CERN-TH-4850/87

New Methods and Results in Conformal QFT_2 and the "String Idea"

Bert Schroer, FU Berlin

Institut für Theorie der Elementarteilchen

Fachbereich Physik

Arnimallee 14, D-1000 Berlin 33

Lecture Notes, Cargèse, July 1987

Abstract

Causal fields in conformal QFT_2 yield a new algebraic structure: the exchange algebra in which "braid" matrices (special Yang-Baxter structures) appear. They are directly related to the possible dimensional spectra of local fields. The Virasoro structure, i.e. the central extension charge (= Casimir energy) is part of the representation theory. An extension of Unruh's idea allows us to calculate correlation functions of arbitrary conformal QFT_2 on higher genus Riemann surfaces directly in terms of the flat space correlations. As a generalization of Hawking's temperature, the new positive definite states are characterized by $3g-3$ "temperatures".

CERN-TH-4850/87

September 1987

I. Introduction and History

Conformal QFT was introduced¹ in the 60's. However, the field theory community in those days largely ignored or rejected conformal invariance since apart from free fields it seemed to be at odds with the principle of Einstein causality. In those days the consequences of relativistic causality combined with assumptions on the energy-momentum spectrum were the main research topic. Apart from certain structural theorems^{2,3} such as TCP and the connection of spin with statistics, the main experimentally testable results were dispersion relations and certain spectral sum rules.

At the beginning of the 70's, when the intimate relation between relativistic quantum field theory and statistical mechanics through the method of analytic continuation ("Euclidean quantum field theory") became widely appreciated, the issue of conformal invariance reappeared on the statistical mechanics side, where the "causality paradox" created less of a nuisance. In this spirit of statistical mechanics Migdal⁴ and Polyakov⁵ proposed the so-called "conformal bootstrap", i.e. the study of the Euclidean Schwinger-Dyson equations with conformal invariant boundary conditions. This programme was mathematically refined with the help of group theoretical ideas and techniques.⁶ However, even with all these refinements it remained too structureless as far as non-perturbative solutions are concerned. On the conceptual side the already mentioned paradox remained a sore, particularly in view of the failure of all attempts (even up to our present days) to soften the postulate of Einstein causality i.e. to replace it by a less stringent (macro) causality postulate. The causality paradox of global conformal transformation was clearly formulated in 1972⁷. After its partial understanding in a very limited setting⁸, it was finally solved in a series of papers^{9,10,11}.

The discovery of a classification scheme for conformal QFT₂ by Belavin, Polyakov and Zamolodchikov¹² (BPZ) with additional important later contributions^{13,14,15} did not rely on any of the global operator methods of the 70's and therefore one may ask why should one be concerned with the "old stuff". The answer is very simple. It has recently turned out that the old methods which are "global" in nature are mathematically as well as conceptually more powerful than the methods¹⁶ of Kac-Feigin-Fuchs-BPZ which are based on an analysis of the Virasoro algebra and

differential equations for the Euclidean correlation functions¹⁵. Not only can one rederive known results concerning, i.e. the quantization of c values for $c < 1$, the affiliated dimensional spectrum and the 3-point normalizations with ease¹⁷ (no Kac-determinant needs to be calculated!), but one finds a new algebra, the “exchange algebra” of the “intertwiners”^{18,19} of the irreducible representation sectors of the energy-momentum tensor (a similar statement holds for loop groups if one uses the current operators). The unitary (positivity) of the field theory turns out to be equivalent to the reality of the (computable) 3-point normalization¹⁷. This was of course already conjectured in the days of the “conformal bootstrap”. At this Summer School I learned from Jörg Frölich and Alain Connes that the “exchange matrices” appearing in this new algebra have the structure of “braids” related to Yang-Baxter matrices and that the eigenvalues of these matrices are connected with cohomology-invariants of certain infinite dimensional (factor) von Neumann algebras. I find it particularly pleasing that all this mathematical and physical wealth appears straight out of the resolution of the “conformal Einstein causality paradox” thus underlining again the great value of deep physical paradoxes.

Most of the analytical insight into quantum field theory, viz. $d=2$ integrable systems, did not come from the Euclidean functional integral approach. The beauty of the Feynman integral and its suggestive power for approximations remains in stark contrast with its analytic unmanageability. This is a regrettable and perhaps only transitory fact of life in theoretical physics. We therefore propose the thesis that also the “string idea” should be investigated by the global operator method rather than by functional integrals. As a prelude to an operator string approach I will discuss the recently proposed technique of constructing expectation values on Riemann surfaces via the “Hawking-Unruh” averaging^{19,20} using Fuchsian groups. This method is not only simpler than the Euclidean approach based on infinite Grassmannians²¹ and Valatin-Bogoliubov-transformations, but it also seems to give more physical insight.

The "Causality paradox" of conformal QFT and its resolution

The causality paradox in conformally invariant QFT is the statement that the simple-minded equivariant conformal transformation law which is obtained by formally integrating up the infinitesimal (Lie-algebra) transformation is in contradiction with Einstein causality of relativistic QFT.

Theorem⁷: In any local field theory with "reverberation" i.e. non-vanishing timelike commutators (for fermions: anticommutators) the global transformation law:

$$U(b) A(x) U^\dagger(b) = (\sigma(b, x))^{-d_A} A\left(\frac{x - bx^2}{\sigma(b, x)}\right)$$

$$\sigma(b, x) = 1 - 2bx + b^2x^2 \quad (1.1)$$

(equivariance under special conformal transformations) is inconsistent with causality:

$$[A(x), A(y)] = 0, \quad (x-y)^2 < 0 \quad (1.2)$$

For the proof one takes a special conformal transformation which transforms the spacelike (1.2) into a timelike separation (a continuous path of such transformations goes through a zero of $\sigma(b, x)$). Since the transformed field apart from its changed argument only contains a c-number in front (or matrix of c-numbers in case of a general field), there is a contradiction with the "reverberation" assumption

To make a long story short^{9,10,11}, the resolution comes from the recognition that the conformal group in Minkowski space is not simply connected. There is a smallest not contractible path c and the corresponding centre element is represented by a non-trivial unitary operator, which apart from the special case $A=0$ energy momentum tensor, does not commute with the local field:

$$Z^n A(x) Z^{-n} \doteq A(x, n) = A_n(x)$$

$$Z = U(\text{minimal non-contr. path}) \quad (1.3)$$

G. Mack introduced the picturesque terminology “heavens and hells” for the index n which ranges over positive and negative integers. A good mathematical formalism is obtained by diagonalizing the unitary operator⁹ (the “line bundle approach”)

$$\text{spec}(\mathcal{Z}) = \{e^{-2\pi\theta i}\}$$

With

$$A_{\xi}^d(x) = \sum_{\xi} \mathcal{Z}^n A^d(x) \mathcal{Z}^{+n} e^{i\pi n(d-2\xi)} \quad (1.4)$$

we obtain the global decomposition of the causal field:⁹

$$A^d(x) = \sum_{\xi} A_{\xi}^d(x) \quad (1.5)$$

The notation for the phase normalization convention is that of reference 9. The irreducible components obey the equivariant law:

$$U(b) A_{\xi}^d(x) U^{\dagger}(b) = \left(\frac{1}{\sigma_{+}(b,x)}\right)^{d-\xi} \left(\frac{1}{\sigma_{-}(b,x)}\right)^{\xi} A_{\xi}^d\left(\frac{x-bx^2}{\sigma(b,x)}\right) \quad (1.6)$$

Here the suffix $\pm$ indicates the boundary value obtained via the Wightman prescription of adding a small positive imaginary time-like ε -component to the vectors x and b and then taking the boundary value $\lim_{\varepsilon \rightarrow 0} \dots$. From the spectrum condition it immediately follows that:

$$A^d(x) |0\rangle = A_{\xi=0}^d(x) |0\rangle, \quad A_0^{d\dagger} = A_{\xi=d}^d \quad (1.7)$$

For the free fields⁸ these two ξ -components are just the (non-local) creation and annihilation parts. Therefore it is reasonable to view the ξ -decomposition (1.5) as a subtle generalization of creation and annihilation operators provided by the global aspects of the conformal group. In order to

learn more about the physical interpretation of ξ we consider 3-point functions of A with two other (composite) fields.

$$\langle 0 | C A B | 0 \rangle = \langle 0 | C_{d_c} A_{\xi} B | 0 \rangle \quad (I.8)$$

and obtain⁹:

$$\xi = \frac{1}{2} (d_A + d_B - d_c) \quad (I.9)$$

.e. the ξ -spectrum contains all the linear combinations of dimensions of those conformally equivariant composite fields which communicate with A via non-vanishing 3-point functions. We will call these fields A, B, C... "quasiprimary"¹². Assuming that these composite fields form a complete set in the Hilbertspace yields the statement that formula (I.9) exhausts the possible ξ -spectrum. In higher point functions the global decomposition (I.5) leads to a sum over ξ -values, i.e.:

$$\langle A B C D \rangle = \sum_{\xi} \langle A_{d_A} B_{\hat{\xi}(\xi)} C_{\xi} D_0 \rangle \quad (I.10)$$

In order to obtain a more detailed insight into the conformal decomposition theory (1.5) as a result of the correct global equivariance of local fields, we now specialize to two dimensions in which case the group theory of the finite-dimensional conformal group simplifies. Adopting the standard notation:

$$u = x + t \quad v = x - t \quad (I.11)$$

for the two lightcone co-ordinates, the 6-dimensional conformal group:

$$PSL(2, R)_L \otimes PSL(2, R)_R \quad (I.12)$$

acts as a $SL(2, \mathbb{R})$ - linear fractional transformation (Möbius-transformation) on the u - v light-cone co-ordinates. Here we ignore all reflection including the physical reflection (T,P,TCP) and the conformal reflection $:x \rightarrow \frac{z}{x^2}$ which intertwine between the two light-cones. On each side one conveniently uses the Iwasawa decomposition¹⁰:

$$SL(2, \mathbb{R}) = M = K A N$$

$$K = \left\{ \begin{pmatrix} \cos \frac{\alpha}{2} & \sin \frac{\alpha}{2} \\ -\sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{pmatrix} \right\} \quad (I.13)$$

$$A = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \right\}, \quad N = \left\{ \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \right\}$$

The existence of the maximal compact subgroup K with its non-trivial universal covering: (not representable by a matrix group)

$$\tilde{K} = \{ \alpha, \alpha \in \mathbb{R} \} \quad (I.14)$$

was the origin of the clash between causality and the “naive” equivariance law (I.1).

