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INTEGRABLE QUANTUM FIELD THEORIES AND CONFORMAL FIELD THEORIES
FROM LATTICE MODELS IN THE LIGHT-CONE APPROACH

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ABSTRACT

The light-cone lattice approach to two-dimensional quantum field theories is generalized to a large class of vertex models with any number of possible states per link and any simple Lie group of symmetry. Starting from a given lattice model, different scaling limits are defined leading to conformal field theories or to massive integrable quantum field theories, for which the lattice Hamiltonian, momentum and currents are constructed. For a large set of models, the complete mass spectrum is also exhibited. Our approach applies equally well to chiral fermionic theories (like the Chiral Gross-Neveu) and to bosonic models like the Principal Chiral Model.

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In recent years much progress has been made in the field of integrable models. By means of the Bethe Ansatz (BA) and its generalizations, it has been possible to construct explicitly the exact eigenvalues and eigenvectors of the transfer matrix of many interesting 2D statistical models. With similar techniques (co-ordinate BA), the Hamiltonian and momentum of various relativistic 2D Quantum Field Theories (QFTs) have been exactly diagonalized [1]. The calculation of finite size corrections to the BA eigenvalues has also allowed one to determine the conformal central charge and scaling dimensions of many gapless statistical models [2], thus connecting integrability in a very promising way to conformal 2D field theory.

In this letter we briefly report on various results obtained in the general scheme introduced in Ref. [3]; this scheme is based on the notion of light-cone lattices and represents a general way to derive and exactly solve relativistic massive QFTs by taking a given scaling limit of gapless 2D integrable vertex models. Roughly speaking, one first rotates the 2D square lattice by 45° , in order to "see it diagonally", that is, in the light-cone co-ordinates $x_{\pm} = x \pm t$, and then considers suitable complex statistical weights such that the given model really becomes a lattice regularization of the Feynman path integral for field theories in 2D Minkowski space-time (see Ref. [3] for details). Let us remark that the original vertex models, being gapless, allow us to obtain massless conformal field theories in a different scaling limit. The possibility of deriving very different QFTs from the same statistical model is a manifestation of the coexistence of many inequivalent field theoretical representations within the non-separable Hilbert space of the lattice theory at vanishing lattice spacing.

Let us consider an integrable lattice vertex model where a q -dimensional vector space is attached to each link of the lattice. The statistical weights at a given vertex are then assembled into a $q^2 \times q^2$ matrix $R(\theta) = (R(\theta)_{\alpha\beta}^{\gamma'\delta'})$, where $\alpha, \beta, \alpha', \beta' = 1, \dots, q$ label the states of the four links entering the vertex (see Fig. 1). θ is one of the variables (which can be complex) parametrizing the weights. It is singled out from the rest, and called the spectral parameter, since the local matrix $R(\theta)$ satisfies the fundamental Yang-Baxter (YB) functional relation in θ :

$$R(\theta-\theta')_{\alpha\beta}^{\gamma\delta} R(\theta)_{\alpha'\gamma}^{\delta'\epsilon} R(\theta')_{\epsilon\delta}^{\delta'\beta'} = R(\theta')_{\alpha'\alpha}^{\gamma\epsilon} R(\theta)_{\epsilon\beta}^{\delta\beta'} R(\theta-\theta')_{\gamma\delta}^{\gamma'\delta'} \quad (1)$$

The conventional row-to-row approach consists of taking the horizontal trace of the horizontal product of the R -matrices along a row (Fig. 2) to form the transfer matrix $\tau(\theta; \{v\})$, whose matrix element then reads:

$$\tau(\theta; \{\nu\})_{\alpha_1 \dots \alpha_N}^{\alpha'_1 \dots \alpha'_N} = \sum_{\beta_1 \dots \beta_N} R(\frac{\theta - \nu_1}{2})_{\beta_1 \alpha_1}^{\alpha'_1 \beta_2} R(\frac{\theta - \nu_2}{2})_{\beta_2 \alpha_2}^{\alpha'_2 \beta_3} \dots R(\frac{\theta - \nu_N}{2})_{\beta_N \alpha_N}^{\alpha'_N \beta_1} \quad (2)$$

