CERN-TH.4889/87

VIRASORO-TYPE ALGEBRAS AND BRST OPERATORS ON RIEMANN SURFACES

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ABSTRACT

We use the Virasoro-type algebras recently introduced by Krichever and Novikov to define a BRST charge on Riemann surfaces of genus g . We discuss the quantum extension of this operator. We briefly sketch the potentialities of this new formalism.

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Recently there have been a few attempts at generalizing the Virasoro algebra to Riemann surfaces of non-trivial genus [1-4]. These attempts emphasize the existence over any Riemann surface of an algebra with the same characteristics as the Virasoro algebra. In Ref. [4], for example, it reduces to the latter when restricted to S^1 . A common feature of these approaches is that these algebras contain, besides the diffeomorphisms, the Teichmüller deformations.

A very interesting development in this direction can be found in two recent papers by Krichever and Novikov [5,6]. The basic ingredient in their approach is the algebra of meromorphic vector fields over a Riemann surface which are holomorphic outside two distinguished points. What seems to be new and remarkable in Refs. [5] and [6] is, first, the fact that this algebra has a universal aspect, but simultaneously contains the information of the genus through a gradation determined by the genus itself. Secondly, moreover, it permits us to treat diffeomorphisms and deformations on the same footing. A third remarkable feature is the existence of a unique discrete basis which lends itself immediately to physical applications, particularly in relation with the operator formalism.

In this paper we use the Krichever-Novikov (KN) algebra to construct, following the Fradkin-Vilkovisky formalism [7], a classical BRST charge associated to any given genus, which describes both diffeomorphisms and deformations. We then study the quantum extension, in particular the nilpotence problem. From our point of view, the existence of a genus-dependent BRST charge can be used in two different directions: either to construct a closed string field theory at fixed genus, or to define a generalization of conformal field theory to Riemann surfaces of genus $g \neq 0$. These two applications will be briefly discussed at the end of the paper.

Let us summarize now the aspects of Refs. [5] and [6] that will be used in this work. On a surface Σ of genus $g \geq 2$, let us consider two distinguished (but generic) points P_+ and P_- with local co-ordinates z_+ and z_- , such that $z_{\pm}(P_{\pm}) = 0$, and the algebra $\mathfrak{m}^{\Sigma}$ of meromorphic vector fields on Σ which are holomorphic outside $P_{\pm}$. As a consequence of the Riemann-Roch theorem, there exists a basis of $\mathfrak{m}^{\Sigma}$ represented by the vector fields e_i which are defined and uniquely determined up to a constant by the following behaviour near P_+ and P_- :

$$e_i = a_i^{\pm} z_{\pm}^{\pm i - g_0 + 1} (1 + o(z_{\pm})) \frac{\partial}{\partial z_{\pm}} \quad (1)$$

Here $g_0 = 3g/2$ and i takes on integral values ($i = \dots, -1, 0, +1, \dots$) for even g and half-integral values ($i = \dots, -\frac{1}{2}, \frac{1}{2}, \dots$) for odd g . Moreover, we can normalize this basis by setting $a_i^{\pm} = 1$. This construction can be extended to $g = 1$.

A basic result of Ref. [5] is that $\mathfrak{m}^\Sigma$ is g_0 -graduated, namely*)

$$[e_i, e_j] = \sum_{j=-g_0}^{g_0} c_{ij}^j e_{i+j-j} \quad (2)$$

The structure constants $c_{i,j}^s$ are determined by the behaviour of the e_i 's at the points $P_\pm$.

The algebra $\mathfrak{m}^\Sigma$ can be decomposed as follows:

$$\mathfrak{m}^\Sigma = \mathfrak{m}_-^\Sigma + \mathfrak{m}_0^\Sigma + \mathfrak{m}_+^\Sigma \quad (3)$$

where $\mathfrak{m}_\pm^\Sigma$ are spanned by $\{e_i\}$ with $\pm i \geq g_0 - 1$ and are the subalgebras of the vector fields holomorphic in $P_\pm$, while $\mathfrak{m}_0^\Sigma$ is the subspace of dimension $3g-3$ for $g \geq 2$, which can be identified with the space tangent to the moduli space. In other words, the vectors e_i , $|i| \leq g_0 - 2$, play the rôle of the generators of complex deformations.