The correct equivariant transformation law for local fields is most elegantly written in terms of the parametrization in which the light-cone co-ordinates are obtained by applying the K subgroup to the origin on each light-cone i.e.:

$$u = \frac{\cos \beta/2 \cdot 0 + \sin \beta/2}{-\sin \beta/2 \cdot 0 + \cos \beta/2} = \tan \beta/2, \quad v = \tan \bar{\beta}/2 \quad (I.15)$$

With this notation the correct equivariance law under $\tilde{K}_L$ reads as follows:

$$U(\alpha) A^{d, \bar{d}}(\beta, \bar{\beta}) U^+(\alpha) = \left| \frac{1}{\cos \frac{\alpha+\beta}{2}} \right|^d A^{d, \bar{d}}(\beta+\alpha, \bar{\beta}) \quad (I.16)$$

Here the field at the new variable $\beta, \bar{\beta}$ is just an abbreviation for (1.3), i.e.

$$A^{d, \bar{d}}(\beta, \bar{\beta}) \doteq Z_L^{[\beta]} Z_R^{[\bar{\beta}]} A^{d, \bar{d}}(u, v) Z_L^{+[\beta]} Z_R^{+[\bar{\beta}]} \quad (I.17)$$

$[\beta]$ = biggest integern in representation:

$$\beta = n\pi + \beta_{red}, \quad |\beta_{red}| < \pi$$

The equivariance law under $U(\bar{\alpha})$ is analogous. Note that the non-trivial “reverberation” aspect is now encoded in the non-triviality ($Z \neq \lambda 1$) of the central generator (I.3). The non-trivial part of the conformal equivariance law (1.14a) together with the trivial laws (which we only write for the left light-cone)

$$U(\lambda) A^{d, \bar{d}}(u, v) U^+(\lambda) = \lambda^d A^{d, \bar{d}}(\lambda u, v) \quad (I.18)$$

$$U(\omega) A^{d, \bar{d}}(u, v) U^+(\omega) = A^{d, \bar{d}}(u + \omega, v)$$

yield upon diagonalization of Z (I.5) the irreducible conformal equivariance properties:

$$Z A_{\xi \bar{\xi}}^{d, \bar{d}}(u, v) Z^+ = e^{-2i\pi(d-\xi)} A_{\xi \bar{\xi}}^{d, \bar{d}}(u, v). \quad (I.19a)$$

$$U(\lambda) A_{\xi \bar{\xi}}^{d, \bar{d}}(u, v) U^+(\lambda) = \lambda^d A_{\xi \bar{\xi}}^{d, \bar{d}}(\lambda u, v) \quad (I.19b)$$

$$U(\omega) A_{\xi \bar{\xi}}^{d, \bar{d}}(u, v) U^+(\omega) = A_{\xi \bar{\xi}}^{d, \bar{d}}(u + \omega, v) \quad (I.19c)$$

$$U(b(\gamma)) A_{\xi \bar{\xi}}^{d, \bar{d}}(u, v) U^+(b(\gamma)) = (1 + \gamma u)^{-(2d, \xi)} A_{\xi \bar{\xi}}^{d, \bar{d}}\left(\frac{u}{1 + \gamma u}, v\right) \quad (I.19d)$$

where

$$(1 + \gamma u)^{(2d, \xi)} = (1 + \gamma u + i\varepsilon)^{-2d + \xi} (1 + \gamma u - i\varepsilon)^{-\xi}$$

Note that the dimension appearing in the formula (I.6) for the scalar field ($d = \bar{d}$) is the total dimension $d + \bar{d} = 2d$.

It is clear that the Einstein causality requirement (1.2) in the spacelike region $u.v. > 0$ will severely restrict the algebraic light-cone structure of the $A_{\xi\bar{\xi}}^{d,\bar{d}}(u, v)$ appearing in the global decomposition.

$$A_{\xi\bar{\xi}}^{d\bar{d}}(u, v) = \sum_{\xi\bar{\xi}} A_{\xi\bar{\xi}}^{d\bar{d}}(u, v) \quad (I.20)$$

In a moment we will see that these irreducible fields $A_{\xi,\bar{\xi}}^{d,\bar{d}}$ factorize, i.e.

$$A_{\xi\bar{\xi}}^{d\bar{d}}(u, v) = A_{\xi}^d(u) \cdot A_{\bar{\xi}}^{\bar{d}}(v) \quad (I.21)$$

The factors will turn out to satisfy a new algebra: the “exchange algebra” in which representation matrices of “braids” will appear. This new algebraic structure is not only sufficient to ensure Einstein causality, but it also seems to be necessary.

We already know the physical interpretation of the new field labels ξ and $\bar{\xi}$. Specializing the two- and three-point analysis of (1.7) and (1.8) to the two-dimensional light-cone split situation one easily obtains:

$$\xi = 1/2 (d_A + d_B - d_C) \quad ; \quad \bar{\xi} = 1/2 (\bar{d}_A + \bar{d}_B - \bar{d}_C) \quad (I.22)$$

where the $d_{B,C}(\bar{d}_{B,C})$ are the (left-right) dimensions of arbitrary (composite) fields which communicate with the field A via non-vanishing 3-point functions. Again the value $\xi, \bar{\xi} = 0$ occurs as a special value of the ξ -spectrum. This value characterizes the “creation” part of the new set of operators. The $A_{0,0}^{d,\bar{d}}$ with its simplified transformation property ($d = h \equiv d - \xi$ in (1.16)) creates the conformal blocks of BPZ and constitutes the only operator which has been used in the Kac-Feigin-Fuchs infinitesimal Virasoro- (or loop)-algebra representation approach. In order to see that all ξ values occur inside composite 3-point functions, we had to assume that the “quasi-primary” (i.e.

Möbius) fields (1.16) generate a complete set of states. Under the same assumption one obtains the validity of the global vacuum expansion for quasiprimary fields:

$$A_{\xi, \bar{\xi}}^{d_1, \bar{d}_1}(u_1, v_1) A_{0,0}^{d_2, \bar{d}_2}(v_2, v_2) |0\rangle = \sum_{d_3, \bar{d}_3} 2 \exp i\pi(d_3 - \bar{d}_3) c_{312} \int du dv K(d_i, \bar{d}_i; u, v) A_{0,0}^{d_3, \bar{d}_3}(u_3, v_3) |0\rangle \quad (1.23)$$

where the sum extends over all quasiprimary fields with dimensions

$$d_3 = d_1 + d_2 - \xi \pmod{\mathbb{Z}} ; \quad \bar{d}_3 = \bar{d}_1 + \bar{d}_2 - \bar{\xi} \pmod{\mathbb{Z}} \quad (1.24)$$

and the kernels have the factorized form

$$K(d_i; \bar{d}_i; u_i, v_i) = K(d_i; u_i) \cdot \bar{K}(\bar{d}_i, v_i) \quad (1.25)$$

$$K(d_i; u_i) = \frac{\Gamma(2d_3) \Gamma(\lambda_3) \Gamma(1-\lambda_2)}{2\pi (u_1 - u_2 - i\varepsilon)^{\lambda_1} (u_1 - u_3 - i\varepsilon)^{\lambda_2 - 1 + 2d_3} (u_2 - u_1 + i\varepsilon)^{1 - 2d_3} (u_2 - u_3 + i\varepsilon)^{\lambda_3}} \quad (1.26)$$

$$\lambda_1 = d_1 + d_2 + d_3 - 1, \quad \lambda_2 = d_1 - d_2 - d_3 + 1, \quad \lambda_3 = -d_1 + d_2 - d_3 + 1$$

With this choice of normalization the global vacuum expansion reproduces the 3-point function:

$$\langle A^{d_3, \bar{d}_3}(u_3, v_3) A^{d_1, \bar{d}_1}(u_1, v_1) A^{d_2, \bar{d}_2}(u_2, v_2) \rangle = \frac{c_{312}}{(u_3 - u_2 - i\varepsilon)^{d_3 + d_1 - d_2} (u_3 - u_2 - i\varepsilon)^{d_3 + d_2 - d_1} (u_1 - u_2 - i\varepsilon)^{d_1 + d_2 - d_3}} (v, \bar{d}) \quad (1.27)$$

where $(v, \bar{d})$ indicates similar factors referring to the v-light-cone. In the calculation we use the 2-point normalization:

$$\langle A_1(u_1, v_1) A_2(u_2, v_2) \rangle = \delta(A_1 = A_2 = A^{d, \bar{d}}) \frac{e^{-i\pi(d - \bar{d})}}{(u_1 - u_2 - i\varepsilon)^{2d} (v_1 - v_2 + i\varepsilon)^{2\bar{d}}} \quad (1.28)$$

