HETEROTIC STRINGS IN SYMMETRIC AND ASYMMETRIC ORBIFOLD BACKGROUNDS

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ABSTRACT

We present a general formalism for constructing four-dimensional string models from $\mathbb{Z}_N$ orbifolds. An explicit form for the generalized GSO projector in the construction of twisted and untwisted states is provided, allowing then the computation of the massless spectrum for any embedding of the point group in the gauge degrees of freedom. Constant background fields, in particular Wilson lines, are considered, and the general constraints on them are presented for both symmetric and asymmetric orbifolds. Several examples are provided, including a general classification of the $\mathbb{Z}_3$ orbifold models with one Wilson line, a systematic search for $SU(3) \times SU(2) \times U(1)^n$ models with three generations, models with continuous symmetry breaking and lower rank, and asymmetric orbifold models with and without Wilson lines.

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1. INTRODUCTION

Orbifolds [1] have proved to be very interesting and rich backgrounds for constructing four-dimensional string models. They provide us with a compactification scheme which shares a level of simplicity similar to that of toroidal compactifications [2] but with more realistic perspectives. Considering the presence of background fields in the tori defining the orbifolds [3], and allowing for the possibility of left-right asymmetric twists [4], orbifold constructions can generate a wide class of models and are believed to provide a unifying framework for four-dimensional strings [5].

The class of models which can be obtained from this construction is parametrized by the background values of the fields g_{mn} , B_{mn} , and A_m^I ($m = 1, \dots, 6$; $I = 1, \dots, 16$) and the symmetric or asymmetric twists which can be performed in the tori defining the orbifolds, together with their different embeddings in the gauge degrees of freedom. These parameters are restricted by consistency conditions, such as modular invariance, but still leave an enormous degree of degeneracy. So far, only a very small class of possibilities has been investigated: some models have been found to present phenomenological virtues [6] but none of them can be claimed to be realistic. A more general and systematic investigation is required; therefore, a detailed study of the formalism for constructing arbitrary orbifold models is needed, in order to explore the possibilities offered by this compactification process.

In this article we present a general prescription for constructing four-dimensional string models from $\mathbb{Z}_N$ orbifold compactifications. An explicit expression for the generalized Gliozzi-Scherk-Olive (GSO) projector in the construction of twisted and untwisted states is provided for an arbitrary $\mathbb{Z}_N$ orbifold, thus allowing the computation of the massless spectrum for any embedding of the point group in the gauge degrees of freedom. Constant background fields, in particular Wilson lines, are considered, and the general constraints on them are given for both symmetric and asymmetric orbifolds. We have found that these conditions are very restrictive in the asymmetric case, and in many cases Wilson lines are not allowed. As an illustration of this formalism, we discuss several examples and possible applications. We describe a computer-generated classification of all the symmetric $\mathbb{Z}_3$ orbifold models with one Wilson line, including several three-generation models with gauge groups containing $SU(7)$, E_6 , $SO(10)$, or $SU(5)$. We also have systematically investigated three-generation models with gauge group $SU(3) \times SU(2) \times U(1)^n$ for two and three Wilson lines turned on. A list of possible models and some examples are given.

Non-Abelian embeddings of the space group in the gauge degrees of freedom allow us to find models with lower rank, such as $SU(3) \times SU(2) \times U(1)$, but unfortunately with unrealistic matter content, starting from the $\mathbb{Z}_3$ orbifold. More interestingly, this construction opens up the possibility of having continuous gauge symmetry breaking, in contraposition to the Abelian case in which the Wilson lines are quantized. We have found that at the same time a gauge field disappears from the massless spectrum, as do also some matter fields, in a sort of Higgs mechanism. It is worth noting that the number of 'Higgses' is most of the time three rather than one.

The outline of the paper is as follows. In Section 2 we present a short introduction to orbifolds and define the possible $\mathbb{Z}_N$ twists which can be performed preserving $N = 1$ supersymmetry. We also give the six-dimensional lattices of semisimple rank-6 gauge groups which have the $\mathbb{Z}_N$ twists as automorphisms. Section 3 is a discussion of the partition functions for $\mathbb{Z}_N$ orbifolds which determine the expression of the generalized GSO projector for both twisted and untwisted sectors. Modular invariance constraints are also derived. In Section 4 we review the formalism for considering constant background fields in the original torus. Wilson lines are introduced using two formulations, namely (Abelian and non-Abelian) space-group embeddings and twisting of the Lorentzian even self-dual lattice $\Lambda_{22,6}$ of toroidal compactifications which is already parametrized by the constant background fields. This later formulation also allows us to consider Wilson lines on asymmetric orbifolds. Section 5 is devoted to explicit models. The classification of $\mathbb{Z}_3$ -based models mentioned above is

presented; also models on symmetric $\mathbb{Z}_4$ and $\mathbb{Z}_7$ orbifolds are given in order to illustrate our formalism in non-trivial cases. A detailed discussion of continuous gauge symmetry breaking in string theories is included. Especially in the non-Abelian embedding constructions, the possibility of having extra-light Higgses from this mechanism is also discussed. Asymmetric orbifold examples are illustrated at the end, with and without Wilson lines. Our conclusions are presented in Section 6. Finally, we have two appendices: the first one is only a list of ϑ -function formulae which are needed in the text; the second is a detailed description of how the generalized GSO projector works in non-trivial cases such as the $\mathbb{Z}_4$ and $\mathbb{Z}_6$ symmetric orbifolds.

2. $\mathbb{Z}_N$ ORBIFOLDS

In the present section we will consider the propagation of heterotic strings on symmetric $\mathbb{Z}_N$ orbifold backgrounds. Most of the discussion can be trivially generalized to the case of asymmetric orbifolds. We will have in mind the $E_8 \times E_8$ heterotic string as our starting point for the ‘modding-out’ operation since it is easier to get semirealistic examples in this case. As we will see in the next section, it is equally possible (and, in fact, more interesting) to mod out the large class of four-dimensional string theories discussed by Narain.

Our world-sheet fields will consist of eight bosonic right-moving fields $X_R^i(\sigma + \tau)$, $i = 1, \dots, 8$ (in the light-cone gauge), and Ramond–Neveu–Schwarz (RNS) right-moving fermions $\lambda^i(\sigma + \tau)$. This completes a right-moving supersymmetric string. Left-moving fields include eight bosons $X_L^i(\sigma - \tau)$ and another sixteen $X_L^I(\sigma - \tau)$ ($I = 1, \dots, 16$) fields without right-handed counterparts [or, equivalently, 32 left-handed world-sheet fermions $\Psi_L^K(\sigma - \tau)$, $K = 1, \dots, 32$]. A four-dimensional string is obtained in the bosonic formulation by compactifying 6 right-moving and 22 left-moving coordinates on a torus $T_R^6 \oplus T_L^{22}$ and then modding this torus by some isometry (this is done in order to decrease the number of supersymmetries and to break the gauge group). This modding-out operation defines an orbifold.

When we mod left- and right-movers in the same way, we obtain a symmetric orbifold. When this is not the case we get an asymmetric orbifold. In all cases the RNS fermions are assumed to be modded in the same way as their world-sheet superpartners in order to conserve world-sheet supersymmetry. In the case of symmetric orbifolds obtained by modding the $E_8 \times E_8$ string, we can write for the orbifold background

$$\mathcal{O} = R_6^{L+R}/S \otimes T_{8 \times 8}^L/G \quad (2.1)$$

where S is the space group of the orbifold, which includes the point group P (automorphisms of the six-dimensional lattice) plus shifts in R^6 : $S = \{(\theta, v^i), \theta \in P, i = 1, \dots, 6\}$. The shifts $(1, v^i)$ include the basis vectors of the six-dimensional lattice, and $T_{8 \times 8}^L$ denotes the $E_8 \times E_8$ root lattice torus which is in general modded by a ‘gauge-twisting group’ G , which again includes automorphisms of the $E_8 \times E_8$ lattice Θ , plus shifts: $G = \{(\Theta, v^I), I = 1, \dots, 16\}$. Orbifolds with non-Abelian point groups are called non-Abelian orbifolds. However, even if the point group P is Abelian, the space group S in general is not. Then if we embed the space group $(\theta, v^i) \rightarrow (\Theta, v^I)$ in a non-Abelian way, we obtain a non-Abelian action in the gauge degrees of freedom even if the point group is Abelian. This was discussed in Ref. [7] and corresponds to the simultaneous presence of twisting-automorphisms and Wilson-line backgrounds. In the present section we will confine ourselves to the effect of $\mathbb{Z}_N$ twists on compactified heterotic strings, and leave the case of additional background fields (Wilson lines and antisymmetric tensor fields) for the next sections.

Here we are interested in twistings θ which leave one unbroken space-time supersymmetry. There are in fact only 13 classes of tangent space $SO(6)$ automorphisms which leave invariant at least

one unbroken supersymmetry. These are the nine elements $\theta = \exp 2\pi i(aJ_{12} + bJ_{34} + cJ_{56})$ with a , b , and c shown in Table 1, where J_{ij} are the $SO(6)$ Cartan generators. To these we have to add the four automorphisms which leave two unbroken supersymmetries. These have $(a, b, c) = (0, 1/n, 1/n)$, where $n = 2, 3, 4, 6$. All these 13 $SO(6)$ elements may appear as diagonal generators of some non-Abelian orbifold, or may generate some Abelian orbifold. To specify the orbifold completely, we also have to choose a lattice in which the automorphism is acting. Different lattices may be chosen with the same point group, e.g. one can have $\mathbb{Z}_3$ acting on an $SU(3)^3$ or an E_6 lattice $\mathbb{Z}_3 [(SU(3)^3]$, $\mathbb{Z}_3 (E_6)$, or $\mathbb{Z}_4 [SO(4)^3]$, $\mathbb{Z}_4 [SU(4)^2]$, etc. Many properties of orbifolds do not depend on the specific lattice but only on the point group. Properties depending on the whole space group (e.g. the presence of Wilson lines) are, however, different.

We will be mainly concerned with six-dimensional Lie-algebra lattices, and the point group will be generated by automorphisms of these lattices. These automorphisms may be inner or outer. Inner automorphisms are given by the Weyl group of the algebra. The Weyl group is generated by elements Γ_i which reflect the vectors with respect to the plane perpendicular to the simple roots e_i :

$$\Gamma_i(x) = x - (e_i \cdot x) e_i \quad (2.2)$$

In general, not all the automorphisms of the six-dimensional lattices have to be in the Weyl group. For example, one may build a $\mathbb{Z}_4$ symmetric orbifold by modding the lattice $SO(4)^3$ by a simultaneous rotation of $\pi/2$ in the first two $SO(4)$'s and π in the third one. This automorphism is not in the Weyl group of $SO(4)^3$, but it leads to a perfectly consistent symmetric orbifold.

An especially interesting Weyl group automorphism is given by the Coxeter element

$$\omega = \Gamma_1 \Gamma_2 \dots \Gamma_6 \quad (2.3)$$

of the corresponding Lie algebra. Lie-algebra lattices modded by Coxeter elements and leading to supersymmetric theories are given in Ref. [8]. They are shown in the second column of Table 1. In the case of the $SO(8)$ lattices in the table, the second of the $\mathbb{Z}_8$ moddings and the $\mathbb{Z}_{12}$ one include outer automorphisms of the Dynkin diagram (of order 2 and 3, respectively) multiplying the Coxeter element.

The twisted strings centre around points of the six tori which are left fixed by the automorphisms of the orbifold. This is necessary for the strings to be consistent with the twisted boundary conditions. For a general symmetric modding $(\theta, v^i; \Theta, v^I)$, one has

$$\begin{aligned} X^i(\sigma=2\pi) &= \Theta^i_j X^j(\sigma=0) + v^i, \quad i=1, \dots, 6 \\ X^I(\sigma=2\pi) &= \Theta^I_J X^J(\sigma=0) + v^I, \quad I=1, \dots, 16 \end{aligned} \quad (2.4)$$

The number of fixed points under θ (modulo the lattice) gives us the degeneracy of the corresponding twisted vacuum and appears in the one-loop partition function as an overall factor, as we will see later on. The number of fixed points under an $SO(6)$ automorphism depends only on the automorphism (not on the specific lattice) and can be calculated using the Lefschetz fixed-point theorem

$$n_{fp} = \chi(\theta) = \det(1 - \theta) \quad (2.5)$$

where θ is written in the vector representation of $SO(6)$. If θ is the Coxeter element of the lattice, we have also

$$\chi(\omega) = |\wedge^W / \wedge_R| = |g| \quad (2.6)$$

where $|\wedge^W / \wedge_R|$ is the index of the root lattice R in the eight lattice W , and $|g|$ is the determinant of the Cartan matrix. Notice that this suffices to compute the fourth column of Table 1. In the case of the $SU(n)$ lattices, the Coxeter element acts in a particularly simple way on the simple roots e_i and the fundamental weights e^{*i} .

$$\begin{aligned} \omega e_i &= e_{i+1}, \quad i \neq r = n-1 \\ \omega e_r &= -e_1 - e_2 - \dots - e_r \end{aligned} \quad (2.7a)$$

$$\omega e^{*i} = e^{*i} - e_i - e_{i-1} - \dots - e_1 \quad (2.7b)$$

Because of the last property in Eq. (2.7b), all the fundamental weights correspond to fixed points (they are all inside a fundamental domain of the torus), and hence there are n fixed points (r weights plus the origin) left fixed by the Coxeter element of an $SU(n)$ algebra. For other groups, the fixed points also belong to the weight lattice; the number of them is in general given by the index $|\wedge^W / \wedge_R|$.

Up to now we have essentially been discussing the properties of the space group S of the orbifold. In general, S will be embedded in the gauge degrees of freedom $(\theta, v^I) \rightarrow (\Theta, v^I)$ by acting with automorphisms Θ and/or shifts v^I on the gauge lattice. In the absence of Wilson-line backgrounds, a Weyl automorphism Θ of the $E_8 \times E_8$ lattice can be equivalently realized through a shift. The converse is not true, and there are shifts which are not equivalent to any Weyl rotation. One can find the list of the 112 inner automorphisms of E_8 in Ref. [9]. If one acts on the gauge degrees of freedom in an $SU(3) \in E_8 \times E_8$ with precisely the same action as on the extra six compactified dimensions + RNS fermions, this is called standard embedding. This can always be done by acting with a shift $(\theta, 0) \rightarrow (1, v^I)$ in the gauge degrees of freedom. Sometimes the standard embedding may be also realized by Weyl rotations $(\theta, 0) \rightarrow (\Theta, 0)$. In the examples of Table 1, only for the $\mathbb{Z}_3$, $\mathbb{Z}_4$, and one of the $\mathbb{Z}_6$ orbifolds is it possible to realize the standard embedding through Weyl rotations. The extraction of the quantum numbers of the massless sector is usually more cumbersome when one realizes the twisting through Weyl rotations. This is why we will mostly consider the twist realization through shifts. Twisting by Weyl rotations is considered in some detail in Ref. [7].