Another basic theorem of Ref. [5] tells us that there exists a unique central extension $\hat{\mathfrak{m}}^\Sigma$ of $\mathfrak{m}^\Sigma$, given by a cocycle $\chi(e_i, e_j)$ such that

$$\chi(e_i, e_j) = 0 \quad \text{for } |i+j| > 2g_0 \quad (4)$$

The extension of (2) reads

$$[e_i, e_j] = \sum_{j=-g_0}^{g_0} c_{ij}^j e_{i+j-j} + t \chi(e_i, e_j) \quad , \quad [t, e_i] = 0 \quad (5)$$

Using $\hat{\mathfrak{m}}^\Sigma$, one can define a generalization of the Verma modules, in which primary states $\phi^{(\lambda)}$ are found, characterized by

$$t \phi^{(\lambda)} = c(\lambda) \phi^{(\lambda)} \quad \text{where} \quad c(\lambda) = -2(6\lambda^2 - 6\lambda + 1) \quad (6)$$

and λ is the analogue of the conformal spin.

In the following we will need also a basis of the meromorphic quadratic differentials holomorphic outside $P_\pm$ [6]. For $g > 1$, this is given by quadratic differentials Ω_i with the following behaviour near $P_\pm$:

$$\Omega_i \sim z_{\pm}^{\mp i + g_0 - 2} (1 + O(z_{\pm})) (dz_{\pm})^2 \quad (7)$$

These can be suitably normalized so that the following orthogonality relations hold:

$$\frac{1}{2\pi i} \oint_{C_{\tau}} \Omega_i(e_j) = \delta_{i,j} \quad (8)$$

Here C_{τ} are contours defined by means of a unique differential of the third kind. They can be interpreted as representing closed string configurations on the Riemann surface at the proper time τ . As $\tau \rightarrow \pm\infty$, they become circles around $P_{\pm}$.

We are now ready to define a BRST charge associated to the KN algebra which generates both diffeomorphisms and deformations. Let us introduce on Σ the energy-momentum tensor $T(P)$, $P \in \Sigma$, represented by a quadratic differential which can be decomposed on the basis $\{\Omega_n\}$ as follows:

$$T(P) = \sum_n L_n \Omega_n(P) \quad , \quad L_n = \frac{1}{2\pi i} \oint_{C_{\tau}} T(P) e_n(P) \quad (9)$$

If $T(P)$ obeys the commutation rules^{*)}

$$[T(P), T(P')] = D_{\tau}(P, P') \cdot T(P') \quad , \quad P, P' \in C_{\tau} \quad (10)$$

where

$$D_{\tau}(P, P') = \sum_n e_n(P) \Omega_n(P') \quad (11)$$

then the coefficients L_n satisfy the algebra (2), as one can easily verify by means of Eq. (8).

Next, following Refs. [7] and [8], we introduce two anticommuting fields: $B(P)$, which is again a quadratic differential, and $\eta(P)$, which is a vector field,

$$B(P) = \sum_n b_n \Omega_n(P) \quad b_n = \frac{1}{2\pi i} \oint_{C_{\tau}} B(P) e_n(P) \quad (12)$$

$$\eta(P) = \sum_n \eta_n e_n(P) \quad \eta_n = \frac{1}{2\pi i} \oint_{C_{\tau}} \eta(P) \Omega_n(P) \quad (13)$$

and assume the anticommutation rules:

$$\{\eta(P), B(P')\} = D_\tau(P, P') \quad (14)$$

Then

$$\{b_n, \eta_k\} = \delta_{n,k} \quad (15)$$

Our definition of the classical BRST charge Q is

$$\begin{aligned} Q &= \frac{1}{2\pi i} \oint_{C_\tau} \left(T(P)\eta(P) - \frac{1}{2} B(P) \{\eta(P), \eta(P)\} \right) = \\ &= \sum_n \left(L_n \eta_n - \frac{1}{2} \sum_k \sum_{j=-g_0}^{g_0} c_{n,k}^j b_{n+k-j} \eta_n \eta_k \right) \end{aligned} \quad (16)$$