Within the assumed completeness assumption of quasi-primary fields, this calculation of the reproducing property of the 3-point function is at the same time the proof for the validity of the global expansion. In contradistinction to the Wilson expansion (which also holds “off vacuum”), it has the group theoretical status of a Plancherel formula¹⁰. Note that as a result of (I.21), which in turn results from the u-v factorization of the 3-point function, the n-point function of the irreducible fields (not the local field!) $A_{\xi, \bar{\xi}}^{d, \bar{d}}$ factorize. This follows from an iterative application of the global decomposition (I.17). This in turn means that the operator factorization (I.18) holds. Note that here and again the power of the Wightman framework for structure arguments as opposed to any ordered expectation values (which are useful in perturbation theory) becomes evident. To the educated reader it should also be clear that the sectors generated by diagonalizing the global operators Z_L and Z_R are identical to the conformal blocks of BPZ. However in obtaining them by the global analysis of the “Einstein causality paradox”, we receive a vast amount of new insight which hitherto was still missing. Let us first try to reproduce and extend known results. The quasiprimary fields come in families¹⁷; because one quasiprimary field globally expands with the energy momentum tensor $T(u)$, $T(v)$ à la (I.20) will automatically generate higher quasiprimary fields $A^{(n)}$

$$T(u_1) A(u_2, v_2) |0\rangle = \sum_n \int du' K^{(n)}(u_1 - u', u_2 - u') A^{(n)}(u', v) |0\rangle \quad (1.29)$$

The beauty and simplicity of the global expansions is that they only deal with quasiprimary fields (and not with their derivatives). It is reasonable to assume that there exists a lowest quasiprimary

field which we will later show to be the primary field of BPZ. Hence we write:

$$c_{312} = c_{312}^{prim} N_{312}^{n_3 n_1 n_2} \bar{N}_{312}^{\bar{n}_3 \bar{n}_1 \bar{n}_2} \quad (I.30)$$

where the N's are the relative normalizations within one family. The hypergeometric techniques which lead from the global expansion to the 4-point formula:

$$\begin{aligned} \langle A_1(x_1) A_2(x_2) A_3(x_3) A_4(x_4) \rangle &= \frac{\left(\frac{u_2-u_4}{u_1-u_4}\right)^{d_1-d_2} \left(\frac{u_1-u_3}{u_2-u_3}\right)^{d_4-d_3}}{(u_1-u_2)^{d_1+d_2} (u_3-u_4)^{d_3+d_4}} \times \frac{\left(\frac{v_2-v_4}{v_1-v_4}\right)^{\bar{d}_1-\bar{d}_2} \left(\frac{v_1-v_3}{v_2-v_3}\right)^{\bar{d}_4-\bar{d}_3}}{(v_1-v_2)^{\bar{d}_1+\bar{d}_2} (v_3-v_4)^{\bar{d}_4+\bar{d}_3}} \times \\ &\sum_{h_0, \bar{h}_0} \exp i\pi(h_0-\bar{h}_0) c_{120}^{prim} c_{034}^{prim} F_{h_0}(x) \bar{F}_{\bar{h}_0}(\bar{x}) \end{aligned} \quad (I.31)$$

with

$$F_{h_0}(x) = x^{h_0} \sum N_{120}^{n_1 n_2 0} N_{034}^{n_0 n_3 n_4} (-x)^{n_0} {}_1F_2(d_2-d_1+d_0, d_3-d_4+d_0; 2d_0)$$

$$x = \frac{(u_1-u_2)(u_3-u_4)}{(u_1-u_3)(u_2-u_4)} \quad d_i = (h_i + n_i) \quad i=1,2,3,4 \quad (I.32)$$

Such formulae are quite old. For special cases (vertex-operator) they appear already in an Appendix of reference 9. They were derived in BPZ and subsequent papers by resummation techniques¹³ starting from Wilson expansion.

Specializing these formulae to the 4-point function: and using the relation

$$\langle A^{h+m, \bar{d}}(x_1) T(u_2) T(u_3) A^{h+n, \bar{d}}(x_4) \rangle \quad (I.33)$$

$$T(u_2) T(u_3) = \frac{c/2}{(u_1-u_3)^4} + \frac{T(u_1) + T(u_3)}{(u_2-u_3)^2} \quad (I.34)$$

+ reg. terms at $u_2 = u_3$

one obtains¹⁷ the statement

$$\text{rat. funct}_N(h) \prod_{p,q=N} (h - h_{p,q}(c)) \geq 0 \quad \text{all } N > 0 \quad (I.35)$$

Applying the Friedan, Qiu, Shenker¹³ argument to this quantity¹⁷ (which is simpler than the Kac determinant), one obtains the desired quantization of c as well as the Kac formula for the dimensions. Furthermore these global techniques allow us to show the absence of ghosts at these quantized values. They also lead to the determination of the multiplicities of quasiprimaries as well as their relative normalization N . For the details of the calculation we refer to a recent publication¹⁷. If all 4 fields are non-canonical, the generalized global expansion away from the vacuum state replaces the formula (I.34). The discussion becomes more subtle since the form of this new global expansion depends on the structure of the exchange algebra and will be presented in a forthcoming paper.

The derivation of the c -quantization resulting from unitarity based on the global expansion technique is in the spirit of an unpublished (1976) paper by Lüscher and Mack as quoted by Todorov²², in which only the region $c < \frac{1}{2}$ was excluded and as yet no complete quantization picture was derived. In passing we would like to mention that in the work of Parisi et al.²³ one finds precursors of the global technique which we used in this derivation.

II. The role of the Diffeomorphism group of the circle and the relation of the operator approach to the BPZ formulation

One of the marvellous discoveries of BPZ¹² is the suggestion that there exist "primary" fields, i.e. local fields which have rather simple (infinitesimal) transformation properties under $Diff_L(R) \times Diff_R(R)$. Consider a conformally invariant classical action, i.e.

$$\frac{1}{2} \int \partial_\mu \varphi \partial^\mu \varphi \, d^2x = \int \partial_u \varphi \partial_v \varphi \, du \, dv \quad (II.1)$$

This is apparently invariant under diffeomorphism:

$$u \rightarrow f(u) \quad , \quad v \rightarrow g(v)$$

Barring anomalies and a spontaneous breakdown à la Nambu and Goldstone, in which case the infinitesimal generators would not exist as a result of bad large-distance behaviour (which is certainly absent in the present case), one should expect the symmetry to be realized by unitary operators in Hilbert space à la Wigner and Weyl. There is no guarantee that the vacuum state will be invariant. In fact, we know that in conformal theories the maximum invariance group of the vacuum is the Möbius subgroup of the ∞ dimensional diffeomorphism group. Therefore this large symmetry is somewhat hidden, i.e. not directly visible in the ground state expectation values. Representation theory of groups in the quantum Hilbert space always has two aspects. One must either study the representation theory of the universal covering group (example: rotation group) or of a suitably central extension (example: Galilei group). The most prominent infinite dimensional groups in two-dimensional physics are the loop group and the diffeomorphism group of the light-cone. The former is simply connected and has a non-trivial central extension, whereas the latter has non-trivial covering as well as a central extension. In both cases it is convenient to use the circular

instead of the light-cone description which is obtained by the isomorphism:

$$u \longrightarrow z = \frac{1 - iu}{1 + iu} \quad v \longrightarrow \bar{z} = \frac{1 - iv}{1 + iv} \quad (II.2)$$

It is physically quite obvious, and can be rigorously proved²³, that the only culprit for the non-simple connectedness is still the rigid Abelian subgroup K contained in the diffeomorphism group. Therefore the tilde in $\widetilde{diff}$ is physically related to causality and the subtle generalization of annihilation and creation operators in (I.5) i.e. to the phase spectrum (dimensional spectrum mode Z). But what is the physical interpretation of the central extension of the diffeomorphism group? Consider the outer automorphism of $SL(\widetilde{2}, R)$ ¹⁹

$$\widetilde{G}_n : h_n^{-1} SL(\widetilde{2}, R) h_n \quad (II.3)$$

with $h_n : z \rightarrow z^n$ (in circular language) being the outer automorphism of level n . For the generators of $SL(\widetilde{2}, R)$ we now use the standard notation

$$L_0 = \frac{1}{2} (P_L + S_L) ; \quad L_{\pm} = \frac{1}{2} (S_L - P_L) \pm i D_L \quad (II.4)$$

where S_L is the light-cone component of the generator of the special conformal transformation (I.6), and D_2 generates the left-handed Lorentz transformation (=scale transformation). L_0 is the generator of the compact subgroup $\tilde{K}$. Defining

$$L_0^{(n)} = U^+(h_n) L_0 U(h_n) \quad (II.5a)$$

$$\hat{L}_n = U^+(h_n) L_{\pm} U(h_n) \quad (II.5b)$$

one finds from the required maximal Abelianess of $L_0, 1$:

$$L_0^{(n)} = \frac{1}{n} L_0 + c_n \mathbb{1} \quad (II.6)$$

i.e. the outer automorphism h_n to a "Casimir energy" of the ground state. With

$$\hat{L}_n = \frac{1}{n} L_n \quad (II.7)$$

one easily finds (from Jacobi identities) that the Casimir effects on different levels can be related. The result is the existence of the Virasoro algebra.