In the standard embedding case there is a simple formula for computing the net number of chiral families (27's of E_6) which is given (as in standard embedding Calabi-Yau compactification) by one half the Euler characteristic χ ,

$$\chi = \frac{1}{N} \sum_{[h,g]=0} \chi(h, g) \quad (2.8)$$

where $\chi(h, g)$ is the number of points left simultaneously fixed by h and g . We will see explicitly how this formula can be understood by considering the one-loop partition function. We have computed the number of net 27's coming from each different twisted sector for the Coxeter orbifolds of Table 1. This is useful if we want to have some idea of what should be changed in the orbifold (e.g.

by making a non-standard embedding or adding Wilson lines) in order to get a more interesting model (e.g. with fewer generations). All examples in Table 1 have extra $(27 + \overline{27})$ pairs except for the $\mathbb{Z}_3$ and $\mathbb{Z}_7$ orbifolds. This is because they are of prime order and all twisted sectors have only fixed points (not fixed tori). The orbifolds of even order have fixed tori in some of the sectors; hence in those sectors they have two invariant space-time spinors with opposite helicity.

The Coxeter orbifolds of Table 1 are examples of orbifolds generated by a single element θ ($\theta^n = 1$). One can also obtain other Abelian orbifolds generated by the direct product of two Abelian symmetries, $\mathbb{Z}_N \times \mathbb{Z}_M$. In this case, apart from the automorphisms of the third column of Table 1, one should consider automorphisms leaving more than one unbroken supersymmetry [i.e. the moddings with $(a, b, c) = (0, 1/n, 1/n)$; $n = 2, 3, 4, 6$]. For example, one can build a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold by modding the lattice $SO(4)^3$ by reflections in the first two $SO(4)$'s [$\theta = (1/2, 1/2, 0)$] and reflections in the last two $SO(4)$'s [$\theta' = (0, 1/2, 1/2)$]. This leaves only one unbroken supersymmetry. If one takes the standard embedding case, one can use formula (2.8) to compute the number of chiral E_6 generations. There is no chiral matter in the untwisted sector since θ, θ' , and $\theta\theta'$ leave fixed tori [$\chi(1, g) = 0$]. From the twisted sector, one gets 48 generations. Notice that the modding in this orbifold could be understood in the fermionic language in terms of periodic and antiperiodic boundary conditions for both the RNS and the internal world-sheet fermions. A generalization of this type of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds to include arbitrary modular invariant backgrounds and asymmetric moddings should give rise to the models discussed in Ref. [5]. A $\mathbb{Z}_3 \times \mathbb{Z}_3$ [$SU(3)^3$] orbifold can also be constructed with 3 untwisted and 81 twisted generations in the standard embedding case^{*}. It should be noted that the number of families in all the Abelian orbifolds with the standard embedding mentioned above is always a multiple of 12 (χ is a multiple of 24).

3. PARTITION FUNCTIONS AND GSO PROJECTORS

In order to study the general properties of the spectrum of an orbifold, it is useful to consider the one-loop partition function. We can easily write down the partition function for an Abelian orbifold generated by a single $\mathbb{Z}_N$ twist (independent considerations of partition functions on orbifolds can be found in [11]). If we assume that we have a standard embedding with $(a, b, c) = (v_1, v_2, v_3)$ and $\tilde{v}^I = (v_1, v_2, v_3, 0 \dots 0)$, $\tilde{v}^{I'} = 0$, we get

$$Z = \frac{1}{N} \sum_{n,m} \eta(n,m) Z_{(\theta^n, \theta^m)} \quad (3.1)$$

where $\eta(n,m)$ are phases fixed by modular invariance requirements, and the contribution of a sector twisted by θ^n (θ^m) in the σ (τ) direction is given by

$$Z_{(\theta^n, \theta^m)} = \int \frac{d^2\tau}{\tau_2^2} \frac{1}{(2\pi\tau_2)} Z_{(\theta^n, \theta^m)}^{(\tau)} \bar{Z}_{(\theta^m, \theta^n)}^{(\bar{\tau})} \quad (3.2)$$

with $\tau = \tau_1 + i\tau_2$ being the one-loop modular parameter. The contribution of the left-movers ('gauge sector') is given by

^{*}) Notice that this spectrum is quite similar to the $N = 2$ superconformal example of Ref. [10].

$$Z_{(\Theta^m, \Theta^n)}(\tau) = \sum_{\alpha, \beta} \eta_{\alpha\beta} \frac{\vartheta[\frac{\alpha+m\nu_1}{\beta+n\nu_1}] \vartheta[\frac{\alpha+m\nu_2}{\beta+n\nu_2}] \vartheta[\frac{\alpha+m\nu_3}{\beta+n\nu_3}] \vartheta[\frac{\alpha}{\beta}]^5}{\vartheta[\frac{\frac{1}{2}+m\nu_1}{\frac{1}{2}+n\nu_1}] \vartheta[\frac{\frac{1}{2}+m\nu_2}{\frac{1}{2}+n\nu_2}] \vartheta[\frac{\frac{1}{2}+m\nu_3}{\frac{1}{2}+n\nu_3}] \eta^7} \times$$

$$\times C(m, n) \sum_{\delta, \epsilon} \eta_{\delta\epsilon} \frac{\vartheta[\frac{\delta}{\epsilon}]^8}{\eta^8} \quad (3.3a)$$

where the $\eta_{\alpha\beta}$'s are constant phases associated with the sum over spin structures. The functions ϑ and η are Jacobi and Dedekind functions. We specify our conventions and show some useful formulae in Appendix 1. The constants $C(m, n)$ are associated with the bosonic oscillators and are given by

$$C(m, n) = \prod_i (-2 \sin(\pi n \nu_i)) \quad \text{if } n \neq 0$$

$$\prod_i (-2 \sin(\pi m \nu_i)) \quad \text{if } n = 0 \quad (3.3b)$$

(the product runs over non-vanishing ν_i 's). The last factor in Eq. (3.3a) corresponds to the extra unmodded E'_8 . For the right-movers, we have

$$\bar{Z}_{(\Theta^m, \Theta^n)}(\bar{\tau}) = \sum_{\alpha, \beta} \bar{\eta}_{\alpha\beta} \frac{\bar{\vartheta}[\frac{\alpha+m\nu_1}{\beta+n\nu_1}] \bar{\vartheta}[\frac{\alpha+m\nu_2}{\beta+n\nu_2}] \bar{\vartheta}[\frac{\alpha+m\nu_3}{\beta+n\nu_3}] \bar{\vartheta}[\frac{\alpha}{\beta}]^5}{\bar{\vartheta}[\frac{\frac{1}{2}+m\nu_1}{\frac{1}{2}+n\nu_1}] \bar{\vartheta}[\frac{\frac{1}{2}+m\nu_2}{\frac{1}{2}+n\nu_2}] \bar{\vartheta}[\frac{\frac{1}{2}+m\nu_3}{\frac{1}{2}+n\nu_3}] \bar{\eta}^3} C^*(m, n) \quad (3.3c)$$

We have not bothered here about constant phases which cancel between left and right bosonic oscillator contributions. The generalization of the above formulae to non-standard embeddings and Wilson-line backgrounds is trivial (you just change the boundary conditions of the world-sheet left-moving fermions). The constant $|C(m, n)|^2$ is in general associated with the number of fixed points of the twisted sectors. In the case of the untwisted partition function this factor is cancelled by a similar factor appearing in the denominators of Eqs. (3.3a) and (3.3c) when we express the ϑ -functions in terms of products.

Here our interest on the orbifold partition functions is related to the generalized GSO projections present in this type of construction [12]. The sum over m and the factor $1/N$ in Eq. (3.1) reflect explicitly the existence of this projection on twist-invariant states. Let us consider, as a first example, the $\mathbb{Z}_3$ orbifold. One finds for the $(1, \theta)$ left-handed partition function $[(0, 0)$ spin structure] the result [recall we have $\nu = (1/3, 1/3, 2/3, \dots, 0)$]

$$Z_{(1, \Theta)}(\tau) = (-\sin \pi/3)^3 \frac{\vartheta[\frac{\tau}{3}]^2 \vartheta[\frac{\tau}{3}] \vartheta[\frac{\tau}{3}]^5}{\vartheta[\frac{\tau}{6}] \vartheta[\frac{\tau}{6}] \eta^3} Z(E'_8) \quad (3.4a)$$

$$= \frac{\prod_n (1+q^{n-1/2})^{10} (1+\lambda q^{n-1/2})^3 (1+\lambda^2 q^{n-1/2})^3}{q^{1/2} \prod_n (1-q^n)^2 (1-\lambda q^n)^3 (1-\lambda^2 q^n)^3} Z(E'_8) \quad (3.4b)$$

$$= \frac{\sum_P q^{P^2/2} e^{2\pi i P \cdot v}}{q^{1/2} \eta^4 \prod_n (1-q^n)^2 (1-\lambda q^n)^3 (1-\lambda^2 q^n)^3} Z(E'_8) \quad (3.4c)$$

where $q = \exp 2\pi i \tau$ and $\lambda = \exp 2\pi i/3$. The form in Eq. (3.4a) is most useful for performing modular transformations, the one in Eq. (3.4b) for seeing the spectrum in the fermionic formulation, and Eq. (3.4c) has a clear interpretation in terms of the bosonic construction. The latter is also more useful to check which massless states survive the generalized GSO projections. Thus if we compute the $(1, \theta)$ right-handed partition function [e.g. $(0,0)$ spin structure] we get essentially something similar to the conjugate of the first factor in Eq. (3.4c). Multiplying both partition functions and summing over $\theta = 1, \theta, \theta^2$ we will get a projector for massless states of the form $P_u = \frac{1}{3}(1 + \Delta + \Delta^2)$ with $\Delta = \exp 2\pi i(p \cdot v - r \cdot v)$. Thus for point-group singlets ($r \cdot v = \text{integer}$), we project on winding states with $p \cdot v = \text{integer}$ (in the bosonic language). This corresponds to the charged $E_6 \times SU(3)$ gauge bosons. The rest of the massless untwisted spectrum suffers the same projection. Something similar happens with the twisted sector. Thus, for example, we find a (θ, θ) partition function for the left-movers $[(0,0)$ spin structure]:

$$Z_{(\theta, \theta)} = \frac{3\sqrt{3}}{\eta^4} \frac{\sum_P q^{(P+v)^2/2} e^{2\pi i (P+v) \cdot v}}{\prod_n (1-q^n)^2 (1-q^{n-1/2} \lambda)^3 (1-q^{n-1/2} \lambda^2)^3} Z(E'_8) \quad (3.5)$$

After multiplying by the right-handed partition function, we have the overall factor twenty-seven corresponding to the number of fixed points. In the sector twisted by θ , we then get a projector on massless twisted states (with vanishing oscillator number) $P_\theta = \frac{1}{3}(1 + \Delta_\theta + \Delta_\theta^2)$ with $\Delta_\theta = \exp 2\pi i[(p+v)v - (r+v)v]$. This corresponds again to the Neveu-Schwarz right-moving contribution (similar results are obtained for the other spin-structures). We can check that the massless right-moving vacuum is a point-group singlet and hence that the GSO-like projection takes place on $(p+v)v = \text{integer}$ -twisted winding states. When oscillators are present, an expansion of the oscillator contributions to the partition function [e.g. denominator in Eq. (3.5)] changes the projector by adding a phase factor to Δ_θ . In the case of the $\mathbb{Z}_3$ orbifold, we have massless 27 's of E_6 for no oscillators ($N_L = 0$). The mass-shell condition is $(p+v)^2 = \frac{4}{3}$ and this guarantees that these states survive projection since $2(p+v)v = (p+v)^2 + \frac{2}{3}$ and $\Delta_\theta = 1$. For $N_L = \frac{1}{3}$ we have massless $\bar{3}$'s. In this case the projector on massless states is modified to $\frac{1}{3}(1 + \alpha\Delta_\theta + \alpha^2\Delta_\theta)$ and the massless condition is $(p+v)^2 = \frac{2}{3}$. Again, for the same reason, these states survive projection. In fact this is a general feature of the $\mathbb{Z}_3$ orbifold: all massless twisted states survive the generalized GSO-like

projection (this is also true for non-canonical embeddings and Wilson-line backgrounds). The same happens also with the other prime-order Abelian orbifold, the $\mathbb{Z}_7$ model.

In the case of $\mathbb{Z}_N$ orbifold where N is not a prime number, the projector on massless states is non-trivial and there are massless twisted states which are eliminated by the projector. This is due to the fact that there are several disjoint orbits of boundary conditions which are not connected by one-loop modular transformations. Let us take the $\mathbb{Z}_4$ orbifold as example (see Appendix 2 for more details). The $(\theta^2, 1)$, (θ^2, θ^2) , and $(1, \theta^2)$ sectors are connected by one-loop modular transformations but they are unrelated to the other sectors. The relative phases $\eta(m,n)$ of the partition function of the separate sectors are determined by the correct projection in the untwisted sector. Thus for $\mathbb{Z}_N$ orbifolds with non-prime N , level matching by itself is not enough to obtain the correct spectrum. Unlike in the case of prime N , a further non-trivial projection appears, which in general also eliminates some massless states. Take, for example, the (θ^2, θ^m) sectors of the $\mathbb{Z}_4$ orbifold. The left-moving partition functions are given by $[(0,0)$ spin structure]

$$Z_{(\theta^2, 1)} = \frac{4 \sum_p q^{(p+2v)^2/2}}{q^{1/2} \eta^4 \prod_n (1-q^n)^4 (1-q^{n-1/2})^4} Z(E'_8) \quad (3.6a)$$

$$Z_{(\theta^2, \theta)} = \frac{2 \sum_p q^{(p+2v)^2/2} e^{2\pi i (p+2v)v}}{q^{1/2} \eta^4 \prod_n (1-q^n)^2 (1-q^n \lambda)^2 (1-q^{n-1/2} \lambda)^2 (1-q^{n-1/2} \lambda^3)^2} Z(E'_8) \quad (3.6b)$$

$$Z_{(\theta^2, \theta^3)} = \frac{4 \sum_p q^{(p+2v)^2/2} e^{2\pi i (p+2v)2v}}{q^{1/2} \eta^4 \prod_n (1-q^n)^4 (1-q^{n-1/2})^4} Z(E'_8) \quad (3.6c)$$

$$Z_{(\theta^2, \theta^2)} = \frac{2 \sum_p q^{(p+2v)^2/2} e^{2\pi i (p+2v)3v}}{q^{1/2} \prod_n (1-q^n)^2 (1-q^n \lambda)^2 (1-q^{n-1/2} \lambda^3)^2 (1-q^{n-1/2} \lambda)^2 \eta^4} Z(E'_8) \quad (3.6d)$$

where $\lambda^4 = 1$ and $v = (1/4, 1/2, 1/2, \dots)$ in the present case. If we multiply these partition functions by their right-moving counterparts (e.g. in the Neveu-Schwarz sector) we get a projector for massless states (for no oscillators acting) of the type

$$P_{\theta^2} = \frac{1}{4} [16 + 4\Delta_{\theta^2} + 16\Delta_{\theta^2}^2 + 4\Delta_{\theta^2}^3] \quad (3.7)$$

where $\Delta_{\theta^2} = \exp 2\pi i[(p+2v)v - (r+2v)v]$ in this case. One can check that there is a twofold right-moving vacuum (corresponding to two opposite 'helicities') with $\exp 2\pi i(r+2v)v = \pm i$. On the other hand, the masslessness condition in the left-handed sector gives massless 27 's ($\overline{27}$'s) for $(p+2v)v = 1/4$ ($-1/4$). Thus we have $\Delta_{\theta^2} = 1$ (-1) for massless 27 's ($\overline{27}$). The overall result is that 10 27 's and 6 $\overline{27}$'s survive out of the original massless 16 ($27 + \overline{27}$) pairs. There will be additional singlets coming from oscillator states. The projectors are trivially modified for the presence of oscillators, as we mentioned for the $\mathbb{Z}_3$ orbifold case.

