformations, take the Poincaré gauge in Eqs. (C.24), (C.28) and (C.30)

$$W_{\alpha\beta\gamma} \longrightarrow e^{\frac{3}{2}\Lambda} W_{\alpha\beta\gamma} \quad (\text{C.35a})$$

$$\frac{2}{3} R \longrightarrow \frac{2}{3} R e^{2\Lambda - \bar{\Lambda}} + \frac{1}{2} e^{2\Lambda} \bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} e^{-\Lambda} \quad (\text{C.35b})$$

$$\frac{1}{3} E_m \longrightarrow e^{\frac{1}{2}(\Lambda + \bar{\Lambda})} \left\{ \frac{1}{3} E_m - \frac{i}{2} D_m (\Lambda - \bar{\Lambda}) + \chi_m^{(2)} \right\} \quad (\text{C.35c})$$

where Z_m^2 is the following Poincaré multiplet

$$\chi_m^{(2)} = -\frac{i}{8} (S_{\alpha} \bar{S}_{\dot{\alpha}})^{-1/2} [(\bar{\chi}_L \gamma_m \chi_R)] \Big|_{\text{POINCARÉ GAUGE}} \quad (\text{C.36})$$

We see that the finite transformations are complicated non-linear transformations. The complications come from terms introduced to invariantize the derivatives with respect S-supersymmetry and conformal boosts.