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12\pi} n(n^2-1) 1 \quad (II.8)$$

The L_n 's thus constructed generate the energy momentum tensor:

$$T(u) = \frac{1}{8\pi} \left(\frac{1}{u+i} \right)^4 \sum_n L_{-n} \left(\frac{u-i}{u+i} \right)^{n+2} \quad (II.9)$$

The c-number functions are a complete set of eigenfunctions of $L_0 = \frac{1}{2}(P+S)$ on the u-line; here L_0 is considered as a ("first quantized") differential operator. Only for $c < 1$ are the tilde in $\widetilde{diff}_{cent}$, the dimensional spectrum and the Casimir effect corresponding to the suffix "cent" rigidly related. Later we will see that the dimensional spectrum in a local conformal QFT_2 is directly related to the algebraic structure of an "exchange algebra". The value of c on the other hand is related to the representation theory. Our "global technology" of the previous section did not use directly the existence of $\widetilde{diff}_{cent}$ and the Virasoro algebra. However, to obtain a better understanding of the concepts behind this formalism, it is indispensable to have $\widetilde{diff}_{cent}$ at the back of one's mind. Only by being aware of the existence of objects which under all G_n transform equivariantly (these are the primary fields¹⁹

$$U(g_n) A_{\xi\bar{\xi}}^{d\bar{d}}(u,v) U^+(g_n) = |g'_n|^d e^{-2i\pi(d-\xi)\theta} A_{\xi\bar{\xi}}^{d\bar{d}}(g_n(u),v) \quad (II.10)$$

where $\theta(u, g_n)$ is a step function which is one in the region where g is negative) does one obtain a profound conceptional insight. Quasiprimary fields on the other hand have additional additive contributions involving lower quasiprimary fields as well as polynomial expressions in the Schwarz derivative of g multiplied by ordinary derivatives of g . Typical examples have been worked out in reference 18. They define the tensor calculus of the central extension, whereas the primary fields do not "feel" this extension.

In order to make contact with the analytic formalism of BPZ¹², it is helpful to recollect the analytic properties of Poincaré invariant quantum field theories and their conformal extension. For $d=2$ this is particularly simple³. As a result of the ordinary spectrum condition, the vacuum expectation values of Lorentz-covariant fields $A_i(u, v)$

$$W_{i_1 \dots i_{n+1}}(u_1, v_1; u_2, v_2 \dots; u_{n+1}, v_{n+1}) = \langle A_{i_1}(u_1, v_1) \dots A_{i_{n+1}}(u_{n+1}, v_{n+1}) \rangle \quad (I.11)$$

are boundary values of analytic functions:

$$W_{i_1 \dots i_{n+1}}(u_1, v_1; \dots; u_{n+1}, v_{n+1}) = \lim_{\varepsilon, \bar{\varepsilon} \rightarrow 0} W_{i_1 \dots i_{n+1}}(\xi_1, \bar{\xi}_1; \dots; \xi_n, \bar{\xi}_n) \quad (II.12)$$

with

$$\begin{aligned} \xi_i &= u_i - u_{i+1} & \text{Im } \xi_i &= \varepsilon_i > 0 \\ \bar{\xi}_i &= v_i - v_{i+1} & \text{Im } \bar{\xi}_i &= -\bar{\varepsilon}_i < 0 \end{aligned}$$

The analytic domain (II.3) is called the "primitive tube" $\mathcal{T}_n$. It contains some of the Euclidean points $x_i = (ix^0, \vec{x}_i)$. It can be extended by complex Lorentz transformations to the Bargmann-Hall-Wightman domain³

$$\mathcal{T}'_n = \bigcup_{A \in L_n(\mathbb{C})} A \mathcal{T}_n \quad (II.13)$$

which also has the name “extended tube”. The analytic functions are still univalued in this region. In the case at hand the complex $L(C)$ transformations just rotate the upper u -plane by an angle, whereas the lower v -plane is acted upon by the opposite rotation³. Einstein Causality (local (anti)commutativity) enlarges the analytic domain to “permuted extended tube”. This domain in which the functions are still univalued is generally not a natural domain: it has a non-trivial holomorphy envelope². The boundary values obtained by approaching the real boundary via different “primitive tubes” belong to different operator orders inside the vacuum expectation values. Note also that the extension via the use of local commutativity leads to regions which do not factorize any more in u - and v -variables. The set of Euclidean points at which quantum field theory makes contact with statistical mechanics constitutes a lower dimensional subset contained in the permuted extended tube. The analytic setting of quantum field theory allows for a slight generalization in $d=2$. Instead of local fields one may work with the more general “dual” fields or “Yang-Baxter” fields. Instead of vanishing (anti)commutators they are characterized by the space-like algebra³:

$$\psi_y(y) \psi_\delta(x) = R_{y\delta}^{\alpha\beta} \psi_\alpha(x) \psi_\beta(y) \quad (\text{II.14})$$

where $x < y$ (left spacelike). For $x > y$ the R matrix is the inverse of (II.14). The Yang-Baxter trilinear relations for R turn out to be a consistency relation of this algebraic structure. Jörg Frölich has pointed out²⁵ that this algebra arises naturally if one tries to construct the intertwining fields in the sense of Haag et al.²⁶ for “observable algebras” in two space-time dimensions. This algebra structure leads to the same “permuted extended tube” domain of local quantum field theory, but this time with a violation of univaluedness (i.e. monodromy around analytic ramification points). The boundary values, at the physical points, which are related to the Hilbert-space interpretation, remain however univalued and physically consistent.

Now we are well prepared to study the “conformal extension”. Starting again from the primitive tube, the use of complex scale- and Lorentz-transformation gives the tilted upper(lower) half-

planes in (II.13) but this time the tilting may be done with two independent angles. The existence of the proper conformal transformation, or equivalently (using another basis), of the compact conformal subgroup $K_{L,R}$ belonging to the “conformal Hamiltonian” L_0 yields another extension. According to our previous discussion, the net result of these extensions may be conveniently described in terms of the two semi-groups¹⁹ $\widetilde{SL}_c^{(L,R)}(2, R)$, the complex extension of the Iwasawa decomposition to $Im\ \alpha > 0$ and $Im\ w > 0$. This causes a branching in the physical region which forced us to change our picture of space-time to the conformal heavens and hells or introduce irreducible global conformal fields in the L-R worlds which carry a “phase” subscript. As will be seen in the next section, the local fields (this also applies to “dual” or “Yang-Baxter” fields (II.14)) are sums of products of the global irreducible fields. To put it in a more picturesque language: the irreducible global light-cone fields, which also may be considered as a subtle generalization of creation and annihilation operators, form the “raw material” for the construction of local as well the soliton fields.

Besides the standard description of conformal quantum field theory (the light-cone picture) there are two further equivalent descriptions.

1) The compact picture

One transforms the u- and v-light-cones onto circles via the (complex) Möbius isomorphism (II.3). The operators live in the same Hilbert space as in the light-cone picture. The global irreducible (“phase”) components fulfil quasi-periodic boundary conditions.

2) The interval (cylinder) picture

Via the logarithmic map one transforms the $z(\bar{z})$ variable of the unit circle to a periodic $R \bmod 2\pi Z$ variable. This is an outer automorphism leading to another Hilbert space (again with positive definite Wightman functions) and a non-vanishing Casimir ground state energy.²⁷ The analytic continuation makes contact with statistical transfer matrices on a strip. The new vacuum has lost the $L_{\pm 1}$ invariance.

The compact picture is most typical for $d=2$ conformal quantum field theories. In earlier works on $d=2$ quantum field theories, such as for example, Klaiber's partial bosonization²⁸ of the massless Thirring model and the subsequent full bosonization of Mandelstam²⁹ and Coleman³⁰, it was not used. The prize for not using it is the somewhat awkward infra-red aspect of a free massless boson field. This compact picture appears for example in the work of Frenkel and Kac³¹. The boson field lives in a Hilbert space which is the direct product of the Fock space with the quantum-mechanical space generated by the zero modes. Although the boson fields live in a (positive definite) Hilbert space, it has no status as a conformal tensor⁸. Dual models and genus $g=0$ string theory are usually formulated in the interval (cylinder) picture.