The above discussion can be easily generalized to an arbitrary $\mathbb{Z}_N$ orbifold. Thus we can write down a general projector for massless (non-oscillator) states of the form

$$P_{\theta^m} = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{\chi}(\theta^m, \theta^n) \Delta_{\theta^m}^n \quad (3.8a)$$

where, for example in the case of periodic boundary conditions, we have

$$\Delta_{\theta^m} = \exp 2\pi i [(p+mv) - (r+mv)] v \quad (3.8b)$$

(we get the same projection for the rest of the spin structures). The coefficients $\tilde{\chi}(\theta^m, \theta^n)$ are constants given by

$$\begin{aligned} \tilde{\chi}(1, \theta^n) &= 1 \\ \tilde{\chi}(\theta^m, \theta^n) &= \chi(\theta^m, \theta^n) \quad \text{for } \chi(\theta^m) \neq 0 \\ &\quad \chi(\theta^m, \theta^n) / 4 \sin^2(n v_j \pi) \quad n \neq 0, m \\ &\quad \prod_i 4 \sin^2(m v_i \pi) \quad n = 0, m \quad \text{for } \chi(\theta^m) = 0 \end{aligned} \quad (3.8c)$$

where i runs only over fermions which are twisted in the σ -direction [there are only two of them if $\chi(\theta^m) = 0$]. The index j refers to the particular fermion which is not twisted in the σ direction. Here χ gives the Euler characteristic of the corresponding set of points left fixed by the argument of χ . This result can be easily obtained from Eqs. (3.3a-c).

Up to now we have mostly been discussing the standard embedding. In this case it turns out that we can choose all the phases in Eq. (3.1), $\eta(m, n) = 1$. This is not the case for a non-standard embedding. Let us assume that our orbifold is twisted by boundary conditions given by $v = (v_1, v_2, v_3)$ in the point-group sector, but it is twisted by $\tilde{v} = (\tilde{v}_1, \dots, \tilde{v}_{16})$ in the gauge degrees of freedom. Then modular transformations starting from any sector of the orbifold give relative phase factors between the different partition functions. All the discussion about the above-mentioned projections can easily be generalized to the case of non-standard embeddings. The only difference is that some of the E_8' world-sheet fermions may in general be twisted, but the point-group structure remains. Then we can easily convince ourselves that the projector in Eqs. (3.8) becomes slightly modified by a phase multiplying $[\Delta_{\theta^m}]^n$:

$$P_{\theta^m}(ns) = \frac{1}{N} \sum_{n=0}^{N-1} e^{-i\pi mn(\tilde{v}^2 - v^2)} \tilde{\chi}(\theta^m, \theta^n) \Delta_{\theta^m}^n \quad (3.9)$$

Of course, in the standard embedding case ($\tilde{v}^2 = v^2$), we recover Eqs. (3.8). A projector of this type is also obtained in the presence of Wilson lines for the twisted sectors, as we will discuss in the next section. Let us finally recall that the projector in Eqs. (3.8) and (3.9) refers to the case with no bosonic oscillator acting on the ground states (although the case with oscillators present is also straightforward). These are, in fact, the most interesting states, leading to chiral families in the specific examples. Armed with the above projectors, we can easily compute the physical massless spectrum of any Abelian $\mathbb{Z}_N$ orbifold with arbitrary embedding and/or Wilson lines. In Appendix 2 we discuss examples of $\mathbb{Z}_4$ and $\mathbb{Z}_6$ to see how we can easily (in the case of standard embedding) compute the number of families without the need to compute the projectors explicitly. The latter are, however, needed in more general cases.

Performing a $\tau \rightarrow \tau + N$ transformation to the $(\theta, 1)$ partition function, we can obtain the modular invariance constraints by requiring this part of the partition function to be invariant. We find the following constraints [13] on v and $\tilde{v}$:

$$\begin{aligned} N(\tilde{v}^2 - v^2) &= 0 \pmod{2} \\ N \sum v_i &= 0 \pmod{2} \end{aligned} \tag{3.10}$$

Note that these two conditions imply $N \sum \tilde{v}_i = 0 \pmod{2}$ also. For the $\mathbb{Z}_N$ orbifolds we are discussing, there are no further constraints coming from higher-loop modular invariance [14].

4. BACKGROUND FIELDS IN SYMMETRIC AND ASYMMETRIC ORBIFOLDS

The effects of constant background fields g_{mn} , A_m , and B_{mn} ($m, n = 1, \dots, 6$) on toroidal compactifications of the heterotic string were considered in Ref. [2]. These fields parametrize a Lorentzian even self-dual lattice of momenta $\Lambda_{22,6}$, generating a very large class of four-dimensional string models. We can perform a modding-out action on the torus in the presence of the background fields in order to obtain an orbifold and generate more interesting four-dimensional models. The consideration of the Wilson lines can be done in two equivalent ways:

- i) We start with a simple torus compactification and embed the whole space group S in the gauge degrees of freedom. In particular, to the six-dimensional translations $(1, e_\alpha)$ we assign shifts in the group lattice given by $(1, a_\alpha^I)$. Since the shifts e_α describe the non-contractible loops of the original torus, we assign a gauge transformation, defined by a_α^I , to every non-contractible loop so that the a_α^I 's represent a set of Wilson lines $A_\mu^I = a_\alpha^I e_\mu^\alpha$ in the torus defining the orbifold.
- ii) We use the formalism of Ref. [4] and twist the lattice of momenta $\Lambda_{22,6}$, which already includes the Wilson lines.

The consideration of the Wilson lines in an orbifold depends crucially on the embedding of the point group in the gauge degrees of freedom. This can be done by either shifts v^I or inner automorphisms Θ on the group lattice. In the absence of Wilson lines, the latter embedding is already included in the former one, since any inner automorphism is equivalent to a shift in the lattice, but not vice versa. When Wilson lines a_α^I are included, both embeddings are in general not equivalent; this is due to the fact that the shifts a_α^I commute with v^I but not with Θ . We will proceed to review these two embeddings in formalism (i).

4.1 Space-group embedding formulation

4.1.1 Abelian embeddings

For a given $\mathbb{Z}_N$ orbifold we will consider first the space group embedding given by $(\theta, e_\alpha^I) \rightarrow (v^I, a_\alpha^I)$. The untwisted partition function will have, apart from the usual projector $(1/N)(1 + \Delta + \Delta^2 + \dots + \Delta^{N-1})$, $\Delta = e^{2\pi i(p \cdot \tilde{v} - r \cdot v)}$, one projector for every Wilson line a_α , given by $(1/N)(1 + \Delta_\alpha + \Delta_\alpha^2 + \dots + \Delta_\alpha^{N-1})$, where $\Delta_\alpha \equiv e^{2\pi i p \cdot a_\alpha}$. This will project out every massless state which does not satisfy

$$p \cdot a_\alpha = n_\alpha \in \mathbb{Z} \tag{4.1}$$

This partition function will also contain contributions from winding states [2], which are relevant for the massless spectrum only for models with enhanced symmetry. The twisted partition function will be split into different sectors according to different shifts of the momenta $p^I + v^I + m_\alpha a_\alpha^I$, $m_\alpha = 0, 1, \dots, N-1$; the massless formulae for these sectors are then

$$\frac{1}{2} (p + v + m_\alpha a_\alpha)^2 + N_L - 1 + \Delta c_\theta = 0 \quad (4.2)$$

where Δc_θ is the modded oscillators' contribution to the zero point energy and is given by

$$\Delta c_\theta = \frac{1}{4} \sum_k \eta_k (1 - \eta_k) \quad (4.3)$$

where η_k is one over the order of the twist for the k^{th} oscillator. Note that this factor is generally different for different twisted sectors.

Therefore we can see that the main effects of the Wilson lines are to project out states of the untwisted sector breaking the gauge group and reducing the matter content, and to differentiate between all the twisted sectors, thus affecting the whole spectrum of the theory.

Consistency requirements impose the following constraints on the Wilson lines:

- i) $(\theta, e)^N = 1$ implies that $Nv + Na \in \Lambda_{16}$ or $Na \in \Lambda_{16}$,
therefore the Wilson lines are also quantized;
- ii) $(\theta e)_j = M_{ji} e_i$ imply that $a_j = M_{ji} a_i + L_j$, $M_{ji} \in \mathbb{Z}$, $L_j \in \Lambda_{16}$
therefore not all of the six Wilson lines are independent;
- iii) modular invariance: since condition (3.10) is a constraint on the vector that shifts the momenta in the twisted sectors, its straight generalization in the presence of Wilson lines is

$$N[(n\tilde{v} + m_\alpha a_\alpha)^2 - n^2 v^2] = 0 \pmod{2}, n=0,1 \quad (4.4)$$

This can be seen as a particular case of non-standard embeddings, but a different one on every twisted sector. Therefore the projector found in Section 3 works equally well when Wilson lines are included.

It can be easily seen that condition (i) is already included in (ii).

4.1.2 Non-Abelian embeddings

Let us now consider the embedding $(\theta, e) \rightarrow (\Theta, a)$, where Θ is an automorphism of the group lattice. The main difference with the previous embedding is that (Θ, a) is in general a non-Abelian group, which allows the possibility of reducing the rank of the gauge group. The Cartan subalgebra states $\alpha^I |0\rangle_L$ of the heterotic string are rotated away by Θ . The invariant states in the untwisted sector, as well as the unrotated ones, are combinations of the type

$$|p\rangle + |\Theta p\rangle + \dots + |\Theta^{N-1} p\rangle \quad (4.5)$$

Some of them play the role of the Cartan subalgebra. As in sub-subsection 4.1.1, we have to project out the states which do not satisfy $p \cdot a_\alpha = n_\alpha \in \mathbb{Z}$. The matter states of the untwisted sector are combinations similar to (4.5), transforming non-trivially under Θ (as $|p\rangle + \omega |\Theta p\rangle + \dots + \omega^{N-1} |\Theta^{N-1} p\rangle$, $\omega^N = 1$).

The conditions for massless states in the twisted sectors are

$$\frac{1}{2} (p_T + m_\alpha a_{\alpha T})^2 + N_L - 1 + \Delta c_\theta + \Delta c_\Theta = 0 \quad (4.6)$$

where Δc_θ and Δc_Θ are the contributions to the zero point energy of the oscillators modded by θ and Θ , respectively; p_T and $a_{\alpha T}$ are the components of p and a_α which are invariant under Θ . This is because there are no winding vectors in the rotated directions. There is an extra degeneracy factor in the twisted sectors given by the square root of the number of fixed points under Θ [4, 15] which in

general belong to the unbroken group weight-lattice and therefore transform non-trivially under it. Level matching restricts the twist Θ for an even orbifold by [4] $L\Theta^{N/2}L = 0 \bmod 2$ for any $L \in \Lambda_{16}$. Note that the components of a_α which are rotated by Θ are completely arbitrary because the constraint $(\Theta, a)^N = 1$ is automatically satisfied, and because modular invariance constrains the vectors that shift the lattice, which are absent in this case. The components $a_{\alpha T}$, however, satisfy all the constraints listed in the previous subsection. The arbitrariness of the rotated components of a_α allows us to vary them continuously and have a smooth symmetry-breaking of the gauge group.

4.2 Twisted lattice formulation

To discuss the role of the background fields in the orbifold case, it is useful to remember their role in toroidal compactifications of the heterotic string^{*)}. There, the Wilson lines a_i and antisymmetric tensor fields b_{ij} parametrize the (22,6) Lorentzian even self-dual lattice of momenta $\Lambda_{22,6}$. The basis vectors for this lattice are

$$\begin{aligned} k^i &= (0, \frac{1}{2} e^{*i}, \frac{1}{2} e^{*i}) \\ \bar{k}_i &= (a_i^T, -b_{ji} e^{*j} - \frac{1}{4} a_j^K a_i^K e^{*j} + e_i, -b_{ji} e^{*j} - \frac{1}{4} a_j^K a_i^K e^{*j} - e_i) \\ \ell_\alpha &= (f_\alpha^T, -\frac{1}{2} f_\alpha^K a_i^K e^{*i}, -\frac{1}{2} f_\alpha^K a_i^K e^{*i}) \end{aligned} \quad (4.7)$$

where e_i and e^{*i} generate the lattice of the six-dimensional torus and its dual, respectively. The vectors f_α^I form a basis for the gauge-group lattice Λ_{16} .

Let us consider a $\mathbb{Z}_N$ twist θ on the six-dimensional part of the lattice $\Lambda_{22,6}$, and a shift v^I on Λ_{16} . If the twist θ treats the left- and right-moving momenta symmetrically, we must obtain the same type of models as considered in subsection 4.1, since the Wilson lines are already included in this formalism. Following Ref. [4] we define the invariant lattice I as the set of vectors in $\Lambda_{22,6}$ which are left invariant under the action of θ . In the case under consideration, it is easy to see that $I = \{L \in \Lambda_{16}, L \cdot a_i = m_i \in \mathbb{Z}\}$. Its dual is $I^* = \{L + n_i a_i, L \in \Lambda_{16}, n_i \in \mathbb{Z}\}$.

We know that $(\theta, \tilde{v})^N$ must leave invariant any vector of the lattice; this implies $N\tilde{v} \in I \subset \Lambda_{16}$. Also, since θ is required to be an automorphism of the whole lattice, i.e. $\theta\Lambda_{22,6} = \Lambda_{22,6}$, it will impose conditions on the allowed values of the parameters of the lattice. Therefore we take every basis vector k^i , $\bar{k}_i$, and ℓ_α , act with θ on their six-dimensional part (left and right), and express the result as a linear combination of them with integer coefficients. We obtain the following conditions,

$$\begin{aligned} \theta_i^j e_j &= N_i^j e_j & N_i^j &\in \mathbb{Z} \\ \theta^{*i}_j e^{*j} &= M^i_j e^{*j} & M^i_j &\in \mathbb{Z} \end{aligned} \quad (4.8)$$

where θ_i^j and θ^{*i}_j are the representations of θ in the bases e_i and e^{*i} , respectively. These equations only tell us that θ is also an automorphism of the lattice defining the six-dimensional torus. For the Wilson lines and antisymmetric tensor, we have

$$a_i = N_i^j a_j + L_i^\alpha f_\alpha \quad L_i^\alpha \in \mathbb{Z} \quad (4.9)$$

*) We acknowledge L. Dixon and K.S. Narain for helping us to understand this point.

$$-2(b\theta^* - \theta b)_{ik} + \frac{1}{2}[a \cdot a(1 - \theta^*) + (1 - \theta)a \cdot a]_{ik} = \tilde{M}_{ik} \in \mathbb{Z} \quad (4.10)$$

$$f_\alpha \cdot a_i (\delta_j^i - \theta^{*i}{}_j) \in \mathbb{Z} \quad (4.11)$$

Equation (4.9) is equivalent to condition (ii) derived in sub-subsection 4.1.1 using the space-group embedding formulation. To obtain information from Eqs. (4.10) and (4.11), we consider them for any power of θ , θ^n , with $n < N$, and add them up. Using the identity $1 + \theta + \theta^2 + \dots + \theta^{N-1} = 0$, we find that they imply the following simpler conditions for the Wilson lines,

$$N a_i^I = L_i^I \quad L_i^I \in \Lambda_{16} \quad (4.12)$$

$$N a_i \cdot a_j = 0 \pmod{1} \quad (4.13)$$

implying that $N a_i \in I$. Since $N v$ also belongs to the invariant lattice, its scalar product with any vector of I^* is an integer; therefore

$$N \tilde{v} \cdot a_i = 0 \pmod{1} \quad (4.14)$$

These conditions, together with the modular invariance constraints [4]

$$N(\tilde{v}^2 - v^2) = 0 \pmod{2} \quad ; \quad L \Theta^{N/2} L = 0 \pmod{2}, N \text{ even} \quad (4.15)$$

are equivalent to the rest of the constraints found in subsection 4.1. It is interesting to see how some conditions come from modular invariance in one formalism but from the group law in the other one. Notice that the lattice $\Lambda_{22,6}$ has information about modular invariance in the toroidal case.