It is easy to verify, using Eqs. (2) and (15) and the Jacobi identity for the structure constants $c_{n,k}^s$, that

$$Q^2 \equiv \{Q, Q\} = 0 \quad (17)$$

In order to obtain the quantum extension $\hat{Q}$ of Q , we have to consider a specific field model over Σ . Henceforth we refer to the closed bosonic string energy-momentum tensor proposed by Krichever and Novikov [6]. They have shown that $:L_n:$ satisfy the algebra (5) with central extension

$$[X(:L_n:, :L_m:)] = d \chi_{n,m} \quad (18)$$

where d is the dimension of the flat target space and $\chi_{n,m}$ is a two-cocycle independent of the dimension of the target space. We now define

$$\hat{Q} = :Q: \quad (19)$$

and the normal ordering for the ghost part is defined as follows: η_k with $k \leq 0$ ($k > 0$) are to be placed on the right (on the left), while $b_k > 0$ ($k \leq 0$) are to be kept on the right (left). With this prescription, we find

$$\{\hat{Q}, \hat{Q}\} = \sum_{i,j} (d \chi_{i,j} + \tilde{\chi}_{i,j}) : \eta_i \eta_j : \quad (20)$$

where $\tilde{\chi}_{i,j}$ is the ghost contribution and can be written as follows:

$$\tilde{\chi}_{i,j} = \sum_{\substack{r,s=-g_0 \\ i+j=r+s}}^{g_0} \left(\sum_{\ell \leq 0} c_{i,\ell}^s c_{j,i+\ell-s}^r \theta_+(i+\ell-s) - \sum_{\ell > 0} c_{i,\ell}^s c_{j,i+\ell-s}^r \theta_-(i+\ell-s) \right) \quad (21)$$

where $\theta_+(n) = 0 \quad n \leq 0, = 1 \quad n > 0$, $\theta_-(n) = 0 \quad n > 0, = 1 \quad n \leq 0$.

Let us analyze now the properties of $\tilde{\chi}_{i,j}$. If $i > 0, j \leq 0$, then by definition $\tilde{\chi}_{j,i} = -\tilde{\chi}_{i,j}$, while, in the sectors $i > 0, j > 0$ and $j \leq 0, i \leq 0$, $\tilde{\chi}_{i,j}$ is antisymmetric, although this property is not apparent in the concise form (21). Because of the constraint $i+j = r+s$ in the summations (21), where as usual $-g_0 \leq r, s \leq g_0$, one obtains

$$\tilde{\chi}_{i,j} = 0 \quad \text{for} \quad |i+j| > 2g_0 \quad (22)$$

Moreover, for $g = 0$ we get $c_{i,j}^s = (j-i)$ and $\tilde{\chi}_{i,j}$ reduces to

$$\tilde{\chi}_{i,j} = -\frac{1}{6} (13i^3 - i) \delta_{i,-j}$$

as expected.

We would now like to be able to conclude that, up to trivial cocycles,

$$\tilde{\chi}_{i,j} = -26 \chi_{i,j} \quad (23)$$

for any genus g , as it holds for $g = 0$. To this end, we have to prove that $\tilde{\chi}_{i,j}$ is a central extension for the KN algebra. Antisymmetry and "locality", Eq. (22), have already been shown. What is left to prove is the Jacobi identity. Once this is done, the property (22) and the uniqueness theorem quoted in correspondence with Eq. (4) tell us that $\tilde{\chi}_{i,j}$ must be proportional, up to trivial cocycles, to $\chi_{i,j}$.