The light-cone and compact pictures, although different in interpretation of their variables, have identical analytic functions²³. Their operator formulae for the Virasoro L_n charges are, however, different. The compact picture yields the simple Fourier formula

$$L_n = \frac{1}{2\pi i} \oint z^{-n-2} T(z) \frac{dz}{z} \quad (II.15)$$

whereas the light-cone formula for L_n is slightly more complicated as the result of the orthonormal system of u -eigenfunctions appearing in (II.9). It is well known from general quantum field theory that commutators of conserved charges with local fields can be represented inside Wightman functions by differences of spatial integrals with the relative time component of the current and the field being once positive and once negative imaginary. This follows from our previous remarks concerning the connection between the signs of $\text{Im } \zeta$ ($\text{Im } \bar{\zeta}$) and the operator orderings. For $d=2$ the two contributions can be combined to yield contour-integrals in ξ and $\bar{\xi}$. These contours may be shifted and in this way the BPZ formulae on (analytically continued) Green's functions emerge. In order to maintain an operator interpretation of these analytically continued expectation values in the process of circulating with one variable around, some authors (see Friedan's lectures) insist on introducing an ordering (time or radial ordering) inside the analytic domain. This is like defining a "tube ordering" for the analytic Wightman functions in general quantum field theory. According to my best knowledge this has never been done before. We will also not use it here because it

would create havoc in connection with the "exchange algebras" of the next section. We will also not employ the somewhat misleading terminology of "radial quantization" for the compact picture. General experience tells us that one should be very economical with the word "quantization". If one wants to use it in this context it should rather be called "angular quantization" since the conformal Hamiltonian L_0 is like an angular momentum but without the univaluedness requirement on wave functions.

III. Exchange algebras

The loop groups as well as the circular diffeomorphism group have a new property which distinguishes them from previously studied groups in physics. Their irreducible representations form families which want to be together. In the infinitesimal Kac-Peterson³² - Feigin-Fuchs¹⁶ - Rocha-Caridi³³ approach to the modular transformation properties of the characters for a fixed c -value are the clearest signal of their relation. In field theoretical language, this property is related to the existence of intertwiners, i.e. field $a(u)$, $b(u)$... which intertwine between these irreducible sectors. A simple, but unfortunately rather trivial illustration of this phenomenon is supplied by the vertex operator. In the compact picture we define

$$V_{\alpha}(z) = : \exp i\alpha \varphi(z) : \quad (\text{III.1})$$

$$\varphi(z) = q + ip \ln z + i \sum_n a_n \frac{z^{-n}}{n}$$

Choosing a torus quantum mechanics and suitably adjusting the real number α the vertex operator will be defined on a finite covering. Although the vertex operator decomposes according to¹⁹

$$V_{\alpha}(z) = \sum_n V_{\alpha}^{(n)}(z) \quad (\text{III.2})$$

where $V_\alpha^{(n)}$ is the contribution acting on the eigenspace of p labelled by n , this additive "charge" structure still yields a trivial situation for the 4-point- (more generally n -point-)function. The ξ -sum in correlation functions always consists of just one term as for free fields. The algebra formed by the vertex operators reads in light-cone co-ordinates ($u_1 > u_2$)

$$\begin{aligned} V_\alpha(u_2) V_\alpha(u_1) &= e^{i\pi\alpha^2} V_\alpha(u_1) V_\alpha(u_2) \\ V_\alpha^+(u_2) V_\alpha(u_1) &= e^{-i\pi\alpha^2} V_\alpha(u_1) V_\alpha^+(u_2) \end{aligned} \quad (\text{III.3})$$

A less trivial example is provided by the Ising model^{18,19}. The interwiners we are looking for are precisely the objects which appeared in the global decomposition theory as a subtle generalization of creation and annihilation operators. In the Ising case the centre element Z leads to 3 sectors. In addition to the vacuum sector one obtains the $h = \frac{1}{16}$ and $h = \frac{1}{2}$ sector¹². The local field which generates the $h = \bar{h} = \frac{1}{16}$ sector is the disorder operator. Its global decomposition reads:

$$\mu(u, v) = a(u) a(v) + \text{h.c.} + \dots \quad (\text{III.4})$$

where the a and a^+ are the only surviving contributions on the ket- resp. bra-vacuum.

$$\begin{aligned} \mu_{0,0}(u, v) &= a(u) a(v) \quad , \quad \mu_{\frac{1}{16}, \frac{1}{16}} = \mu_{00}^+ \\ &< m \mu \mu > \neq 0 \end{aligned} \quad (\text{III.5})$$

As a consequence of:

where $m = \bar{\psi}\psi$ is the mass operator, there must exist another irreducible component $b(u)$ which has the same dimension as a but a different phase:

$$b(u) = b_{-\frac{3}{8}}^{\frac{1}{16}}(u) \quad (\text{III.6})$$

$$\text{i.e.} \quad \bar{Z} b(u) Z^+ = \exp - 2i\pi \frac{7}{16} b(u)$$

Such a field cannot appear in the Kac-Feigen-Fuchs-BPZ approach since

$$b(u) |0\rangle = 0 \quad (\text{III.7})$$

However, b as well as a^+ can be applied after a has acted on the vacuum:

$$a^+(u_1) a(u_2) |0\rangle \in \text{Vacuum sector of } \mathbb{Z} \quad (\text{III.8a})$$

$$b(u_1) a(u_2) |0\rangle \in e^{-\frac{2i\pi}{2}} \text{ sector of } \mathbb{Z} \quad (\text{III.8b})$$

Two creators a applied to the vacuum vanish. The exotic rules distinguish the non-trivial conformal creation operators from ordinary free field operators which belong to the representation $c = \frac{1}{2}$. Since no other sector besides $\frac{1}{16}$, $\frac{1}{2}$, 0 exists, there is no need to introduce more light-cone fields. So one expects on each light-cone two 4-point functions:

$$F = \langle a^+(u_1) a(u_2) a^+(u_3) a(u_4) \rangle \quad (\text{III.9a})$$

$$G = \langle a^+(u_1) b^+(u_2) b(u_3) a(u_4) \rangle \quad (\text{III.9b})$$

and a more general 2^{n-1} $2n$ -point function.

The knowledge of the mixed (order-disorder) correlation function on one line¹⁸ is sufficient to determine all mixed n -point functions of the light-cone conformal creation and annihilators a, a^+ ,

$$\begin{pmatrix} a^+(u') & a(u) \\ b(u') & a(u) \end{pmatrix} = \text{Vertex phase} \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\frac{\pi}{2}} \end{pmatrix} \begin{pmatrix} a^+(u) & a(u') \\ b(u) & a(u') \end{pmatrix}$$

$$\begin{pmatrix} a(u') & a^+(u) \\ b^+(u') & b(u) \end{pmatrix} = \text{Vertex phase} \begin{pmatrix} e^{-i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} \\ e^{i\frac{\pi}{4}} & e^{-i\frac{\pi}{4}} \end{pmatrix} \begin{pmatrix} a(u) & a^+(u') \\ b^+(u) & b(u') \end{pmatrix}$$

$$b(u') b^+(u) = \text{Vertex phase} \quad b(u) b^+(u') \quad (\text{III.10})$$

Here the vertex is the phase appearing in (III.4) for $\dim V = \frac{1}{16}$. The exchange algebra serves to relate any $2n$ point function in a non-canonical order into the "ordered" function. The last relation in (III.10) only becomes effective in higher than 4-point functions. It is fairly easy to see that the calculated $2n$ -point function for the a - b operators of the Ising model are positive definite¹⁹. It is very plausible that exchange algebras (different from Virasoro or loop algebras) have only one irreducible vacuum representation. A rigorous argument will be presented in a future publication.

Let us now try to generalize the algebraic aspect of the Ising model to the Wess-Zumino-Witten model for $SU(2)$. The suggestion from the analysis of the current algebra is that there are $k+1$ sectors being distinguished by the lowest angular momentum j with $2j < k^{34}$. Furthermore, these sectors correspond precisely to the different centre sectors Z which feature in our global Einstein causality discussion. The local WZW field is therefore of the form $(\alpha, \beta = \pm \frac{1}{2})$:

$$g^{\alpha\beta}(u,v) = \sum_{j,\sigma,\mu} a_{(j,\sigma)}^{\alpha}(u) \tilde{a}_{(j,\mu)}^{\beta}(v) + \dots \quad (\text{III.11})$$

Here the sum extends over all half-integers $j \leq 2k$ and σ and μ stands for $\pm$. The interpretation is that the operator $a_{(j,\pm)}^{\alpha}(a)$ acts only on the Hilbert space H_j belonging to the j^{th} sector and it maps into $H_{j\pm\frac{1}{2}}$ according to the sign $\pm$. Time-reversal invariance requires us to identify the $\tilde{a}$'s with the time-reversed a 's. It is evident that the exchange relations ($u > u', 0 < j < 2k$)

$$a_{(\frac{1}{2},-)}^{\beta}(u') a_{(0,+)}^{\alpha}(u) = \eta a_{(\frac{1}{2},-)}^{\alpha}(u) a_{(0,+)}^{\beta}(u') \quad (\text{III.12a})$$

$$a_{(j+1,+)}^{\beta}(u') a_{(j,+)}^{\alpha}(u) = \eta \omega_j a_{(j+1,+)}^{\alpha}(u) a_{(j,+)}^{\beta}(u') \quad (\text{III.12b})$$

$$\begin{pmatrix} a_{(j-\frac{1}{2},+)}^{\beta}(u') & a_{(j+\frac{1}{2},-)}^{\alpha}(u) \\ a_{(j+\frac{1}{2},-)}^{\beta}(u') & a_{(j+\frac{1}{2},+)}^{\alpha}(u) \end{pmatrix} = \eta \begin{pmatrix} \alpha_j & \beta_j \\ \beta_j & \delta_j \end{pmatrix} \begin{pmatrix} a_{(j,+)}^{\alpha}(u) & a_{(j+\frac{1}{2},-)}^{\beta}(u') \\ a_{(j+1,-)}^{\alpha}(u) & a_{(j+\frac{1}{2},+)}^{\beta}(u') \end{pmatrix} \quad (\text{III.13})$$

$$a_{(\frac{k}{2}-\frac{1}{2},+)}^{\beta}(u') a_{(\frac{k}{2},-)}^{\alpha}(u) = \eta \tilde{\omega} a_{(k/2-\frac{1}{2},+)}^{\alpha}(u) a_{(k/2,-)}^{\beta}(u') \quad (\text{III.14})$$

with $\tilde{a}$ satisfying the analogous relation with the complex conjugate factors and matrices (from time reversal) are equivalent with causality:

$$g^{\alpha\dot{\beta}}(u,v) g^{\alpha'\dot{\beta}'}(u',v') = g^{\alpha'\dot{\beta}'}(u',v') g^{\alpha\dot{\beta}}(u,v) \quad (\text{III.15})$$

$$\text{for } (u - u')(v - v') > 0.$$

if and only if

$$|\eta| = 1 = |\omega_j| = |\tilde{\omega}|$$

$$\begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix}^* \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix} = \mathbb{1} \quad (\text{III.16})$$

These conditions just follow by projecting the causality relation into the j respectively the Z sectors.