In the present formalism, the massless spectrum in the untwisted sector is found as in the case without Wilson lines, but the momenta have to belong to the invariant lattice, hence the condition $p \cdot a = m \in \mathbb{Z}$. For the twisted sectors the momenta belong to the shifted lattice $I^* + v$, which in this case corresponds to $\Lambda_{16} + v + n_i a_i$, as found before. The degeneracy factor for the twisted sectors is given by [4]

$$D = \sqrt{\frac{\det' (1 - \Theta)}{|I^*/I|}} \quad (4.16)$$

Note that for a Wilson line of order m , we have $|I^*/I| = m^2$, thus explaining in this way the splitting of the twisted sectors with their respective multiplicity factor found in subsection 4.1.

Finally, note that Eqs. (4.10) tell us that not only the Wilson lines but also the antisymmetric tensor fields are quantized. We generally take $b_{ij} = 0$, since this does not affect the massless spectrum. We can also consider other values for b_{ij} ; in particular, fixing the radius of the torus simultaneously, they can be used to enhance the symmetry to a rank-22 group, as in the case of toroidal compactifications. A similar analysis can be performed for the non-Abelian embeddings. In this case there are no shifts, the embedding Θ is restricted to be an automorphism of $E_8 \times E_8$ of the same order as θ , and, since the total lattice is not a group lattice, Θ is not equivalent to a shift, the Wilson lines rotated by Θ are arbitrary, and the rank can be reduced.

4.3 Wilson lines on asymmetric orbifolds

In the previous sections we have seen how the consideration of Wilson lines enriches enormously the structure of possible four-dimensional string models obtained from symmetric orbifolds. Using the formalism of the previous subsection, we will now consider the possibility of turning on Wilson lines when we perform independent twists on the left- and right-moving momenta, θ_L and θ_R [4].

We will restrict ourselves to the case where the toroidal compactification has been done in $1/2\Lambda_R$, where Λ_R is the root lattice of a simply laced Lie group of rank 6. The b_{ij} field is chosen to be $-1/4g_{ij}$, $i > j$; $1/4g_{ij}$, $i < j$, where g_{ij} is the Cartan matrix. The symmetry is enhanced to a rank-22 gauge group. The six-dimensional left- and right-moving momenta p_L and p_R belong to the weight lattice Λ_W , and $p_L - p_R \in \Lambda_R$ [1]. The basis vectors can now be written as

$$\begin{aligned} \kappa^i &= (0, e^{*i}, e^{*i}) \\ \bar{\kappa}_i &= (a_i^T, \tilde{c}_{ij} e^{*j} + e_i, \tilde{c}_{ij} e^{*j}) \\ l_\alpha &= (f_\alpha^T, -f_\alpha \cdot a_i e^{*i}, -f_\alpha \cdot a_i e^{*i}) \end{aligned} \quad (4.17)$$

where e_i and e^{*i} generate the root lattice and the weight lattice, respectively, and $\tilde{c}_{ij} = -1/2a_i \cdot a_j + c_{ij}$, where $c_{ij} = 2b_{ij} - 1/2g_{ij} \in \mathbb{Z}$.

The constraints on the existence of asymmetric orbifolds are much more restrictive than in the symmetric case. As in the previous subsection we will check consistency by requiring the twists (θ_L, θ_R) to be automorphisms of the lattice $\Lambda_{22,6}$.

The twists θ_L and θ_R are restricted to satisfy

$$(\theta_L^* - \theta_R^*)^i_j e^{*j} = N^{ij} e_j \quad N^{ij} \in \mathbb{Z} \quad (4.18)$$

$$(\theta_L)_i^j e_j = \tilde{N}_i^j e_j - \tilde{c}_{ij} N^{jk} e_k \quad \tilde{N}_i^j \in \mathbb{Z} \quad (4.19)$$

For the Wilson lines we have the following conditions:

$$a_i = \tilde{N}_i^j a_j + \tilde{L}_{i\alpha} f_\alpha \quad (4.20)$$

$$\tilde{c}_{ij} (\theta_L + \theta_R)_k^j = 2M_{ik} - 2L_{\alpha i} f_\alpha \cdot a_k + 2\tilde{N}_i^j \tilde{c}_{jk} \quad (4.21)$$

$$2f_\alpha \cdot a_i \left[1 + \frac{1}{2}(\theta_L + \theta_R)_k^j\right]_k^i = 2\tilde{M}_{\alpha k} + 2\tilde{N}_\alpha^i (a_i \cdot a_k + \tilde{c}_{ik}) \quad \tilde{M}_{\alpha k}, \tilde{N}_\alpha^i \in \mathbb{Z} \quad (4.22)$$

$$-(f_\alpha \cdot a_i) N^{ik} = \tilde{N}_\alpha^k \quad \forall \alpha \quad (4.23)$$

Equations (4.20) are the analogues of Eqs. (4.9) and tell us the number of independent Wilson lines we can have. From conditions (4.21) and (4.22) we can extract information in a similar way to that of the last subsection. We consider them for any power n of θ_L and θ_R and sum from $n = 1$ to $n = \Omega$, where Ω is the minimum common factor of the orders of θ_L and θ_R . We then obtain

$$\Omega a_i = L_i \in \Lambda_{16} \quad (4.24)$$

$$\Omega a_i \cdot a_j \in \mathbb{Z} \quad (4.25)$$

So far we have not encountered any fundamental difference between the symmetric and the asymmetric cases. In fact the asymmetric conditions reduce to the symmetric ones for $\theta_L = \theta_R$ as they should. However, we have not used the last constraint (4.23), which restricts the possible Wilson lines. If $f_\alpha \cdot a_i \in \mathbb{Z}$, this equation is satisfied automatically. The question is, Is it possible to satisfy Eq. (4.23) with $f_\alpha \cdot a_i \notin \mathbb{Z}$? — otherwise a_i are just trivial. In order to answer this question, we have to consider two possibilities: i) N^{ik} is an invertible matrix, and ii) N^{ik} is not invertible. If N^{ik} is invertible, we can solve for $f \cdot a_i$ to get

$$\begin{aligned} f_\alpha \cdot a_i &= -\tilde{N}_\alpha{}^\kappa (N^{-1})_{\kappa i} \\ &= -\tilde{N}_\alpha{}^\kappa g_{\kappa j} (\theta_L - \theta_R)_\kappa{}^j g_{ji} \end{aligned} \quad (4.26)$$

Since $N^{ik} \in \mathbb{Z}$, a necessary condition for $f_\alpha \cdot a_i$ to be an integer is that $(\det N)^{-1} \in \mathbb{Z}$ or

$$\det g / \det (\theta_L - \theta_R) \in \mathbb{Z} \quad (4.27)$$

If N^{ik} is not invertible, there are several invariant directions, for which $\theta_L = \theta_R$. In these directions, $f_\alpha \cdot a_i$ is not restricted by Eq. (4.23) and the Wilson lines a_i could be introduced, satisfying only the same constraints as in the symmetric case. For the non-invariant directions, we can again solve for $f_\alpha \cdot a_i$ as we did before.

An indirect criterion which we can use when it is possible to add Wilson lines in an orbifold is the following. The degeneracy factor in an asymmetric orbifold (assuming there are no invariant directions) is given by $D = \sqrt{\det (1 - \theta) / |I^* / I|}$. If the invariant lattice I is known in the absence of Wilson lines, then adding a Wilson line of order $m \in \mathbb{Z}$ [i.e. $a = (1/m)L$, $L \in \Lambda_{16}$] changes the invariant lattice to $I_N = \{L \in I; L \cdot a \in \mathbb{Z}\}$ and the dual is $I_N^* = I^* + na$.

This changes the index

$$|I_N^* / I_N| = m^2 |I^* / I|$$

and the degeneracy factor is reduced to

$$D_N = \frac{1}{m} D$$

Therefore, if D is not a multiple of m , it is impossible to add Wilson lines.

Let us consider the case where $\theta_L = 1$ and θ_R is given by the Coxeter element in any of the lattices of Table 1. Equation (4.27) is in this case

$$\det g / \det (1 - \theta) \in \mathbb{Z} \quad (4.28)$$

From volume considerations ($|\Lambda_R| \approx 1/v \approx 1/\sqrt{g}$) we know that $\det g = |\Lambda_W / \Lambda_R| = |I^* / I|$. Thus formula (4.28) is equal to D^{-1} , which in all these cases can be easily evaluated to be $D = 1$.

Therefore from both arguments we can see that it is impossible to add Wilson lines to all these asymmetric orbifolds. We will consider a positive example in the next section.

5. EXPLICIT MODELS

We will now turn to the construction of explicit models using the formalism developed in the previous subsections. As we can see, there is an enormous amount of four-dimensional string models that can be constructed in this way with, say, $N = 1$ supersymmetry. From each of the nine possible twists listed in Table 1, we can select several lattices where they can act. For each of them we can make different embeddings of the point group in the gauge degrees of freedom. For each embedding we can have several independent Wilson lines; the amount of models obtained by adding each Wilson line grows geometrically. For instance, in the $\mathbb{Z}_3$ orbifold there are four different embeddings. Adding one Wilson line, the total number of different models is $\geq 10^3$, as we will see later. Adding a second and third Wilson line increases the number to an estimate of $\geq 10^9$ different models. Wilson lines also make an explicit difference between lattices with the same point group, which would lead to equivalent models if they are not present. To this list we have to add all the models constructed by non-Abelian embeddings and by left-right asymmetric twists, and also the ones with a $\mathbb{Z}_n \times \mathbb{Z}_m$ twist, which are supposed to include the models of Refs. [5]. Note that for symmetric twists, the radius of the orbifold can be varied continuously and, in particular, chosen to enhance the gauge symmetry, for a particular value of the b_{ij} field.

A general classification of all these models is extremely difficult and, in general, not very illuminating. Rather than do this, we have explored different sources of models that are probably interesting from the phenomenological point of view. This can be done if we have some idea of what can be expected from a particular construction. We have also tried to look for properties which are as model-independent as possible. In this section we present the results of our investigation without claiming to have complete phenomenological success, but with the idea of providing interesting examples of how to construct models using the present formalism.

5.1 Wilson lines on symmetric $\mathbb{Z}_N$ orbifolds

In the foregoing text we have seen that the introduction of Wilson lines into the Abelian embedding affects the spectrum of a given model in three major ways: The Wilson lines break the gauge group; they can kill part or all the matter in the untwisted sector; and they split the twisted sectors, changing their matter content. We can use these properties to look for interesting models, starting from any of the $\mathbb{Z}_N$ orbifolds listed in Table 1. If we use the number of generations (three) as a guide to eventually interesting models, the best candidate is the $\mathbb{Z}_3$ orbifold [with the $SU(3)^3$ or any other allowed lattice] because there is an overall factor of 3 coming from the right-moving part of the untwisted sector matter. This is due to the fact that we twist the three complex coordinates by the same angle, and also, in a less relevant fashion, that the degeneracy factor in the twisted sectors is one or a multiple of three. Nevertheless, looking at the table we can see that there are some orbifolds with three generations in the untwisted sector, and a non-standard embedding and/or addition of Wilson lines can lead to three family models. This is, however, more difficult to control than in the $\mathbb{Z}_3$ orbifold. In the following sub-subsections we will present several examples of $\mathbb{Z}_N$ orbifolds with non-standard embeddings and/or Wilson lines.

5.1.1 General classification of $\mathbb{Z}_3$ models with one Wilson line

As it is well known, there are four non-trivial embeddings that can be done in the $\mathbb{Z}_3$ orbifold, and to each of these models we can add one, two, or three Wilson lines. We have made a general computer classification of all the one-loop modular-invariant models that can be obtained from the $\mathbb{Z}_3$ orbifold by adding one Wilson line (and arbitrary embeddings turned on). Some of the results are

shown in Table 2. Because of space limitations, we present only that part of a model which comes from one E_8 sector of the theory, and only the untwisted matter. A complete model will have a gauge group which will be the direct product of two of the possible subgroups shown in the table. Notice that for a given gauge group there are several possible untwisted matter contents, depending on the specific example. Furthermore, even if two models have the same gauge group and untwisted-matter content in a given E_8 sector, they may have different twisted-matter content depending on the specific action of v^I and a^I in the second E_8 . As a whole, in this way we should obtain $\sim 10^3$ different models.

Table 2 shows that we can obtain up to 18 rank-8 subgroups of E_8 , but there are many E_8 subgroups which do not appear. Owing to the fact that both twists and Wilson lines are of order 3, the number of charged generators and matter fields is always a multiple of three. This is a generic feature in the $\mathbb{Z}_3$ orbifold which makes it quite natural to obtain three-generation models. Each model shown in the table has (when chiral) a number of generations which is a multiple of three. In fact, in Table 2 you can see that there are three-generation models with gauge groups $SU(5) \times SU(2)^2 \times U(1)^2$, $SO(10) \times SU(2) \times U(1)^2$, $SU(7) \times U(1)^2$, $SU(6) \times SU(3) \times U(1)$, and $E_8 \times U(1)^2$. Thus all the most interesting grand unifying groups can be obtained with three generations of quarks and leptons. The smallest subgroup one can obtain from a given E_8 is $SU(3)^3 \times U(1)^2$. In order to get even smaller groups such as $SU(3) \times SU(2) \times U(1)^5$, we would have to add one or two Wilson lines. We can use the examples in Table 2 as the starting point to obtain models with a second Wilson line and smaller gauge groups, as we will show below. Let us finally comment that the specific shifts $\tilde{v}^I$ and Wilson lines a^I shown in the table are just examples; there are many equivalent a 's and v 's leading to the same given model.

5.1.2 $SU(3) \times SU(2) \times U(1)^n$ models with three generations

The total number of models obtained for the Abelian embedding of the $\mathbb{Z}_3$ orbifold with one Wilson line is of the order of 10^2 – 10^3 . The addition of one or two extra Wilson lines will give rise to an enormous number of models ($\geq 10^6$ – 10^9) for this particular orbifold. Rather than make a complete classification, we can impose further physical constraints on those already imposed ($D = 4$, $N = 1$ supergravity). We have explored the possibility of obtaining models with three generations and gauge group $SU(3) \times SU(2) \times U(1)^n$. Note that since this mechanism does not reduce the rank, we cannot avoid the extra $U(1)$'s. The result of the investigation is positive. In fact we can impose more physical constraints, such as the absence of extra colour triplets which are dangerous for proton decay. We can follow a systematic procedure to obtain these models, and this we will now explain.