The Jacobi identity is rather complex, so we write it in a compact form, as follows:

$$\begin{aligned} & \frac{1}{2\pi i} \oint_{C_\tau} \oint_{C_\tau} \left(\left\{ \eta(P), [D_\tau(P, P'), \{\eta(P), \eta(P)\}]_+ \right\} D_\tau^-(P', P) - \right. \\ & \left. - \left\{ \eta(P), [D_\tau(P, P'), \{\eta(P), \eta(P)\}]_- \right\} D_\tau^+(P', P) \right) = 0 \quad (24) \end{aligned}$$

where $[]_{\pm}$ means the commutator with values in the subspace spanned by the $e_n(P)$ with $n > 0$ or $n \leq 0$ respectively. Moreover, $D_{\tau}^{\pm}(P, P')$ is defined as in Eq. (11) where the summation extends to the $n > 0$ or $n \leq 0$ respectively.

A direct numerical proof of Eq. (24) is being carried out. In this letter, we content ourselves with a general argument for the validity of Eqs. (23) and (24).

The KN algebra can be restricted to any C_{τ} , giving rise to an algebra which, for large enough τ , is isomorphic to the Virasoro algebra. Even more, this restriction is dense in the subalgebra of vector fields which can be holomorphically extended to an annulus containing C_{τ} . Let us denote by $\bar{e}_i$ the restriction of e_i to C_{τ} . Then, if e_n are the generators of the Virasoro algebra, we can express $\bar{e}_i$ as

$$\bar{e}_i = \sum_n \alpha_i^n e_n$$

The algebra of the $\bar{e}_i$'s is

$$[\bar{e}_i, \bar{e}_j] = \sum_{n=-g_0}^{g_0} c_{i,j}^n \bar{e}_{i+j-n} + t \chi(\bar{e}_i, \bar{e}_j)$$

where $\chi(\bar{e}_i, \bar{e}_j) = \chi(e_i, e_j)$. Now the restriction of the $\hat{Q}^2$ is

$$\hat{Q}^2 = \sum_{i,j} (d \chi_{i,j} + \tilde{\chi}_{i,j}) : \bar{\eta}_i \bar{\eta}_j : = \sum_{i,j} (d-26) \lambda_{i,j} : \xi_i \xi_j : \quad (25)$$

where $\xi_i = \sum_k \alpha_i^k \bar{\eta}_k$ are ghost operators corresponding to the Virasoro algebra and $\lambda_{i,j}$ is the Virasoro central extension. From Eq. (25), it follows that $\tilde{\chi}_{i,j} = -26 \chi_{i,j}$. As a consequence of this argument, we can conclude that $\hat{Q}$ is nilpotent in $d = 26$. It follows that in field theory over Riemann surfaces there are no anomalies but the universal ones.

Let us come to possible applications of this result. First, we can regard the Riemann surface of genus g with two punctures as the g -th loop-order propagator for interacting closed strings. Having defined a nilpotent BRST charge $\hat{Q}$, we can think of this propagator as the one implied by the following action [9]:

$$S = (\Phi_0 | \hat{Q} \oplus \hat{Q}^* | \Phi_{-1}) \quad (26)$$

Here $\hat{Q}^*$ is the complex conjugate of Q in the sense of Ref. [6], $\Phi_0(\Phi_{-1})$ is a closed

string functional with ghost number 0 (-1) and (|) denotes a suitable pairing. Of course, in order to obtain the propagator, one must fix the large gauge symmetry of (26).

Another application concerns a generalization of conformal field theory on a Riemann surface of genus g . Indeed, by using the charge $\hat{Q}$ we can write down Ward identities for diffeomorphisms and deformations. We simply have to insert the equation

$$S_\eta \phi = [\hat{Q}, \phi]$$

where ϕ is, e.g., a primary field, into a correlation function and express the left-hand side by means of the formalism of Ref. [5]. Due to Eq. (3), we are going to find distinct contributions from the diffeomorphisms and deformations [10].

ACKNOWLEDGEMENTS

We thank E. Arbarello, M. Cornalba, E. Miller, C. Reina, J. Stasheff, R. Stora and M. Tonin for useful discussions and N. Sanchez for pointing out to us Refs. [5] and [6]. One of us (P.C.-R.) would like to thank J. Stasheff for having supported his stay at the Institute for Advanced Study (Princeton) during the elaboration of this paper.

FOOTNOTE

*) The reader should keep in mind that the geometric brackets of vector fields, the classical Poisson brackets and the quantum commutators are indicated by the same commutator symbol.

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