The triple consistency relations ("braid" or Yang-Baxter relations) between three a 's are:

$$A_{12} A_{23} A_{12} = A_{23} A_{12} A_{23} \quad (\text{III.17})$$

where the various possible sectors yield $0 < j < k$:

$$1) \quad A_{12} = \omega_j, \quad A_{23} = \omega_{j-1/2} \quad (\text{III.18})$$

$$2) \quad A_{12} = \begin{pmatrix} \alpha_{1/2} & \beta_{1/2} \\ \gamma_{1/2} & \delta_{1/2} \end{pmatrix}, \quad A_{23} = \begin{pmatrix} 1 & \\ & \omega_{1/2} \end{pmatrix} \quad (\text{III.19})$$

$$3) \quad A_{12} = \begin{pmatrix} \omega_{j-1/2} & \alpha_j & \beta_j \\ & \gamma_j & \delta_j \end{pmatrix}, \quad A_{23} = \begin{pmatrix} \alpha_{j-1/2} & \beta_{j-1/2} \\ \gamma_{j-1/2} & \delta_{j-1/2} \\ & & \omega_j \end{pmatrix} \quad (\text{III.20})$$

$$4) \quad A_{12} = \begin{pmatrix} \omega_{k/2-1/2} & \\ & \tilde{\omega} \end{pmatrix}, \quad A_{23} = \begin{pmatrix} \alpha_{k/2-1/2} & \beta_{k/2-1/2} \\ \gamma_{k/2-1/2} & \delta_{k/2-1/2} \end{pmatrix} \quad (\text{III.21})$$

The non-trivial general solution of these relations is:

$$\omega_j = \omega, \quad \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix} = D_j R_j D_j^{-1} \quad (\text{III.22})$$

where D_j is an arbitrary real diagonal matrix and R_j is the special unitary matrix:

$$R_j = \frac{1}{S_j(\omega)} \begin{pmatrix} -(-\omega)^{2j+1} & \sqrt{-\omega S_{j-1/2} S_{j+1/2}} \\ \sqrt{-\omega S_{j-1/2} S_{j+1/2}} & 1 \end{pmatrix} \quad (\text{III.23})$$

with

$$S_j(\omega) = \sum_{i=0}^{2j} (-\omega)^i \quad \text{and} \quad S_{k/2+1/2}(\omega) = 0, \quad \omega = -e^{-\frac{2i\pi p}{k+2}} \quad (\text{III.24})$$

where p and $k+2$ have no common divisors.

In the two-point function the isoscalar contribution yields:

$$\langle a_{(\frac{1}{2},-)}^{\frac{1}{2}}(u) \bar{a}_{(0,+)}^{-\frac{1}{2}}(u') \rangle = - \langle \bar{a}_{(\frac{1}{2},+)}^{-\frac{1}{2}}(u) a_{(0,+)}^{\frac{1}{2}}(u') \rangle \quad (\text{III.25})$$

whereas the dimension $d_{\frac{1}{2}}$ of the a shows up in the phase:

$$\langle a_{(\frac{1}{2},-)}^{\frac{1}{2}}(u') \bar{a}_{(0,+)}^{-\frac{1}{2}}(u) \rangle = e^{2i\pi d_{\frac{1}{2}}} \langle a_{(\frac{1}{2},-)}^{\frac{1}{2}}(u) \bar{a}_{(0,+)}^{-\frac{1}{2}}(u') \rangle \quad (\text{III.26})$$

Hence the η in the exchange algebra fulfils

$$\eta = - \exp i 2\pi d_{\frac{1}{2}} \quad (\text{III.27})$$

Let $d_{1/2}$ be the dimension of the composite field which links the vacuum sector with the $j = \frac{1}{2}$ sector. According to the Z selection rule for the 3-point function of two elementary with one composite field we have:

$$\eta \omega = \exp i\pi (2d_{\frac{1}{2}} - d_1) \quad (\text{III.28})$$

More generally one obtains for the j^{th} sector composite field the phase rule:

$$(\eta \omega)^{2j-1} = \exp i\pi (d_{j-\frac{1}{2}} + d_{\frac{1}{2}} - d_{j+\frac{1}{2}}) \quad (\text{III.29})$$

The solution of this recursion gives:

$$d_j = j(j+1) \frac{p}{k+2} \quad (\text{III.30})$$

We obtain dimensional trajectories labelled by p . At this point we do not know to which value of the Virasoro charge our trajectory belongs. Comparison with the standard representation theory yields of course $p=1$. It would be nice if one could prove on purely algebraic grounds that the case $p \in \mathcal{N}, p > 1$ belongs to a direct product of $SU(2)$ with a flavour space. An analogous derivation of the Kac formula for the minimal families from the exchange algebra has been given elsewhere by Rehren.³⁵ A complete understanding of the minimal models is more subtle than in the $SU(2)$ case. Whereas for $SU(2)$ the value p appears to have the interpretation of a flavour-quantum number, the p for the minimal models appear to label soliton sectors. The understanding of this corresponding c -value goes one step in the direction of the representation theory of the exchange algebra. One needs to construct the energy-momentum tensor as a composite field from its characteristic commutation relations with the exchange fields. Its “strength”, which is determined by these commutation relations, will determine c . It is perfectly conceivable that the same braid matrices (e.g. $SU(2)$ and the minimal model) will lead to different c -values. There exists an interesting connection with the analysis of modular invariance of Cardy²⁷ and Capelli, Itzykson and Zuber⁴⁸. The relevant question in connection with our exchange algebra construction is whether there exist other possibilities of selecting a ’s from the $(k+2)$ sectors (say a subset of a ’s), such that causality still holds with a different formula from (III.13) and a different exchange algebra (III.14-16). In this way the very mysterious ADE classification⁴⁸ may correspond to a different exchange algebra with different “braid” matrices. We intend to address this problem in a forthcoming publication. Here we content ourselves with the general observation that dimensions are directly related to algebra whereas the Virasoro charge only shows up in the representation. The structure of the exchange algebra has repercussions for the form of the global operator expansion “off vacuum” which one needs in order to complete the bootstrap programme. This question will also be dealt with in a separate publication. Note that the eigenvalues of the diagonal and non-diagonal exchange matrices are equal.

The non-trivial spectral value is expected to have a relation to the cohomology of the exchange algebra. The only well-known studied algebras are those generated by causal fields restricted to,

say, a compact space-time region. These local von Neumann algebras are known³⁶ to be factors of type III in the terminology of Murray - von Neumann as refined by Powers and Connes³⁷. Since the local fields are sums of products of the exchange algebra fields, viz. in the Ising model:

$$\mu(x) = a(u) a(v) + h.c. + b(u) b(v) + h.c. \quad (III.31)$$

it follows that the exchange algebra fields restricted to a u -interval are also of type III, since algebras of the form $\sum I \times I$ or $\sum II \times II$ cannot yield type III local algebras.

Confessing my almost complete ignorance of these matters, I find it nevertheless tempting to speculate that the restricted exchange algebras may be of type III_λ with a value λ outside the positive Powers set³⁵:

$$\lambda = \exp 2i\pi \frac{1}{m+1} \quad (III.32)$$

This problem clearly needs further and more competent investigations.

IV. The possible physical destiny of two- dimensional conformal invariant QFT and the “string idea”

Two-dimensional conformal QFT and its systematic classification with the help of non-Lagrangian operator methods certainly has a mathematical depth which is not easily matched by any other known scheme in theoretical physics. Diffeomorphism groups (and loop groups), their representations and interwiners certainly lead to surprising new concepts. However, the only present-day physical application as a framework of classifying two-dimensional universality classes of critical systems of condensed matter is somewhat limited. One interesting slight generalization is to obtain a classification of integrable $d=2$ QFT by leading the critical point. Apart from some subtleties, one would expect such an approach to be within reach. The continuous part of the diffeomorphism group breaks down, but discrete subgroups¹⁹ of the centre $Z_L \times Z_R$ as well as the spacelike Yang-Baxter structures of (non-observable) intertwining fields (as discussed in Jörg Fröhlich’s lecture) should remain. The S-matrix of integrable models certainly fulfils all the properties of an S-matrix bootstrap, i.e. fits into the conceptional framework which leads to the Veneziano model and its string theoretical reinterpretation. The main difference with the Veneziano-type S-matrix lies in the fact that zero mass particles are completely absent in the spectrum. Can a generalized string idea with a $d=2$ target space perhaps produce such S-matrices?