Imposing the output for the gauge group, we want the invariant roots to be $\pm(1, -1, 0, \dots, 0)$ defining $SU(2)$, and $\pm(0, 0, 1, -1, 0, \dots, 0)$, $\pm(0, 0, 0, 1, -1, 0, \dots, 0)$, $\pm(0, 0, 1, 0, -1, 0, \dots, 0)$ defining $SU(3)$. This fixes the $SU(2)$ and $SU(3)$ regions in the eight-dimensional vectors. As we will explain later, we want the three quark doublets to come from the untwisted sector. We then find that we need at least two Wilson lines to achieve this: one with the first five entries different from zero but without breaking $SU(3) \times SU(2)$ ($p \cdot a = n \in \mathbb{Z}$, with p being any of the roots listed above), and/or the other one with the first five entries vanishing and breaking the symmetries left over from the last three entries. A third Wilson line should also satisfy one of the two conditions. This will allow us to have the right gauge group and matter content in the untwisted sector. The twisted sectors will satisfy the mass formulae (4.2), and the shifts $(v + n_i a_i)$ are equivalent to one of the four shifts corresponding to the four possible embeddings of the $\mathbb{Z}_3$ orbifold. Therefore it is straightforward to compute the matter for the twisted sector: just find the equivalent shift and use the matter content for that shift, found in Ref. [3], decomposed in terms of $SU(3) \times SU(2)$ representations. From this we obtain the following information. The only way of obtaining a $(3, 2)$ representation is if the shift $(v + na)$ is equivalent to the standard embedding's shift, which gives a **27** in the twisted sector. In this case we would have the **27** decomposed in terms of $SU(3) \times SU(2)$, and would also have the

unwanted extra colour triplets. This is why we insisted that the (3,2) should come from the untwisted sector. The twisted sectors, however, have to provide the quark singlets in order to cancel the (3,2) anomalies. The extra triplets can be avoided by appropriate choice of the hidden-sector part of the Wilson lines. However, in this process it appears unavoidable to have many (~ 12) extra SU(2) doublets. A list of possible shifts and Wilson lines that satisfy the above-mentioned constraints is provided in Table 3. To each of them can be added several hidden-sector parts which satisfy modular invariance, and can be chosen so as to give no extra colour triplets from the twisted sectors. Also, a third Wilson line can be added, chosen from any of the present list. Several examples, including the massless spectrum, are provided in Table 4.

Even though these models are very close to the Standard Model, we cannot claim complete phenomenological success. Besides having many extra doublets, a careful analysis of the different U(1) quantum numbers has not been performed, and it is necessary to know the number (n) of U(1)'s that couple to quarks and leptons, the possible effects of the anomalous U(1) which usually appears in these constructions, and the role of the U(1)'s mixing the hidden and observable sectors. We content ourselves with having a good control over this construction in order to obtain interesting models. A detailed phenomenological analysis of these or other similar models would be justified after making a more general exploration of this construction of four-dimensional string models.

Notice that even though we have imposed very restrictive physical constraints, there is still a fairly large number of models that satisfy them. There are several ways of adding the hidden-sector part to the list of Table 3 in order to satisfy modular invariance (this can be found by using the complete list of models with one Wilson line, from sub-subsection 5.1.1). There are also many possible third Wilson lines which can be added to each model and which satisfy the requirements mentioned above. Finally, models with extra colour triplets cannot be completely discarded, because they may couple in such a way as not to spoil the proton decay rate. It is not impossible, then, that after a careful analysis of these models there may be one (or many) with realistic qualifications.

5.1.3 Other orbifolds: $\mathbb{Z}_7$ and $\mathbb{Z}_4$

The phenomenological possibilities of Abelian orbifolds other than $\mathbb{Z}_3$ have not been explored in the literature. Table 1 gives us some hints to follow when searching for interesting models. There are three orbifolds in that table with three generations in the untwisted sector. This suggests that by having non-standard embeddings and/or Wilson lines, we may get three-generation models. In the case of the $\mathbb{Z}_8$ and $\mathbb{Z}_{12}$ orbifolds, the computation of the massless spectrum in the presence of non-standard embeddings is cumbersome, although straightforward if we use the methods discussed in the previous sections. We will content ourselves here with a discussion of the $\mathbb{Z}_7$ orbifold, which is simpler because it is of prime order. We also comment briefly on the $\mathbb{Z}_4$ orbifold as an example of even-order modding.

The symmetric $\mathbb{Z}_7$ orbifold can be constructed by modding the six-dimensional SU(7) root lattice by its Coxeter element. This action ω is an order-7 automorphism, which leaves seven fixed points (the six fundamental weights plus the origin). Also, ω^3 and ω^2 are of order 7, and one has, as a whole, three twisted sectors, each with a degeneracy factor 7 (see Table 1). The sectors twisted by ω^4 , ω^5 , and ω^6 will give rise to the antiparticles. In standard embedding we have an action in an E_8 through a shift $\tilde{v}^1 = \frac{1}{7}(1, 2, -3, \dots, 0)$. It can easily be checked that the unbroken gauge group is $E_6 \times U(1)^2 \times E_8$, and we have untwisted matter transforming as $3(27 + 1)$ under E_6 (plus the moduli). From each of the three twisted sectors we get seven E_6 generations, each one of the seven sitting on a different fixed point. The matter content in the three twisted sectors is the same because $\tilde{v}$, $2\tilde{v}$, and $3\tilde{v}$ are shifts which are equivalent in the root lattice of E_8 . The total number of generations is 24. One may look for non-standard embeddings to reduce the number of families. The gauge shifts $\tilde{v}$ are subject to the modular invariance constraint

$$7\tilde{v}^2 = 0 \pmod{2} \quad (5.1)$$

They also verify $7\tilde{v} \in \Lambda_{8 \times 8}$, and one can always choose the length of $\tilde{v}$ in each E_8 subgroup smaller than or equal to one. There seem to be 44 non-standard embeddings compatible with modular invariance (compared with 3 in the $\mathbb{Z}_3$ orbifold case!). They lead to a variety of models [16] with different gauge groups, the smallest E_8 unbroken subgroup being $SU(5) \times SU(3) \times SU(2) \times U(1)$. Unlike the standard embedding case, $\tilde{v}$, $2\tilde{v}$, and $3\tilde{v}$ are in general not equivalent in the $E_8 \times E_8$ lattice, and the sectors twisted by θ , θ^2 , and θ^3 lead to different matter (always with a multiplicity factor 7).

A three-generation model is obtained by choosing the non-standard embedding $\tilde{v} = \frac{1}{7}(1, 2, -3, \dots, 0) \times \frac{1}{7}(1, 2, -3, \dots, 0)$. This breaks $E_8 \times E_8$ down to $E_6 \times U(1)^2 \times E_6' \times U(1)^2$ and leads to untwisted matter in a $3[(27, 1) + (1, 1) + (27', 1') + (1', 1')]$ representation. In the twisted sectors θ^n we get massless states for

$$\frac{(p + n\tilde{v})^2}{2} + N_L - \frac{5}{7} = 0 \quad (5.2)$$

and we can easily check that only singlets are obtained from them. Thus as a whole we have (in each given E_8 sector) an $E_6 \times U(1)^2$ model with three generations, all coming from the untwisted sector. Other non-standard embeddings give rise to four generations. One may consider the embedding $\tilde{v} = \frac{1}{7}(1, 1, 6, \dots, 0) \frac{1}{7}(2, \dots, 0)$. This breaks the gauge group down to $SO(10) \times SU(3) \times U(1) \times [SO(14)' \times U(1)']$. In the first sector we have untwisted matter transforming as $(16, 3) + (16, 1)$ under $SO(10) \times SU(3)$. From the twisted sectors we get only singlets and $\mathbf{10}$'s of $SO(10)$.

As we have said, the smallest subgroup of E_8 that may be obtained through non-standard embeddings is $SU(5) \times SU(3) \times SU(2) \times U(1)$. If we want to obtain things like $SU(3) \times SU(2) \times U(1)^5$, we have to add Wilson lines. In fact in this orbifold you can only add one Wilson line. This is a consequence of the constraints on Wilson lines, discussed in Section 4, which reflect the topology of the orbifold. Equation (2.7b) and the embedding of the space group in the gauge degrees of freedom also imply that in each of the three twisted sectors the seven fixed points are no longer equivalent since they have different associated gauge shifts. In each θ^n ($n = 1, \dots, 3$) sector we have for the shift associated with each of the seven fixed points ($m = 0 - 6$),

$$p^I \longrightarrow p^I + n v^I + m a^I \quad (5.3)$$

Thus we have a different gauge transformation acting in each of the $3 \times 7 = 21$ twisted sectors. This makes the computation of the massless spectrum a bit cumbersome, although the formalism is very suitable for a computer search. The Wilson line a^I may be any of the ~ 40 shifts which are modular-invariant; so that considering all kinds of non-standard embeddings and Wilson lines, we expect to obtain $\sim 10^3 - 10^4$ models from the $\mathbb{Z}_7$ orbifold. We have not as yet attempted a systematic analysis, but we can give a new three-generation model by adding a Wilson line on the standard embedding $\mathbb{Z}_7$ orbifold. Take as the gauge shift

$$a = \frac{1}{7}(1, 2, -3, 0, \dots, 0) \frac{1}{7}(1, 2, -3, 0, \dots, 0)$$

The projection on $p^I \cdot a^I = m \in \mathbb{Z}$ kills all non-singlet untwisted matter fields. Concerning the twisted sectors, all of them but three (which correspond to the fixed point with respect to θ , θ^2 , θ^3 at the origin) get shifted momenta as in Eq. (5.3). This means that only those three give rise to interesting

matter (three 27 's) and the rest only to singlets. Thus we have a new three-generation model with gauge group $E_6 \times U(1)^2 \times [E'_6 \times U(1)^2]$.

Table 1 shows that the $\mathbb{Z}_4$ orbifold does not look very promising if we want to get three generations; the number 3 appears nowhere in the massless spectrum. We consider it here in order to show how the addition of Wilson lines is sensitive to the whole space group of the orbifold and not only to the point group. As we mentioned in Section 2 (see also Appendix 2), a $\mathbb{Z}_4$ $[SO(4)^3]$ orbifold can be built by rotating the first two $SO(4)$ lattices by $\pi/2$ and the third one by π . A $\mathbb{Z}_4$ $[SU(4)^2]$ orbifold can also be built by modding an $SU(4)^2$ root lattice by its Coxeter element. In the first case the action is not in the Weyl group, but in the second case it is. The number of fixed points is, of course, the same in both cases: $2 \times 2 \times 4 = 16$ in $\mathbb{Z}_4$ $[SO(4)^3]$ and $4 \times 4 = 16$ in $\mathbb{Z}_4$ $[SU(4)^2]$. The massless spectrum for the standard embedding case is computed in Appendix 2 and is the same for both orbifolds (this is also true for the non-canonical embeddings). However, in the presence of Wilson lines these two $\mathbb{Z}_4$ orbifolds behave differently. Take first the $\mathbb{Z}_4$ $[SU(4)]$ example. Using the conditions found in Section 4, we can see $a_1 = a_2 = a_3$ and $a_4 = a_5 = a_6$, and we can have one independent Wilson line in each of the two $SU(4)$ lattices. We can also check that $4a_1$ and $4a_4 \in \Lambda_{8 \times 8}$. If we repeat the same exercise for the $\mathbb{Z}_4$ $[SO(4)^3]$ example, we find that there are up to four independent Wilson lines, $a_1 = a_2$, $a_3 = a_4$, a_5 , and a_6 . Furthermore, they have to be of order two: $2a_1, 2a_3, 2a_5, 2a_6 \in \Lambda_{8 \times 8}$. Thus the models that we may obtain by adding Wilson lines to these two orbifolds are in general quite different. This is usually the situation for $\mathbb{Z}_N$ orbifolds with different lattice.

5.2 Continuous symmetry breakdown

In string theory the mechanism of gauge-symmetry breakdown can be divided into two classes:

- i) Those where the background values (of, for example, gauge fields in extra dimensions) are quantized; and
- ii) Those where these values can be varied smoothly.

The quantization in the first case comes from the symmetries (e.g. modular invariance) of the theory, and, typically, non-vanishing values will be of the order of the Planck mass. A change of these background fields from one quantized value to another will give a new theory in a different topological sector, and even the chiral structure of the model can be altered. Examples of such a mechanism are the shifts v representing the orbifold twist in the gauge group, or the Wilson lines in the Abelian embedding discussed in Section 4. This mechanism is needed in the breakdown of $E_8 \times E_8$ in order to obtain a chiral model in four dimensions. In terms of a field theory, this type of gauge-symmetry breakdown will not resemble a Higgs mechanism—the relevant gauge bosons are simply projected from the spectrum. This, however, is not true in case (ii), where the values can be smoothly varied; the chiral structure of the theory cannot be changed, and with respect to gauge-symmetry breakdown one expects to get small gauge-boson masses by slightly changing the background values from the symmetric configuration. No projection of states occurs and this should resemble a Higgs mechanism. Examples of such a mechanism are provided by changing Wilson lines in the non-Abelian embedding discussed in Section 4. We think that a smooth mechanism should be responsible for the breakdown of the weak interaction group $SU(2) \times U(1)$. We are accustomed to discuss this question in the framework of a low-energy effective field theory, assuming that there is a non-vanishing vacuum expectation value (v.e.v.) of a scalar field out of which one component serves as the longitudinal degree of freedom of the, now massive, gauge bosons W and Z with mass given by this v.e.v. Since m_W is small compared with m_P , we think that this v.e.v. is not quantized but can be smoothly varied, i.e. it corresponds to mechanism (ii). We do not yet understand why m_W is so small compared with m_P , but in this framework we can at least parametrize such a small number.

Having made this assumption we shall now investigate such a mechanism in the currently known consistent four-dimensional string models and look for regularities. All models with some pretensions to being phenomenological can be understood in terms of a generalized compactification of the $D = 10$ heterotic string to four dimensions $D = 4$. We would like to point out here that our bosonic formulation for the extra coordinates is very suitable for a discussion of a continuous change of parameters, whilst such a discussion is notoriously difficult in the fermionic formulation or in the covariant lattice approach.

In such a smooth breakdown of gauge symmetries we should be able to explicitly observe a Higgs mechanism: a matter field (a scalar Higgs) combines with the massless gauge boson to form a massive gauge boson. Our investigation shows a curious result. In most of the cases, not only one but three such scalar fields become massive. The origin of this Three-Higgs Rule (THR) lies in the fact that one compactifies three complex dimensions.