In Eq. (2) we have introduced arbitrary inhomogeneity parameters $\nu_1, \dots, \nu_N$ and chosen a convenient rescaling by a factor of $\frac{1}{2}$ between τ and R . The fundamental YB algebra (1) implies that, for any choice of ν_j

$$[\tau(\theta; \{\nu\}), \tau(\theta'; \{\nu\})] = 0, \quad \forall \theta, \theta' \quad (3)$$

Hence the transfer matrix is the generating function for all the conserved commuting charges of the theory [we recall that, in this row-to-row context, $H = -\text{Re} \log \tau$ is the non-local lattice Hamiltonian^{*)}]. The exact BA diagonalization of $\tau(\theta; \{\nu\})$ shows that all the critical properties of the model at hand do not depend on $\nu_1, \dots, \nu_N$.

R matrices solving the YB algebra and depending rationally or trigonometrically on the spectral parameter θ are in close correspondence with simple Lie algebras [4]. In the rational case, a given Lie algebra $\hat{G}$ or group G is selected by requiring that $R(\theta)$ be invariant under the (global) group action, in some specific irreducible representation, over the spin space attached to the links. The large θ expansion of $R(\theta)$ then reads, quite generally [5],

$$R(\theta) \sim P \left\{ 1 + \frac{1}{i\theta} (\pi + \lambda) + O\left(\frac{1}{\theta^2}\right) \right\} \quad (4)$$

where P is the exchange operator $P_{\alpha\beta}^{\alpha'\beta'} = \delta_{\beta}^{\alpha'} \delta_{\alpha}^{\beta'}$, λ is a numerical constant,

$$\pi = \sum_{AB} K^{AB} t_A \otimes t_B \quad (5)$$

t_A , $A = 1, \dots, \dim G$ are the (Hermitian) generators in the given irrep. and K^{AB} is proportional to the inverse of the Killing form of G . Apart from the trivial constant shift λ , the θ^{-1} term in Eq. (4) is just the classical r -matrix characterizing the Lie-Poisson bracket structure of a classically-integrable model with G as symmetry group [1]. Therefore we can regard Eq. (4) as a semiclassical expansion ($\theta \sim O(\hbar^{-1})$) of the quantum R -matrix.

A rational R -matrix is in certain cases the isotropic limit of a trigonometric R -matrix. The latter is invariant only under products of $U(1)$ subgroups of the

*) Local lattice Hamiltonians can also be obtained in many cases, by means of the so-called trace formulae [1], in the limit $\theta \rightarrow 0$.

original group G . Nevertheless, the structure of the R-matrix and of the corresponding BA equations are both deeply linked with the Dynkin diagram of the Lie algebra $\dot{G}$ [4]. Let us recall the general form of the excitation spectrum of the generic model with the row-to-row transfer matrix $\tau(\theta; \{v\})$ of Eq. (2) [2b]:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log \frac{\Lambda_e(\varphi, \theta; \{v\})}{\Lambda_{\max}(\theta; \{v\})} \underset{\varphi + \theta \rightarrow \pm \infty}{\sim} \frac{m_e}{i\pi} e^{\mp \kappa(\varphi + \theta)} + O(e^{\mp 2\kappa(\varphi + \theta)}) \quad (6)$$

Here $\Lambda_\ell(\phi, \theta; \{v\})$ is the contribution of a hole with rapidity ϕ in the ℓ th branch to the BA eigenvalue of $\tau(\theta; \{v\})$. The parameters κ and m_ℓ do not depend on $\{v\}$ and are given in the Table for all models associated to simple Lie algebras (in the fundamental representation). We limit ourselves, in this context, to the rational, fully symmetric case, although very similar results can be obtained also in the trigonometric, anisotropic case, for a certain range of the anisotropy factor. The possibility, in the thermodynamic limit $N \rightarrow \infty$, of locating a hole at infinite rapidity shows that the energy spectrum associated to Eq. (6) is indeed gapless.