A more exciting possibility was hinted at in Polyakov’s work³⁶. $D=2$ conformal quantum field theories combined with “some sort of string idea” may lead to an analytic understanding of certain higher dimensional non-trivial quantum field theories such as for example the $d=3$ Ising model. Again there is no gravitation in such models. This possibility is in my opinion much more realistic and interesting than hunting down a theory of everything (Toe) at the scale of the Planck mass (which, as we heard at this summer school, has meanwhile changed into a “quasiclassical approximation” of something). It should be explored by all means. Judging from past experience with attempts to obtain analytic classification schemes (see remarks made in the introduction), a flexible operator (i.e. algebraic) approach seems to be most promising for preliminary studies of this idea.

There appears to be agreement among physicists that a good understanding of conformal theories on Riemann surfaces and their behaviour under Teichmüller deformations is helpful for an understanding of the “string idea”. One approach to conformal QFT and Riemann surfaces is to start from the Polyakov functional integral. The mathematical formalism is that of, say, a free field interacting with an external (Euclidean) metric field. The general formalism of dealing with such external field problems was developed about 20 years ago³⁹. Through a correspondence with Fulling this formalism, which leads to Valatin-Bogolubov transformed “vacua”, became popular in semiclassical quantizations of general relativity in particular in connection with the Hawking effect. Mathematicians recently refined certain aspects of the formalism, in particular its application to Riemann surfaces. The details can be found in L. Alvarez-Gaumé’s lectures³⁸. We believe that this Euclidean formalism is mathematically complicated and physically not very revealing. In order to obtain what we believe is a more adequate description, it is interesting to draw some parallels of the Hawking-Unruh acceleration transformation with the Riemann-surface situation. Unruh found³⁹ that in order to understand flat space quantum field theories in a uniformly accelerated system it is not necessary to solve the QFT in the corresponding changed external metric, but rather to reinterpret the flat- space correlation function via the acceleration transformation in a Rindler half-space, with the, say 0-3 light-cone “wedge” region being the horizon. One finds that the transformed correlation functions restricted to the Rindler world are temperature correlation functions with the Hawking temperature⁴⁰. Of course such a procedure would be meaningless for a true black hole; in that case one has to quantize the matter theory in the Schwarzschild background. It turns out that the Riemann surface case is more like the Unruh situation²⁰ than like that of a black hole. In the Polyakov functional integral formulation this analogy is only indirectly indicated through the fact that the world-sheet metric was at the beginning just a Lagrangian parameter. Let us look at the operator formalism in the simple case of a free Abelian current $I(u)$ (a (1,0)

primary conformal field). Define in the Fock-space the following Unruh-like Fuchsian average²⁰

$$J^G(u) = \lim_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \sum_{\substack{g_i \in G \\ i=1}}^N U(g_i) J(u) U^\dagger(g_i) \quad (\text{IV.1})$$

Here G is a discontinuous Fuchsian subgroup of the Möbius group belonging to genus $g > 1$. There are simple arithmetic examples of such groups which are suitable for explicit calculations. For operators with scale dimension $d < 2$ as in the case at hand, the resulting series for the two-point functions:

$$\langle J^G(u) J^G(u') \rangle = \lim_{N \rightarrow \infty} \sum_i \langle J(u) U(g_i) J(u') \rangle \quad (\text{IV.2})$$

$$\doteq \langle J(u) J(u') \rangle_G$$

are only conditionally convergent. The case $g=1$ requires the use of a translation semi-group; by combining terms pair-wise, and using in addition to the $(1, \tau)$ lattice also the "mirror" lattice, one can easily see positivity and express the result in terms of the Weierstraß p - function²⁰. The Fuchsian case $g > 1$ is conceptually simpler since one works with the unitary group representers. Positivity of the expression (IV.1) before taking the lines is manifest. The only problem is a clever grouping of terms in order to obtain convergence for the Selberg-type representation of the resulting distribution (hyperfunction). The sum for Fuchsian groups belonging to non-compact genus g Riemann surfaces is conceptually and computationally simpler than that of compact surfaces. In the former case, the Dirichlet region representing the Riemann surface has a free line on the u -light cone (the "infinity" of the non-compact surface) on which positivity holds. In the lattice case the points of the light-cone represented by (hyperbolically equidistant) sequences in the upper half plane transplanted by the elements of the Fuchsian group to the Dirichlet region give limiting

point sets. (Julia sets) of possible Hausdorff dimension larger than one. Therefore in this case positivity does not hold on a submanifold but rather on such a general set. This set cannot be shifted by a symmetry operation as in the $g=1$ case, because on general Riemann surfaces there are no continuous symmetries. The higher point functions retain their free field structure: they are sums over products of the two-point functions. Using an argument of Verlinde and Verlinde⁴³, one finds that the analytically continued two-point functions are the same as those obtained as ratios of θ -functions with the help of the Jacobian map⁴⁰. Meanwhile other uses of the Unruh idea have appeared in the literature⁴⁴. Corresponding transformations in those cases cannot be interpreted as global deformations within the symmetry group of the vacuum, which in the conformal case consists of the two Möbius groups. In such cases one does not know to use the flat space quantum field theory and it is highly questionable whether they give rise to global Hawking temperatures. In doing these Fuchsian averages with spinors, one has to recall two facts.

1.) Flat space free spinors permit two natural expectation values

$$a) \quad \langle 0 | \psi(u) \psi^\dagger(u') | 0 \rangle \quad (IV.3a)$$

$$b) \quad \langle \sigma | \psi(u) \psi^\dagger(u') | \sigma \rangle \quad (IV.3b)$$

where

$$|\sigma\rangle = \sigma(t=i; x=0) |0\rangle$$

is a state obtained by the application of the (dis)order operator $(\mu)\sigma$. On the cylinder these expectation values are called Ramon-Schwarz expectation values.

2.) Spinors see part of the covering group $\tilde{M}$ (and therefore $\tilde{G}$) of the Möbius-group.

Taking both (1) and (2) into account, one ends up with 2^{2g} different expectation values (spin structures) from the Fuchsian averages. It is reasonable to express this result in the following way: there exists only one spinor operator, the "Wigner spinor" of the field theoretic textbook; however, on Riemann surfaces one has to consider 2^{2g} different expectation values on the Wigner spinor.

In order to continue, we need some more mathematical concepts. For finite temperature correlation functions (i.e. $g=1$), the Gelfand-Segal-Naimark⁴⁴ reconstruction of the Hilbert space and the operators (in our case by $I(u)$) has a non-trivial, rather big commutant (for $g=0$ the commutant is just $\lambda 1$), which turns out to be anti-isomorphic to von Neumann algebra⁴³. As a result of the compactness of the conformal Hamiltonian L_0 , the v. Neumann algebra turns out to be a factor of type I. If one wants the Virasoro algebra, in particular L_0 , to have anything to do with the remaining symmetry for $g=1$, the correct formula is not

$$L_0 = \frac{1}{2\pi i} \oint \bar{z}^{-2} T(\bar{z}) \frac{d\bar{z}}{\bar{z}} \quad (\text{IV.4})$$

but rather

$$L_0 = \frac{1}{2\pi i} \oint \bar{z}^{-2} T(\bar{z}) \frac{d\bar{z}}{\bar{z}} \quad - \text{ (similar contribution from commutant) } \quad (\text{IV.5})$$

This makes the representation of L_0 be unbounded below, i.e. not of Feigin-Fuchs type. I do not expect that for $g > 1$ this situation is going to improve.

Now consider some very important subgroups and subsets of the flat space $\widetilde{diff}_{cent}$. The already-mentioned "level n-groups" G_n have the property ($u_2 > 0$, $f(+\infty) = +\infty$).

$$\frac{1}{C_n} < \frac{f(u_1+u_2) - f(u_1)}{f(u_1) - f(u_1-u_2)} < C_n \quad (\text{IV.6})$$

where the smallest c_n measures the distance to a Möbius transformation and $c_n = 1$ for Möbius transformations. The c_n increases with n . Transformations with this bound are called quasi-symmetric. The quasi-symmetric subgroup of diff can be formally generate by the G_n

$$f(u) = \prod_n g_n(u) \quad , \quad g_n \in G_n \quad (\text{IV.7})$$

in a suitable topology. This subgroup forms (if subjected to normalization see below) what is known in mathematics as the universal Teichmüller space of Bers⁴⁴. Q. Teichmüller-spaces belonging to Riemann surfaces are $3g-3$ parametric subsets of *normalized* quasi-symmetric transformations (in addition to $f(+\infty) = +\infty$ also $f(o) = o$, $f(1) = 1$

$$T_g = \left\{ f \text{ normalized} \mid f \circ f^{-1}: \text{Möbius} \right\} \quad (\text{IV.8})$$

They form a contractible (as a result of the normalization) $3g-3$ parametric complex subspace T_g which, loosely speaking, depends only on the genus of G (the T 's of different G 's with the same genus are biholomorphic equivalent) and admits the inner automorphism of G as a “modular” symmetry group.