Let us start our discussion with a simple example: toroidal compactification of the heterotic superstring from ten to four dimensions. The mass spectrum is then given by

$$\frac{1}{4} m_L^2 = N_L - 1 + \frac{1}{2} L^2$$

$$\frac{1}{4} m_R^2 = N_R + \frac{1}{2} \tilde{L}^2$$

with $m_L^2 = m_R^2$ and $(L, \tilde{L})$ forming a Lorentzian, integer, even self-dual lattice with signature $(22, 6)$ (see subsection 4.2 for the explicit form of this lattice). This defines a class of models connected continuously to each other by 132 parameters corresponding to the Lorentz rotations in the coset $SO(22, 6)/SO(22) \times SO(6)$. The 132 parameters can be understood as being background values of the metric g_{ij} (radii and angles of the torus), of the antisymmetric tensor field b_{ij} , and of the Wilson lines a_i^j on the non-contractible loops on the torus. By choosing special values of g_{ij} and b_{ij} we can enhance the Kaluza-Klein gauge group $U(1)^6$ to any simply laced group of rank 6. Furthermore, special values of the Wilson lines allow the appearance of a general rank-22 group such as $O(44)$. Since the background fields can be smoothly varied, a slight departure from the symmetric values leads to a smooth breakdown of the gauge group, potentially down to $U(1)^{22}$; i.e. gauge bosons become massive. To discuss the Higgs mechanism on the torus, it is enough to observe that the models in $D = 4$ have $N = 4$ supersymmetry, i.e. $N = 4$ gauge multiplets become massive in the process. In $N = 1$ language, the $N = 4$ multiplet contains an $N = 1$ gauge multiplet as well as three $N = 1$ scalar multiplets, and as a result, three Higgs multiplets become massive in the process of gauge-symmetry breakdown. It could be argued that this is just a result of $N = 4$ supersymmetry which will not appear in realistic models. The mechanism, however, is more general and comes from the fact that $(10 - 4)/2 = 3$ which is also at the origin of $N = 4$ supersymmetry.

Let us now consider the $\mathbb{Z}_3$ orbifold. The THR has first been observed in the $SU(3)^3$ example of Ref. [7]. The smooth mechanism was a change in the values of background fields that allowed the rank of the gauge group to be lowered, as discussed in sub-subsection 4.1.2. For special values of Wilson lines the gauge group in this example was $SU(3)^3 \times U(1)^2$. Departing from this symmetric value, $U(1)^2$ was broken, and simultaneously six matter multiplets disappeared from the massless spectrum, thus confirming the THR. The same results are true for all the four different non-Abelian embeddings of the space group into the gauge group in the framework of the $\mathbb{Z}_3$ orbifold. As an example, let us discuss a model with gauge group $SU(3) \times SU(2) \times U(1)$.

We start with the E_8 lattice written in $SU(3)^4$ basis and rotate the first two $SU(3)$'s simultaneously by 120° . This breaks E_8 to $O(14) \times U(1)$, with 3×78 matter fields in the untwisted sector. As a Wilson line let us take $a_1 = \frac{1}{3}(\varrho_1, \varrho_1, 0, \varrho_1)$ and the rotated $a_2 = \theta a_1$, where ϱ_1 is a simple root of $SU(3)$. This leaves the third $SU(3)$ complete, and the remaining oscillators as well as two combinations of the weight states combine to $SU(2) \times U(1)$. If we change the first entry slightly

away from $1/3Q_1$, the two combinations of the weight states disappear from the massless spectrum, and $SU(3) \times SU(2) \times U(1)$ is broken to $SU(3) \times U(1)^2$. An inspection of the matter fields shows that THR is valid, as it is for all cases of this mechanism with Wilson lines in the non-Abelian embeddings of the $\mathbb{Z}_3$ orbifold.

A second smooth mechanism for breaking gauge symmetries in orbifolds is related to the six compactified dimensions. Consider, for example, a toroidal compactification with an $E_6 \times E_8 \times E_8$ gauge group. The $\mathbb{Z}_3$ twist breaks E_6 to $SU(3)^3$, which in turn can be spontaneously broken by changing the radii of the underlying torus or b_{ij} background configurations. If we work out these cases explicitly, THR seems to be valid, i.e. all these smooth mechanisms of gauge-symmetry breakdown in the framework of the $\mathbb{Z}_3$ orbifold have three Higgses for each massive gauge boson.

The same rule is found to be true for all smooth breakdown mechanisms in the $\mathbb{Z}_N$ orbifold, where N is a prime number. In the case of $N = 7$, we consider $SU(7) \subset E_8$, and the gauge group is broken down to $SO(8) \times SU(3) \times U(1)^2$. Wilson lines can break this group down smoothly to $SU(2) \times U(1)$, and every massive gauge boson drags three matter fields with it.

The THR seems to be valid for the breakdown of the gauge symmetries related to the six compactified dimensions for any Abelian orbifold. A counterexample, however, can be found, for example for a $\mathbb{Z}_4$ orbifold with non-Abelian embedding of the space group. Take $SU(4) \times SU(4)$ as subgroup of E_8 and rotate them simultaneously by $\mathbb{Z}_4$, which breaks E_8 to $O(12) \times SU(2) \times U(1)$. Wilson lines could break this group down to $O(4)$, i.e. 64 generators could be broken. An inspection of the matter states shows that only 54 of these states are accompanied by three Higgses, whilst the other 10 violate THR. This example is certainly not realistic, but it shows that THR cannot be a definite prediction in orbifold models. The simple origin of the rule, namely the fact that one compactifies from $D = 10$ to $D = 4$, nevertheless makes it quite attractive, and it could very well be that this is true in realistic models. We might also speculate that there is a connection with the fact that we have three generations of quarks and leptons.

It also remains to be seen whether such a rule might generalize beyond torus and orbifold compactification. The only known smooth mechanism at the moment is the blowing-up procedure from an orbifold to a Calabi-Yau manifold. Also, here we usually find several Higgses per gauge boson, although for different reasons. The whole situation is therefore not conclusive.

None the less, we feel that such a discussion of spontaneous gauge-symmetry breakdown is much more restrictive than in the low-energy-field theory, and that it is therefore necessary to study it in the framework of the full string theory.

5.3 Asymmetric $\mathbb{Z}_3$ (E_6) orbifold

We will now consider, as an illustrative example, an asymmetric $\mathbb{Z}_3$ twist in the E_6 lattice. The Coxeter element $\omega = \Gamma_1, \dots, \Gamma_6$ is a rotation of order 12 in the lattice, whose eigenvalues are $\alpha, \alpha^4, \alpha^5, \alpha^7, \alpha^8$, and α^{11} , where $\alpha = e^{2\pi i/12}$. We define the $\mathbb{Z}_3$ twist to be $\theta_R \equiv \theta = \omega^4$ and $\theta_L = 1$; θ has two triple-degenerate eigenvalues, α^4 and α^8 .

In the absence of Wilson lines, the invariant lattice is $E_8 \times E_8 \times E_6$ and the degeneracy factor is

$$D = \sqrt{\frac{\det(1 - \Theta)}{|I^*/I|}} = 3 \quad (5.4)$$

since $\det(1 - \theta) = 27$ and $|I^*/I| = |E_6^W/E_6^R| = |\mathbb{Z}_3| = 3$, where $E_6^{R,W}$ represent the root and weight lattices of E_6 . Note that from the discussion in subsection 4.3, this value of D will in principle allow the introduction of an order-3 Wilson line, which we will consider below. If we combine the twist θ with a shift of order 3, v in the $E_8 \times E_8$ part of the lattice, we can generate chiral models similar to

those in the case of the $\mathbb{Z}_3$ orbifold, i.e. with four different embeddings. The massless spectrum can easily be obtained and it is provided in Table 5 for each of the embeddings.

Let us now consider the addition of Wilson lines in these models. Conditions (4.20) to (4.25) are satisfied for $a_1 = a_2 = \dots = a_6$ and $a^2 = 2M/3$, $M \in \mathbb{Z}$. Therefore, as suggested by the value of the degeneracy factor, we can add one Wilson line of order 3.

Note that in the presence of this Wilson line the invariant lattice is reduced to all the vectors $L \in E_8 \times E_8 \times E_6$, so that $L \cdot a = m \in \mathbb{Z}$; this, as usual, gives us the untwisted sector. The three twisted sectors come from the shifted dual lattice $\Gamma^* + v$, where $\Gamma^* = E_8 \times E_8 \times E_6 + na$ ($n = 0, 1, 2$), and the degeneracy factor is now $1/3 D = 1$ for each twisted sector. Using this information we can easily compute the spectrum for a given Wilson line. Some examples are given in Table 6, where we can see the different spectra and how, in each case, the non-Abelian anomalies cancel in a non-trivial way.

Tables 5 and 6 show some generic features of asymmetric orbifolds with respect to their possible phenomenological applications. One of them is that the ‘effective’ rank of the gauge group is large, usually larger than for symmetric orbifolds. By this we mean the following. Very often in string models one can include the interesting physics (supposed to contain the Standard Model) in a gauge group sector separated from other group factors (‘hidden sector’). In this way the rank of the gauge group coupling to ‘interesting matter’ is usually smaller than the actual rank of the model (16 or 22). This seems more difficult to obtain in the asymmetric orbifold case, as Tables 5 and 6 exemplify. This is due not only to the fact that the gauge symmetry is enlarged (by an E_6 factor in the present example) but also because there appear massless particles with non-trivial quantum numbers with respect to subgroups coming from both E_8 ’s.

We could also analyse other cases of asymmetric twists and Wilson lines: for instance, choosing $\theta_L = \omega$ and $\theta_R = \theta = \omega^4$ in the same E_6 lattice, there are two directions in the six-dimensional space for which $\theta_L = \theta_R$ (eigenvalues α^4 and α^8). As mentioned in subsection 4.3, for these directions we have a symmetric twist, and Wilson lines can be added which satisfy the usual constraints for symmetric orbifolds. Note, however, that the situation is far more restrictive than for symmetric orbifolds, as can be reflected by not having Wilson lines on any of the asymmetric versions of the $\mathbb{Z}_N$ orbifolds. Without Wilson lines it is difficult to break the gauge group sufficiently and to control the massless particle content. We have been unable to find any phenomenologically attractive asymmetric orbifold. We also think that symmetric orbifolds seem much more suitable for this purpose.

6. CONCLUDING REMARKS

Orbifolds provide a unifying framework for the construction of four-dimensional string theories. In this paper we have discussed a general and systematic approach for classifying these models. The parameters are given by the values of classical background fields g_{mn} , B_{mn} , and A_m^I , as well as by a symmetric or an asymmetric space-time twist and the different embeddings in the gauge group. The number of consistent four-dimensional models is so large ($\geq 10^9$ for a symmetric $\mathbb{Z}_3$ orbifold and the Abelian embeddings) that they cannot all be constructed explicitly. However, using the formalism provided in this article, and in particular the explicit expression for the generalized GSO projectors, we can in principle construct any model based on the $\mathbb{Z}_N$ and $\mathbb{Z}_N \otimes \mathbb{Z}_M$ orbifolds. (Non-Abelian orbifolds present further complications and have to satisfy extra constraints from higher-loop modular invariance [14].)

Without having guidance from further constraints which could come from considerations of non-perturbative string effects, the way to proceed is to apply some ‘phenomenological’ requirements and select those models which resemble the $SU(3) \times SU(2) \times U(1)$ Standard Model with three families of quarks and leptons. Our results show that in this respect the orbifold approach is especially efficient for finding potentially interesting models from the phenomenological point of

view. Using our general classification of the $\mathbb{Z}_3$ orbifold models with one Wilson line, we have found several three-generation models with symmetries containing $SU(5)$, $SO(10)$, and E_6 , and a relatively large class of models with three generations and $SU(3) \times SU(2) \times U(1)^n$ gauge group, obtained from the addition of two or three Wilson lines on $\mathbb{Z}_3$. Also, three family models were constructed using other orbifolds, e.g. $\mathbb{Z}_7$. We see that many interesting models can be quite easily constructed, although at the moment none of them is fully realistic. Of course, we have not yet been able to give a complete classification; still more detailed investigations are necessary. Nearly all the phenomenologically interesting models have been obtained in the framework of symmetric orbifolds, and it remains to be seen whether asymmetric orbifolds could also be interesting. They have the property that some of the ‘moduli’ of the underlying orbifold seem to be fixed by consistency. Apart from the fact that we do not really know whether this is desirable, we want to point out that this question has to be investigated carefully, and separately, in every model. The absence of chiral massless singlets does not, in general, guarantee that the radii of compactification are fixed. Very often in asymmetric orbifolds these dilatons are not gauge singlets but form part of gauge non-singlet chiral multiplets. Then the flatness of the dilatons’ potential is disguised as flat directions of the scalar potential of non-singlet fields. This happens, for example, in the case of the asymmetric $\mathbb{Z}_3$ orbifold (e.g. first model in Table 5). There are no massless dilaton singlets. However, we can realize the action in one of the E_8 ’s in terms of Weyl rotations of an $SU(3)^3$ subgroup of E_8 . We can further interpret these six left-handed coordinates as our six extra dimensions (instead of the ones in the E_6 lattice). Then we can see that the dilaton-like states $|0\rangle_R \otimes \alpha_L^\dagger |0\rangle$ form part of a $(27, 3)$ under $E_6 \times SU(3) \in E_8$. Thus there are massless dilatons in these models but they are no longer singlets.

We have also argued that a smooth mechanism of gauge-symmetry breakdown should be responsible for the breakdown of $SU(2) \times U(1)$. In the classification of the known mechanism for such a smooth breakdown, we found that in most models three Higgses are needed instead of one in the field theory formalism. This rule has its origin in the fact that we have to compactify space-time from 10 to 4 dimensions, and it remains to be seen whether it persists in realistic models.

Many questions have to remain open at the moment. Of course, we would like to obtain a complete classification of four-dimensional string models, but the number of models is so large that perhaps we must first impose more physical constraints before we proceed. The next way to constrain the system more tightly would be to investigate the pattern of Yukawa couplings [17, 18] in models that have a satisfactory particle spectrum. At the same time we probably need a more complete analysis of the four-dimensional models with lowered rank, as well as of the question of supersymmetry breakdown.

Acknowledgements

We wish to express our thanks to L. Dixon for illuminating discussions. The useful conversations we had with A. Font, C. Muñoz, K.S. Narain and M.H. Sarmadi are also gratefully acknowledged.

APPENDIX 1

THETA FUNCTIONS AND MODULAR TRANSFORMATIONS. NOTATION.