Gapless lattice models exhibit conformal invariant behaviour for distances much larger than the lattice spacing. Therefore, conformal field theories arise in the limit

$$a \rightarrow 0 \quad , \quad \theta \text{ fixed} \quad (7)$$

The finite-size corrections to the free energy and excitation energies yield the conformal properties of the model, that is, the central charge c and the scaling dimension $\Delta, \bar{\Delta}$ [6]. In the BA approach, c has been exactly calculated for all fundamental models (i.e., models in which the link vector space supports the fundamental representation) associated to simple and simply-laced Lie algebras [2], and for the spin S $SU(2)$ symmetric models [7]. These results indicate the general formula

$$c = \frac{x \dim \dot{G}}{x + \hat{h}} \quad (8)$$

for the central charge. Here $\hat{h}$ is the dual Coxeter number of the Lie algebra $\dot{G}$, while x stands for the total number of fundamental R-matrices "fused" to produce the R-matrix of the model [we have $x = 1$ for the fundamental models and $x = 2S$ for the S -spin $SU(2)$ symmetric model]. The Sugawara construction of the Virasoro algebra given in Ref. [8] also fulfils Eq. (9), with x the "level" of the original Kac-Moody algebra over $\dot{G}$. We believe that the gapless integrable lattice theories associated to Lie algebras provide, through their long-distance behaviour, an alternative construction of the conformal algebra (a construction which starts from a manifestly interacting lattice model, since the S-matrix is exactly calculable and non-trivial).

Let us now briefly introduce the light-cone lattice approach. Given a model with a rational or trigonometric R-matrix $R(\theta)$, in the absence of inhomogeneity parameters v_j , one considers a transfer matrix $U(\theta)$ quite different from Eq. (2). Rather than being given by the product along a given row, $U(\theta)$ is given by the product $U_R(\theta)U_L(\theta)$ of the two diagonal-to-diagonal transfer matrices (N is assumed to be even)

$$\left[U_R(\theta)\right]_{\alpha_1 \dots \alpha_N}^{\alpha'_1 \dots \alpha'_N} = \prod_{j=1}^{N/2} R(\theta)_{\alpha_{2j-1} \alpha_{2j}}^{\alpha'_{2j-2} \alpha'_{2j-1}} \quad (9a)$$

$$\left[U_L(\theta)\right]_{\alpha_1 \dots \alpha_N}^{\alpha'_1 \dots \alpha'_N} = \prod_{j=1}^{N/2} R(\theta)_{\alpha_{2j-1} \alpha_{2j}}^{\alpha'_{2j} \alpha'_{2j+1}} \quad (\alpha_{j+N} \equiv \alpha_j) \quad (9b)$$

and can be depicted as in Fig. 3. It is clear that, on the infinite lattice, no difference exists in the statistical sense between the row-to-row and diagonal-to-diagonal viewpoints. On the lattice rotated clockwise by 45° , however, the new transfer matrix $U(\theta)$ can again be used to define the time evolution. Actually, since U_R and U_L commute, one can define two light-cone lattice evolutions by

$$\frac{i}{2}(H \pm P) = \frac{i}{a} \log U_{R,L}(\theta) \quad (10)$$

where H and P are respectively the Hamiltonian and the momentum [with appropriate (length) $^{-1}$ dimension, a being the lattice spacing], and $U_{R,L}(\theta)$ are assumed to be unitary (a property which can be achieved, by a judicious choice of the reality of θ and/or the anisotropy factor, for any rational or trigonometric R-matrix).

The problem is now to diagonalize $U_{R,L}$ exactly. To this end, as it turns out, the YB algebra (1) satisfied by $R(\theta)$ is perfectly sufficient. The only extra condition required is that $R(\theta)$ be a regular solution of Eq. (1), namely

$$R(\theta)_{\alpha\beta}^{\beta'\alpha'} = \delta_{\alpha}^{\beta'} \delta_{\beta}^{\alpha'} \quad (11)$$

Consider then the row-to-row transfer matrix $\tau(\theta; \{v\})$ of Eq. (2), with the particular choice of inhomogeneities (recall that N = even):

$$v_j = (-)^{j+1} v, \quad j = 1, 2, \dots, N \quad (12)$$

Thanks to Eq. (11), one quickly verifies

$$U_R(\nu)^\dagger = \tau(-\nu; \{\nu\}) \quad (13a)$$

$$U_L(\nu) = \tau(\nu; \{\nu\}) \quad (13b)$$

Indeed, with the alternating inhomogeneities (12), in $\tau(\pm\nu, \{\nu\})$ half of the R-matrices have vanishing argument, so that, e.g.,