All these rich mathematical structures are “virtually” there in the flat space conformal field theory, but they only become “activated” through the Fuchsian average and the subsequent Teichmüller deformations. The latter can in principle be put in Virasoro-algebra-like calculations; the equation (IV.8) with the subgroup Ansatz (IV.7) is infinitesimally a Virasoro-algebra-like equation involving elements of arbitrary high level. To do Teichmüller variations on Fuchsian groups via Virasoro techniques should be interesting for mathematicians.

It is clear that our operator formalism after Teichmüller variations will give expectation values which depend on the Teichmüller parameters (they present themselves as a generalization of the Hawking temperature) and which will have equivariant transformation properties under modular transformations.

These modular properties just follow from the definition of $M_g < T_g$. As an illustrative example let us return to (IV.2) which after application of the Teichmüller variations reads:

$$W(u, u'; \tau_1, \dots, \tau_{3g-3}) = \langle J(u) J(u') \rangle_{G(\tau_1 \dots \tau_{3g-3})} \quad (\text{IV.9})$$

Here the subscript denotes the Unruh average for the Teichmüller deformed Fuchsian group. The elements of M_g describe an inner automorphism of G and hence lead to modular invariance:

$$\begin{aligned} \langle J(u) J(u') \rangle_{G(u_1, \dots, u_{3g-3})} &= \sum_i \langle J(u) U(f(u_1, \dots, u_{3g-3})) U(g_i) U^\dagger(f(u_1, \dots, u_{3g-3}))(u') \rangle \\ &= \langle J(u) J(u') \rangle_{G(u'_1 \dots u'_{3g-3})} \end{aligned} \quad (\text{IV.10})$$

if the two Teichmüller-deformed Fuchsian groups are identical sets, i.e. $\text{Mod}_g \simeq \text{Aut}(G_g)$.

In order to reconstruct the partition function of a conformal field theory on a Riemann surface one follows the procedure of Verlinde and Verlinde⁴³. One first represents the conformal energy-momentum tensor as a composite field in terms of the local fields of the model. This requires the construction of a "normal product" via a space-time limiting procedure⁴³ for the "would be" partition function. Its modular invariance is guaranteed as a result of the modular transformation properties of the 1-point functions. The special case $g=1$, which requires an averaging using a semi-group, is of considerable interest. In this case one hopes to make contact with the amazing results of Capelli, Itzykson and Zuber⁴⁸. These authors found that the use of modular invariance of partition functions represented as sums over products of characters allows for a classification in terms of A D and E Dynkin diagrams. Partition functions are more universal than local fields. They correspond in some sense to a maximal set of (bosonic) local fields, i.e. a set which is stable under "soliton completion". Let us illustrate this concept in an example. For the Ising model, the operators $m(x)$ and $T(u)$, $T(v)$ form a set which is closed under Wilson expansions as well as under the global

expansions considered in this paper. They are, however, not “maximal” since the Hilbert space can be enlarged by adding another (soliton) sector. The enlarged space is irreducible with respect to the operator-algebra generated by the (dis)order operator $(\mu)\sigma$. This new operator together with m (and the identity operator) yields a maximal set of primary fields. Besides the trivial possibility of forming tensor products with other independent local fields, there is no further natural enlargement. Maximal finite families of primary local fields only exist for $c < 1$.²⁷ Trying to understand the CIZ classification within the framework of flat space local field theory amounts to comprehending the existence of a finite number of possibilities of maximal local fields which can be formed from the same “raw material” given in terms of the global irreducible light-cone fields. Their “exchange algebra” structure regulates the possibilities of forming maximal local sets. In terms of local von Neumann algebras, one has to construct type III_1 local algebras from type III “exotic” (possibly III with a phase) von Neumann algebras. From this “raw material” one may also construct dual fields or Fröhlich’s intertwining fields which would again belong to “exotic” local von Neumann algebras. From the viewpoint of modular properties, the partition functions of “string theories” are distinguished within the set of all conformal partition functions. The left- and right-hand pieces of the total string partition functions are characterized as being modular forms of specific weights with no anomalies⁴⁹ (these phases are not “anomalous” within conformal field theories). This absence of anomalies is related to the statement that in Polyakov⁵⁰, an understanding of the “Liouville issue” and its related quantization seems to be crucial in order to use the functional integral for strings as a means to obtain analytic insight into higher-dimensional quantum field theory. Polyakov proposed that the 3d Ising model should be analysed as a string theory. The metric free operator approach proposed here seems to be ideally suited for exploring the “string idea” in quantum field theory. The Riemann surface formalism only makes contact with the string idea in the g -perturbation expansion. In this approximation one expects the Liouville issue to appear if one looks at the modifications which are necessary in order to liquidate the “projective” left-right anomalies which occur in the Unruh averages of the (Teichmüller-deformed) Fuchsian groups. In order to obtain a non-perturbative starting point, a “universal” averaging which does not distinguish a special

genus seems to be necessary. This may either consist of using the groups of all quasi-symmetric diffeomorphisms (the Ber's universal Teichmüller space) or performing an ascending sequence $T_g > T_{g+1} > T_{g+2} > \dots$ (say starting from $g=2$) of averages. An understanding of the relations between Teichmüller deformed Fuchsian groups for ascending genus g and the possibility of splitting such groups (corresponding to the pinching of surfaces) is clearly asked for. The objects which should make contact with the higher dimensional quantum field theory are expected to be spatial integrals over two dimensional fields of dimension one (vertex operators in the standard string approach). The presently-known string picture associates scattering amplitudes in "target space" with such quantities. In terms of the original two-dimensional "source space" these objects are non-local diffeomorphism invariants upon integration over the (ascending) Teichmüller parameters. The hope is that these universal quantities may turn out to be asymptotes (in the sense of scattering theory) of a new local field theory in target space. An operator approach, as suggested here, is not only capable of permitting a formulation of the desired locality properties in target space, but, judging from experience with systematic classifications of integrable and conformal quantum field theories, may lend itself to a classification of local "target theories". Algebraic approaches, as we already mentioned in the introduction, seem to be presently more powerful than those based on Euclidean functional integrals and Lagrangians.

The somewhat vague circumscription in the use of the word string as "the string idea" in this work has been chosen on purpose. I want to stay in complete harmony with Witten's statement that the "core ideas about strings remain elusive". On the physical side one hope may be that the string idea may help to obtain non-perturbative analytical insight into (some) non-trivial higher dimensional quantum field theory. Stringlike ideas may turn out to be the correct vehicle by which the amazing analytic understanding and classification of $d=2$ quantum field theories can be transported into a higher dimensional "target" spaces. We certainly do not intend to suggest to use it for a "Toe" (theory of everything) at 10^{19} GeV. In this connection I would like to recollect an episode of more than 14 years ago. At that time Gursey and Radicati had the modest idea that the group $SU(6)$ may be useful in the spectral classification and transition amplitudes of hadronic

physics. Within a short time this idea turned into a hot fashion. At the height of this fashion, when I was visiting Feza Gursey in Ankara, a press conference was given at an APS meeting in New York. Various physicists talked about the $SU(6)$ idea as being an important discovery in combining Relativity with Quantum Physics. There appeared a half-page report on this press conference in Time Magazine, of which I happened to take a copy with me. Two days later Tullio Regge appeared in Ankara. In greeting him, Feza immediately asked: "Please tell me what is the important discovery which was the main topic of the press conference?" Tullio: "some gentlemen made a very important discovery indeed". Feza (very excited): "What is it?" Tullio: "They discovered that Pauli is dead".

I am convinced that the string idea is a more important discovery than that in this episode. This I maintain even in plain view of the physical disaster of the "mechanical string". By mechanical string I mean that string which came from the attempt to find a nice geometrical scheme for the S-matrix guessed by Veneziano and its subsequent unitarization. These mechanical string ideas always produce zero mass objects which were later identified with gravitons and gauge particles. Many people believed that in this way one obtained a unique quantum theory including quantum gravity. As we heard recently, this theory is neither unique (there are millions of "theories of everything") and there are serious doubts about whether it solves "quantum gravity" in target space (it seems to do this in the $d=2$ source space) although it may yield classical restriction for the target metric in the form of an Einstein equation. I think that the real physical destiny of the (suitably generalized, i.e. non-mechanical) string idea lies in the classification of local quantum field theory. As we saw in section III, Einstein causality in conformal QFT_2 allows for a beautiful, non-perturbative and very non-trivial "exchange algebra" classification in terms of "braid" matrices. Anybody who worked on the principles of general quantum field theory cannot help but be impressed by this later indication of the ideas of the fathers of non-Lagrangian nonperturbative QFT. At least in $d=2$, quantum field theories do not like to be classified and understood by Lagrangians and Euclidean functional integrals. The insight rather comes through representation theory of new operator algebras. Wouldn't it be nice if a string idea could be used to

carry this deep non-perturbative insight to obtain a classification of S-matrices of Einstein causal quantum field theories? The crucial test for such an idea would be the connection of conformal QFT_2 with known S-matrices fulfilling the boot-strap principles in $d=2$. Many such S-matrices are known and they have a more complicated form than the Veneziano amplitude. In particular they do not have tachyons nor zero mass particles. I intend to return to these fascinating questions in future publications.

I am indebted to M. Karowski, K.H. Rehren, T.T. Truong, and R. Stora for interesting and helpful discussion. I thank M. Jacob for the hospitality extended to me during a short visit to the CERN Theoretical Physics Division.

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