We closely follow the notation of Ref. [19]. Useful formulae can be found in Ref. [20]. We have for the ϑ functions with rational characteristics θ, ϕ :

$$\vartheta\left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix}\right](\tau) = \sum_n q^{(n+\theta)^2/2} e^{2\pi i (n+\theta)\phi} \quad (\text{A1.1})$$

where $q = \exp 2\pi i \tau$, τ being the one-loop modular parameter. The following identities can be used:

$$\vartheta\left[\begin{smallmatrix} \theta+m \\ \phi+n \end{smallmatrix}\right] = e^{2\pi i \theta n} \vartheta\left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix}\right] \quad \forall m, n \in \mathbb{Z} \quad (\text{A1.2})$$

The expression of these functions in product form is given by

$$\vartheta\left[\begin{smallmatrix} \theta \\ \phi \end{smallmatrix}\right](\tau) = \eta(\tau) e^{2\pi i \theta \phi} q^{\left(\frac{\theta^2}{2} - \frac{1}{24}\right)} \prod_{n=1}^{\infty} (1 + q^{n+\theta-\frac{1}{2}} e^{2\pi i \phi}) (1 + q^{n-\theta-\frac{1}{2}} e^{-2\pi i \phi}) \quad (\text{A1.3})$$

where η is Dedekind's function

$$\eta(\tau) = e^{i\pi\tau/12} \prod_{n=1}^{\infty} (1 - q^n) \quad (\text{A1.4})$$

For the three non-vanishing spin structures, we obtain

$$\begin{aligned} \vartheta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] &= \prod_{n=1}^{\infty} (1 - q^n) (1 + q^{n-\frac{1}{2}})^2 \\ \vartheta\left[\begin{smallmatrix} 0 \\ \frac{1}{2} \end{smallmatrix}\right] &= \prod_{n=1}^{\infty} (1 - q^n) (1 - q^{n-\frac{1}{2}})^2 \\ \vartheta\left[\begin{smallmatrix} \frac{1}{2} \\ 0 \end{smallmatrix}\right] &= 2 q^{\frac{1}{8}} \prod_{n=1}^{\infty} (1 - q^n) (1 + q^n)^2 \end{aligned} \quad (\text{A1.5})$$

The Dedekind η function is related to the previous ones by:

$$\eta^3 = \frac{1}{2\pi} \vartheta'\left[\begin{smallmatrix} \frac{1}{2} \\ \frac{1}{2} \end{smallmatrix}\right] = -\frac{1}{2} \vartheta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] \vartheta\left[\begin{smallmatrix} 0 \\ \frac{1}{2} \end{smallmatrix}\right] \vartheta\left[\begin{smallmatrix} \frac{1}{2} \\ 0 \end{smallmatrix}\right] \quad (\text{A1.6})$$

All these functions have definite transformation properties under the modular actions $\tau \rightarrow \tau + 1$:

$$\mathfrak{J}\left[\begin{smallmatrix} \Theta \\ \Phi \end{smallmatrix}\right](\tau+1) = e^{-i\pi(\Theta^2-\Theta)} \mathfrak{J}\left[\begin{smallmatrix} \Theta \\ \Theta+\Phi-\frac{1}{2} \end{smallmatrix}\right](\tau) \quad (\text{A1.7})$$

$$\eta(\tau+1) = e^{i\pi/2} \eta(\tau) \quad (\text{A1.8})$$

and $\tau \rightarrow -1/\tau$

$$\mathfrak{J}\left[\begin{smallmatrix} \Theta \\ \Phi \end{smallmatrix}\right](-1/\tau) = e^{2\pi i \Theta \Phi} (-i\tau)^{1/2} \mathfrak{J}\left[\begin{smallmatrix} \Phi \\ -\Theta \end{smallmatrix}\right](\tau) \quad (\text{A1.9})$$

$$\eta(-1/\tau) = (-i\tau)^{1/2} \eta(\tau) \quad (\text{A1.10})$$

Under these one-loop modular transformations, the partition-function measure $d^2\tau/\tau_2^2$ is invariant ($\tau_2 = \text{Im } \tau$).

APPENDIX 2

$\mathbb{Z}_4$ AND $\mathbb{Z}_6$ EXAMPLES

Here we consider these two examples of even-order symmetric orbifolds in order to illustrate the text in Sections 2 and 3. One can build a $\mathbb{Z}_4$ orbifold by taking three copies of the $SO(4)$ root lattice (two perpendicular roots of equal length) and modding them, by a simultaneous rotation, by $\pi/2$ in the first two $SO(4)$'s and by π in the third one. This corresponds to $v = (1/4, 1/4, 1/2, 0)$ and $\tilde{v} = (1/4, 1/4, 1/2, 0, 0, 0, 0, 0) \times (0)$ in standard embedding. The $SO(4)$ lattice has two fixed points under a $\pi/2$ rotation [with coordinates $(0, 0)$ and $(1/2, 1/2)$ for roots normalized to unity] and four fixed points [with coordinates $(0, 0)$, $(0, 1/2)$, $(1/2, 0)$, and $(1/2, 1/2)$] under a π rotation. As a whole, this orbifold has $2 \times 2 \times 4 = 16$ fixed points. Let us compute the massless spectrum.

From the untwisted sectors, we get the supergravity and dilaton multiplets plus the gauge bosons. Charged gauge bosons correspond to $E_8 \times E_8$ root vectors with $p^I \cdot v^I = \text{integer}$. We can easily check that the gauge group is then

$$E_6 \times SU(2) \times U(1) \times E_8' \quad (A2.1)$$

Concerning non-singlet untwisted matter, we can obtain projection-invariant states from objects of the form $|a\rangle_R \times |p^I\rangle_L$, where $|a\rangle_R$ denotes the vacuum states for the extra six modded coordinates (or spinors). Two of them transform as $\lambda = \exp 2\pi i/4$ under P , two as λ^3 , and two as λ^2 . The first two give particles and antiparticles with gauge quantum numbers

$$2(27, 2) + 2(1, 2) \quad (A2.2)$$

The third one has quantum numbers

$$(27, 1) + (\bar{27}, 1) \quad (A2.3)$$

We also have six massless chiral singlets from the 'moduli' of the orbifold. This completes the massless untwisted sector.

In the sector twisted by θ , the degeneracy of the ground state is given by Eq. (3.8a) and is equal to the number of fixed points, 16. In bosonic language, the mass formula for left-movers is given by

$$\frac{1}{4} m_L^2 = \frac{(p+v)^2}{2} + \tilde{N}_L - 11/16 \quad (A2.4)$$

For $N_L = 0$ we get massless states for $p^2 = 2$, $p \cdot v = -1/2$. They correspond to a $(27, 1)$. For $N_L = 1/4$, we get massless states for $p^2 = 2$, $p \cdot v = -3/4$. They correspond to $16 \times 2 = 32$ copies of a $(1, 2)$. These are the states which are associated with a possible 'reparation of the singularities'. Summing up, we get $16[(27, 1) + 2(1, 2)]$ from the sector twisted by θ . All these particles survive projection. Let us see this in some detail.

In the sector twisted by θ , the massless states' projector has the form $P_\theta = 1/4(16 + 16\Delta_\theta + 16\Delta_\theta^2 + 16\Delta_\theta^3)$ where, for example, the Ramond right-handed sector $\Delta_\theta = \exp 2\pi i[(p+v)v - (r+w+v)v]$, with $w = (1/2, 1/2, 1/2, 1/2)$. The massless twisted sector is obtained for $(r+w+v) = (-1/4, -1/4, 0, 1/2)$ [i.e. for $r = (-1, -1, -1, 0)$]. For this state we have $\exp 2\pi i(r+w+v)v = \exp(-i\pi/4)$, and for the left-moving phase 27 we have $(p+v)v = -1/8$, so that $\Delta_\theta = 1$ and all states survive. An easy way to check the chirality of this 27 with respect to the

untwisted sector is by comparing the fourth entry of $(r + w + v)$ with the one of $(r + w)$ in the untwisted sector. In this way we can see that these 27 's have the same chirality as the ones in the untwisted sector.

The sector twisted by θ^2 is more difficult since the degeneracy factors in the partition function are different for each of the four subsectors (θ^2, θ^m) , $m = 0, 1, 2, 3$. The massless states' condition is now

$$\frac{1}{4} m_L^2 = \frac{(p+2v)^2}{2} + \tilde{N}_L - \frac{3}{4} = 0 \quad (\text{A2.5})$$

For $\tilde{N}_L = 0$, we get massless 27 's for $p^2 = 2$ with $p \cdot 2v = -1$. We obtain $\bar{27}$'s for $p^2 = 4$ with $p \cdot 2v = -2$. The projector has now the form shown in Eq. (3.7) in the main text. In the Ramond-Ramond sector we would have $\Delta_{\theta^2} = \exp 2\pi i[(p+2v)v - (r+2v+w)v]$. The massless twisted right-handed states are obtained for $(r+2v+w) = (0, 0, \frac{1}{2}, \frac{1}{2})$ or $(0, 0, -\frac{1}{2}, -\frac{1}{2})$ which are associated with vacua of opposite helicity. The right-handed phase is $\exp 2\pi i(r+2v+w)v = \pm i$, respectively. We will then get ten massless 27 's for $\Delta_{\theta^2} = 1$ and six massless 27 's for $\Delta_{\theta^2} = -1$ surviving out of the 16 $(27 + \bar{27})$ original pairs (the θ^2 twist leaves 16 fixed tori invariant). Notice that in the above discussion, if this particular form of the lattice were not used, the same massless spectrum would be obtained for a $\mathbb{Z}_4$ $[\text{SU}(4)^2]$ orbifold.

In the standard embedding case there is an easy rule for obtaining the above results, using just the Euler characteristic formula of Eq. (2.8). The number of generations of the sector twisted by θ is always given by $\chi(\theta)$ and leads to chiral families (16 in this case). In the sector twisted by θ^2 , there are 16 fixed tori with coordinates in the first two $\text{SO}(4)$ tori given by $(a; b)$, with a, b equal to any of the four points $(0, 0), (0, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2})$. Of these 16 tori which are left invariant by θ^2 , four of them [with coordinates $(0, 0; 0, 0), (0, 0; \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}; 0, 0), (\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2})$] are also left fixed by θ . They correspond to the four chiral 27 's coming from the (θ^2, θ) and (θ^2, θ^3) contributions to the Euler characteristic formula. From the remaining twelve tori one can form six linear combinations which are left fixed also by θ . They correspond to six 27 's and six $\bar{27}$'s which complete the spectrum left invariant by the projector.

Let us briefly discuss a $\mathbb{Z}_6$ orbifold. Take an $\text{SU}(3)^3$ lattice and twist it by 60° in the first two $\text{SU}(3)$ lattices and 120° in the third one. This corresponds to a $v = (\frac{1}{6}, \frac{1}{6}, -\frac{1}{3})$. Notice that this automorphism is not in the Weyl group. It is easy to check that it has an unbroken gauge group (in standard embedding) $E_6 \times \text{SU}(2) \times \text{U}(1) \times E_8'$. There is untwisted matter with quantum numbers $2(2, 27) + (1, 27)$ under $\text{SU}(2) \times E_6$. In the sector twisted by θ , all states survive projection and we have three (= No. of fixed points) 27 's. Notice that in this orbifold, these three fixed points correspond to the two fundamental weights of the third $\text{SU}(3)$ plus the origin. In the sector twisted by θ^2 there are also 27 's, and the projector has the form

$$P_{\theta^2} = \frac{1}{6} (27 + 3 \Delta_{\theta^2} + 27 \Delta_{\theta^2}^2 + 3 \Delta_{\theta^2}^3 + 27 \Delta_{\theta^2}^4 + 3 \Delta_{\theta^2}^5) \quad (\text{A2.6})$$

However, it can easily be checked that for the massless states we always have $\Delta_{\theta^2} = 1$, and we get 15 27 's (no $\bar{27}$'s since there are no fixed tori). Finally, the sector twisted by θ^3 leads to fixed tori, and the projector is of the form

$$P_{\theta^3} = \frac{1}{6} (16 + \Delta_{\theta^3} + \Delta_{\theta^3}^2 + 16 \Delta_{\theta^3}^3 + \Delta_{\theta^3}^4 + \Delta_{\theta^3}^5) \quad (\text{A2.7})$$

It can be seen that for the massless states $\Delta_{\theta^3} = 1$ for the 27 's (+ antiparticles) and $\Delta_{\theta^3} = \exp 2\pi i/3$ for the $\bar{27}$'s (+ antiparticles). Thus we get six 27 's plus five $\bar{27}$'s from this sector. Notice how these

results agree with those shown in Table 1. Again, for the standard embedding case, these numbers can be obtained by using the rule explained for the $\mathbb{Z}_4$ case. The sector twisted by θ always gives $\chi(\theta)$ **27**'s, three **27**'s in this case. In the θ^2 sector, of the 27 points left fixed by θ^2 , 3 are also left fixed by θ . This gives $\chi(\theta^2, \theta) (= 3)$ **27**'s. Of the remaining 24, we can form $12 = 24/2$ linear combinations left invariant by θ . They give 12 further **27**'s, so that as a whole we have 15 **27**'s from the θ^2 sector. Finally, the θ^3 sector will have **27**'s and $\overline{\mathbf{27}}$'s, since it leaves $16 = 4 \times 4$ fixed tori. Of those, there is one torus [with zero coordinates in the first two SU(3) tori] which is also invariant under θ . It leads to a **27** state. With the remaining 15 tori, we can form $5 = 15/3$ linear combinations which are θ -invariant. They lead to five $(\mathbf{27} + \overline{\mathbf{27}})$ pairs. Thus we again reproduce the spectrum described above. With this rule we can reproduce all the results in Table 1. However, for the case of non-standard embeddings and/or Wilson-line backgrounds we will have to build, in general, the corresponding projectors (unless the orbifold is of prime order).

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Table 1

Symmetric Z_N Coxeter orbifolds

	Lattice	(a,b,c)	No. of fixed pts.	Untwisted No. 27	Twisted sectors No. of 27						No. of chiral 27	No. of (27 + $\overline{27}$)	Gauge group $\times E_6'$
					θ	θ^2	θ^3	θ^4	θ^5	θ^6			
Z_3	$SU(3)^3$	$1/3(1,1,-2)$	27	9	27						36	0	$E_6 \times SU(3)$
Z_4	$SU(4)^2$	$1/4(1,1,-2)$	16	4	16	4					24	7	$E_6 \times SU(2) \times U(1)$
Z_6	$SU(3) \times G_2^2$	$1/6(1,1,-2)$	3	5	3	15	1				24	5	$E_6 \times SU(2) \times U(1)$
Z_6	$SU(6) \times SU(2)$ $SU(3) \times SO(8)$	$1/6(1,2,-3)$	12	2	12	6	4				24	11	$E_6 \times U(1)^2$
Z_7	$SU(7)$	$1/7(1,2,-3)$	7	3	7	7	7				24	0	$E_6 \times SU(1)^2$
Z_8	$SO(8)$ $SO(5) \times SO(9)$	$1/8(1,2,-3)$	4	3	4	10	4	3			24	3	$E_6 \times U(1)^2$
Z_8	$SO(4) \times SO(8)$	$1/8(1,3,-4)$	8	2	8	4	8	2			24	4	$E_6 \times U(1)^2$
Z_{12}	E_6 F_4 $SU(3) \times SO(8)$	$1/12(1,4,-5)$	3	3	3	3	2	9	3	1	24	3	$E_6 \times U(1)^2$
Z_{12}	$SO(4) \times F_4$	$1/12(1,5,-6)$	4	2	4	2	8	2	4	2	24	7	$E_6 \times U(1)^2$

Table 2

Classification of $\mathbb{Z}_3$ orbifold models with one Wilson line.
Models with three generations are marked with an asterisk.

Number of charged generators	Example of shift (v) and Wilson line (a)	Gauge group from first E_8	Untwisted matter
18	$3v = (2, 1, 1, 1, 1, 0, 0, 0)(2, 0, \dots, 0)$ $3a = (0, 1, 1 - 1, 0, 1, 2, 0)(1, 1, 1, 1, 0, \dots, 0)$	$SU(3)^3 \times U(1)^2$	$3[(3, 1, \bar{3}) + (1, \bar{3}, 3) + (\bar{3}, 3, 1)]$
	$3v = (2, 1, 1, 1, 1, 0, 0, 0)(2, 0, \dots, 0)$ $3a = (0, 1, 0, 0, -1, 2, 0, 0)(0, \dots, 0)$	$SU(3)^3 \times U(1)^2$	$3[(3, 3, 3) + 3(1, 1, 1)]$
24	$3v = (2, 0, \dots, 0)(1, 1, 0, \dots, 0)$ $3a = (0, 1, 1, 1, 1, 0, 0, 0)(0, 0, 1, 1, 0, 0, 0, 0)$	$SU(4)^2 \times U(1)^2$	$3[(6, 4) + (1, 6)]$
	$3v = (2, 1, 1, 1, 1, 0, 0, 0)(2, 0, \dots, 0)$ $3a = (2, -1, -1, -1, -1, 0, 0, 0)(0, 2, 0, \dots, 0)$		$3[(\bar{4}, 1) + (1, \bar{4}) + (4, 4)]$
	$3v = (2, 1, 1, 0, \dots, 0)(2, 1, 1, 0, \dots, 0)$ $3a = (1, -1, -1, 1, 1, 1, 0, 0)(0, \dots, 0)$	$SU(3)^4$	$3[(3, 3, 3, 1)]$
	$3v = (2, 1, 1, 0, \dots, 0)(2, 1, 1, 0, \dots, 0)$ $3a = (1, 1, 0, 2, 0, \dots, 0)(0, \dots, 0)$	$SO(8) \times U(1)^4$	$3[(8_s + 8_v + 8_a) + 3(1)]$
	$3v = (2, 1, 1, 1, 1, 0, 0, 0)(2, 0, \dots, 0)$ $3a = (0, \dots, 0, 2)(0, 1, 1, 0, \dots, 0)$	$SU(5) \times SU(2)^2 \times U(1)^2$	$3[(\bar{10}, 1, 1) + (5, 2, 1) + (5, 1, 2)]$
30	$3v = (2, 0, \dots, 0)(2, 1, 1, 1, 1, 0, \dots, 0)$ $3a = (-1, 0, 0, 1, 1, 1, 0, 0)(1, 2, 0, -1, -1, 0, 1, 0)$	$SO(8) \times SU(3) \times U(1)^2$	$3[(8_s + 8_v + 8_a, 1) + (1, 3)]$
	$3v = (2, 1, 1, 0, \dots, 0)(0, \dots, 0)$ $3a = (1, 1, 0, 1, 1, 0, \dots, 0)(1, 1, 0, \dots, 0)$	$SU(6) \times U(1)^3$	$3[15 + 6 + 6]$
	$3v = (2, 0, \dots, 0)(1, 1, 0, \dots, 0)$ $3a = \frac{1}{2}(-1, 3, 1, \dots, 1)\frac{1}{2}(1, 1, \dots, 1)$		$3[20 + 4(1)]$
36	$3v = (2, 1, 1, 1, 1, 0, 0, 0)(2, 0, \dots, 0)$ $3a = (0, \dots, 2, 1, 1)(0, \dots, 0)$	$SU(6) \times SU(3) \times U(1)$	$3[(15, 1) + (6, 1)]$
	$3v = (2, 1, 1, 1, 1, 0, 0, 0)(2, 0, \dots, 0)$ $3a = (0, 1, 1, 0, \dots, 0)(2, 0, \dots, 0)$		$3[(15, 3)]$
	$3v = (2, 1, 1, 1, 1, 0, 0, 0)(2, 0, \dots, 0)$ $3a = (0, 2, -1, 0, 0, 1, 1, 1)(1, 1, 1, 1, 0, 0, 0, 0)$		$3[(6, 3)]$
42	$3v = (2, 1, 1, 1, 1, 0, 0, 0)(2, 0, \dots, 0)$ $3a = (0, 2, -2, 0, \dots, 0)(0, 2, 0, \dots, 0)$	$SU(7) \times U(1)^2$	$3[35 + 7]$
	$3v = (2, 1, 1, 1, 1, 0, 0, 0)(2, 0, \dots, 0)$ $3a = (1, 1, 0, -1, -1, 0, 0, 0)(1, 1, 0, \dots, 0)$		$3[21]$
	$3v = (2, 0, \dots, 0)(2, 1, 1, 1, 1, 0, 0, 0)$ $3a = \frac{1}{2}(5, 1, 1, 1, 1, 1, 1, 1)\frac{1}{2}(3, 1, 1, 1, -1, 1, 1, 1)$		$3[7 + \bar{7} + 1]$
	$3v = (2, 0, \dots, 0)(2, 1, 1, 1, 1, 0, 0, 0)$ $3a = (0, 1, 1, 0, \dots, 0)(1, 1, 1, -1, 0, \dots, 0)$	$SO(10) \times SU(2) \times U(1)^2$	$3[(16, 2) + (10, 1)]$
	$3v = (2, 1, 1, 0, \dots, 0)(2, 1, 1, 0, \dots, 0)$ $3a = (1, -1, 1, 1, 0, \dots, 0)(0, 1, 0, 1, 0, \dots, 0)$		$3[(10, 2) + (1, 1)]$
	$3v = (2, 0, \dots, 0)(2, 1, 1, 1, 1, 0, \dots, 0)$ $3a = (2, 1, 1, 0, \dots, 0)(2, 1, 0, 0, 0, 1, 0, 0)$		$3[(16, 1) + (1, 2)]$

Table 2 (cont.)

Number of charged generators	Example of shift (v) and Wilson line (a)	Gauge group from first E_8	Untwisted matter
60	$3v = (2, 0, \dots, 0)(2, 1, 1, 1, 1, 0, 0, 0)$ $3a = (1, 1, 0, \dots, 0)(2, 0, \dots, 0)$	$SO(12) \times U(1)^2$	$3[(32 + 1)]$
	$3v = (2, 0, \dots, 0)(2, 1, 1, 1, 1, 0, 0, 0)$ $3a = (0, 2, 0, \dots, 0)(2, 1, -1, -1, 0, 0, 0, 1)$		$3[12]$
* 72	$3v = (1, 1, 0, \dots, 0)(2, 0, \dots, 0)$ $3a = (1, 0, 1, 0, \dots, 0)(1, 1, 1, 1, 0, 0, 0, 0)$	$E_6 \times U(1)^2$	$3[27]$
	$3v = (1, 1, 0, \dots, 0)(2, 0, \dots, 0)$ $3a = (-1, -2, 1, 0, \dots, 0)(0, \dots, 0)$		$6[1]$
	$3v = (2, 1, 1, 1, 1, 0, 0, 0)(2, 0, \dots, 0)$ $3a = (2, 1, 1, 1, 1, 0, 0, 0)(2, 0, \dots, 0)$	$SU(9)$	-
78	$3v = (2, 1, 1, 0, \dots, 0)(0, \dots, 0)$ $3a = (0, \dots, 0)(2, 1, 1, 0, \dots, 0)$	$E_6 \times SU(3)$	$3[(27, 3)]$
	$3v = (2, 1, 1, 0, \dots, 0)(0, \dots, 0)$ $3a = (2, 1, 1, 0, \dots, 0)(0, \dots, 0)$		-
84	$3v = (2, 0, \dots, 0)(2, 1, 1, 1, 1, 0, 0, 0)$ $3a = (2, 0, \dots, 0)(1, 0, 0, 0, 0, 1, 0, 0)$	$SO(14) \times U(1)$	-
126	$3v = (1, 1, 0, \dots, 0)(2, 0, \dots, 0)$ $3a = (1, 1, 0, \dots, 0)(-1, 1, 1, 1, 0, \dots, 0)$	$E_7 \times U(1)$	-
240	$3v = (0, \dots, 0)(2, 1, 1, 0, \dots, 0)$ $3a = (0, \dots, 0)(0, 0, 0, 2, 1, 1, 0, 0)$	E_8	-

Table 3

Shifts and Wilson lines on one E_8 which satisfy the constraints
imposed to give three families of $SU(3) \times SU(2) \times U(1)^n$

	3_v	3_{a_1}	3_{a_3}
1	(1, 1, 0, 0, 0, 1, 1, 0)	(1, 1, 1, 1, 2, 0, 0, 0)	(0, 0, 0, 0, 0, 2, 0, 0)
2	(1, 1, 0, 0, 0, 1, 1, 0)	(1, 1, 1, 1, 1, 1, 0, 0)	(0, 0, 0, 0, 0, 2, 0, 0)
3	(0, 0, 1, 1, 1, 0, 0, 1)	(2, 1, 1, 1, 1, 0, 0, 0)	(0, 0, 0, 0, 0, 2, 0, 0)
4	(0, 0, 1, 1, 1, 0, 0, 1)	(1, 1, 1, 1, 1, 1, 0, 0)	(0, 0, 0, 0, 0, 2, 0, 0)
5	(1, 1, 0, 0, 0, 2, 0, 0)	(1, 1, 1, 1, 2, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 2, 0)
6	(1, 1, 1, 1, 1, 0, 0, 1)	(1, 1, -1, -1, -1, 1, 0, 0)	(0, 0, 0, 0, 0, 1, 1, 2)
7	(1, 1, 1, 1, 1, 0, 0, 1)	(-1, -1, 1, 1, -2, 0, 0, 0)	(0, 0, 0, 0, 0, 2, 0, 0)
8	(0, 0, 1, 1, 1, 0, 1, 2)	(2, 1, 1, 1, 1, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 2)
9	(2, 1, 1, 1, 1, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 0, 2)	(1, -1, 1, 1, 1, 0, 1, 0)

Table 4

Some models with three families of $SU(3) \times SU(2) \times U(1)^n$ and no extra colour triplets

	1	2	3
3v 3a ₁ = 3a ₂ 3a ₃ = 3a ₄	(2, 1, 1, 1, 1, 0, 0, 0)(2, 0, 0, 0, 0, 0, 0, 0) (2, 1, 1, 0, 0, 1, 1, 0)(0, 1, 1, 1, 0, 0, 0, 0) (0, 0, 2, 0, 0, 0, 0, 0)(-1, 0, 0, 1, 2, 1, 1, 0)	(1, 1, 0, 0, 0, 1, 1, 0)(2, 1, 1, 1, 1, 0, 0, 0) (1, 1, 1, 1, 2, 0, 0, 0)(2, 0, 0, 0, 0, 0, 0, 0) (0, 0, 0, 0, 0, -2, 0, 0)(0, 0, 2, -1, 1, 1, 1, 0)	(1, 1, 1, 1, 1, 0, 0, 0)(1, 1, 1, 1, 1, 1, 0, 0) (1, 1, -1, -1, -1, -1, 1, 0, 0)(0, -1, -1, 1, 1, 1, 1, 0) (0, 0, 0, 0, 0, 1, 1, 2)(0, -1, 1, 1, 1, -1, 1, 0)
Gauge group	$[SU(3) \times SU(2) \times U(1)^5] \times [SU(2)^2 \times U(1)^6]$	$[SU(3) \times SU(2) \times U(1)^5] \times [SU(2)^2 \times U(1)^6]$	$[SU(3) \times SU(2) \times U(1)^5] \times [SU(3) \times SU(2)^2 \times U(1)^4]$
Matter content	$3[(3, 2) + 2(3^*, 1) + (1, 2) + (1, 1)] + 12(1, 2) + \text{hidden matter} + \text{singlets}$	$3[(3, 2) + 2(3^*, 1) + (1, 2) + (1, 1)] + 12(1, 2) + \text{hidden matter} + \text{singlets}$	$3[(3, 2) + 2(3^*, 1) + (1, 2) + (1, 1)] + 12(1, 2) + \text{hidden matter} + \text{singlets}$

Table 5

The four $E_8 \times E_8$ embeddings for the asymmetric $Z_3 (E_6)$ orbifold

Shift	Gauge group	Untwisted sector	Twisted sector
$\frac{1}{3}(1, 1, 2, 0, \dots, 0)^{\frac{1}{3}}(0, \dots, 0)$	$E_6 \times SU(3) \times E'_8 \times E_6$	$3(27, 3; 1)$	$3[(27, 3; 1) + (1, \bar{3}; 27) + (1, \bar{3}; \bar{27})]$
$\frac{1}{3}(1, 1, 0, \dots, 0)^{\frac{1}{3}}(2, 0, \dots, 0)$	$E_7 \times U(1) \times SO(14)' \times U(1)' \times E_6$	$3[(56, 1') + (1, 1') + (1, 14') + (1, 64')]$	$3[(56, 1') + (1, 1') + (1, 14') + (1, \bar{64}')] + 3(1, 1'; 27) + 3(1, 1'; \bar{27})$
$\frac{1}{3}(1, 1, 1, 1, 2, 0, 0, 0)^{\frac{1}{3}}(2, 0, \dots, 0)$	$SU(9) \times SO(14)' \times U(1)' \times E_6$	$3[(84, 1') + (1, 14') + (1, 64')]$	$3[(36, 1') + (\bar{9}, 14')]$
$\frac{1}{3}(1, 1, 2, 0, \dots, 0)^{\frac{1}{3}}(1, 1, 2, 0, \dots, 0)$	$[E_6 \times SU(3)]^2 \times E_6$	$3[(27, 3) \times (1, 1)' + (1, 1) \times (27, 3)']$	$3[(27, 1) \times (1, \bar{3})' + (1, 3) \times (27, 1)']$

Table 6

Examples of asymmetric Z_3 (E_6) models with one Wilson line

Shift	Wilson line	Gauge group	Untwisted sector	Twisted sectors
$\frac{1}{3}(1,1,2,0,\dots,0)\frac{1}{3}(0,\dots,0)$	$\frac{1}{3}(0,\dots,0,1,1)\frac{1}{3}(2,0,\dots,0)$	$[SU(6) \times SU(3) \times U(1)] \times [SO(14) \times U(1)]' \times E_6$	$3[(3,15;1)]$	$3\{[(\bar{15} + 6 + 6), 3;1] + [(1, \bar{3}; 27) + (1, \bar{3}; 27)] + 6[(1, 15) + (\bar{3}, 1) + (3, 6)] + [(1, \bar{6}) + (\bar{3}, 1); 14']\}$
$\frac{1}{3}(1,1,1,2,0,0,0)\frac{1}{3}(2,0,\dots,0)$	$\frac{1}{3}(0,\dots,2)\frac{1}{3}(0,1,1,0,\dots,0)$	$[SU(5) \times SU(2)^2 \times U(1)^2] \times [SO(10) \times SU(2) \times U(1)^2]' \times E_6$	$3[(10,1,1) + (5,2,1) + (5,1,2)]$	$3[(10,1,1) + (5,2,1) + (5,1,2) + (1,2,1) + (1,1,2) + 2(1,1,1)] + [(\bar{5},1,1) + (1,2,1) + (1,1,2); 14'] + 2[(10 + 5), 1] + 2[(\bar{5}, 2) + (2' + 1')]$
$\frac{1}{3}(1,1,2,0,\dots,0)\frac{1}{3}(0,\dots,0)$	$\frac{1}{3}(1,1,2,0,\dots,0)\frac{1}{3}(1,1,2,0,\dots,0)$	$E_6 \times SU(3) \times E_6' \times SU(3)' \times E_6$	–	$3[(\bar{27}, 3; 1, 1; 1) + (1, \bar{3}, 1, 1; 27) + (1, \bar{3}, 1, 1; \bar{27}) + (\bar{27}, 1, 1'; \bar{3}; 1) + (1, 3; 27, 1; 1) + (1, 1; 27, \bar{3}, 1) + (1, 1, 1, 3; 27) + (1, 1, 1, 3; \bar{27})]$