$$\begin{aligned} \tau(\nu; \{\nu\})_{\alpha_1 \dots \alpha_N}^{\alpha'_1 \dots \alpha'_N} &= \sum_{\beta_1 \dots \beta_N} \delta_{\beta_1}^{\alpha'_1} \delta_{\alpha_1}^{\beta_2} R(\nu)_{\beta_2 \alpha_2}^{\alpha'_2 \beta_3} \dots \delta_{\beta_{N-1}}^{\alpha'_{N-1}} \delta_{\alpha_{N-1}}^{\beta_N} R(\nu)_{\beta_N \alpha_N}^{\alpha'_N \beta_1} \\ &= \prod_{j=1}^{N/2} R(\nu)_{\alpha_{2j-1} \alpha_{2j}}^{\alpha'_{2j} \alpha'_{2j+1}} = U_L(\nu)_{\alpha_1 \dots \alpha_N}^{\alpha'_1 \dots \alpha'_N} \end{aligned} \quad (14)$$

by extensive use of Eq. (11).

The central relations (13) allow us to write down immediately the energy and momentum spectrum on the diagonal lattice; indeed, combining Eqs. (6), (10) and (14), one obtains that a hole of rank ℓ and rapidity ϕ contributes, relative to the ground state, the energy and momentum

$$\varepsilon(\varphi) \underset{\theta \rightarrow \infty}{\sim} m_\ell \frac{e^{-\kappa\theta}}{\pi a} \operatorname{ch} \kappa\varphi + O(e^{-2\kappa\theta}) \quad (15a)$$

$$p(\varphi) \underset{\theta \rightarrow \infty}{\sim} m_\ell \frac{e^{-\kappa\theta}}{\pi a} \operatorname{sh} \kappa\varphi + O(e^{-2\kappa\theta}) \quad (15b)$$

We now consider the following scaling limit [to be compared with Eq. (7)]

$$a \rightarrow 0, \quad \theta \rightarrow \infty, \quad \mu \equiv \frac{e^{-\kappa\theta}}{\pi a} \text{ fixed} \quad (16)$$

One then recognizes in Eqs. (15) the relativistic spectrum of massive particles with masses μm_ℓ and physical rapidity $\kappa\phi$. μ is thus the renormalized mass scale of the theory. Equation (16) has the characteristic structure of dimensional transmutation in asymptotically free field theory, with a^{-1} the UV cut-off and $\theta = \theta(g)$ a function of the bare coupling constant, diverging as $g \rightarrow 0$. This similarity will soon be clarified, when the problem of recognizing the Lagrangian field theory corresponding to the scaling limit (16) of a given lattice vertex model will be

discussed. For the moment, let us observe that the S-matrix for all the above massive particles can be exactly calculated by means of the Bethe Ansatz. In the scaling limit it depends solely on unconstrained rapidity differences and satisfies all the standard properties of a relativistic QFT S-matrix, like analyticity, crossing and a well-defined bound state pole structure.

In Ref. [3], the light-cone vertex model with a two-dimensional link vector space and trigonometric $U(1)$ -invariant R-matrix was considered. In the statistical language, it corresponds to the symmetric six-vertex model [1]. In this case, it is possible to construct explicitly the regularized canonical bare fields on the lattice, and to show that in the scaling limit (16) the Massive Thirring Model (MTM) emerges [3,9]. An analogous explicit local construction of the fundamental lattice fields for other light-cone vertex models would be very interesting to obtain. Here we explicitly construct the lattice currents for a large class of rational and trigonometric models.

The basic ingredient is the semiclassical expansion (4) of the R-matrix. To be specific, consider the rational $SU(m)$ symmetric R-matrix

$$R(\theta) = \frac{1 + i\theta P}{1 + i\theta} \quad (17)$$

where the exchange operator P , $P_{\alpha\beta}^{\alpha'\beta'} = \delta_{\alpha}^{\beta'} \delta_{\beta}^{\alpha'}$, can be written

$$P = \frac{1}{m} + \pi \quad ; \quad \pi = 2 \sum_{A=1}^{m^2-1} T^A \otimes T^A \quad (18)$$

in terms of the Hermitian $m \times m$ generator T^A (normalized by $\text{tr} T^A T^B = \frac{1}{2} \delta^{AB}$). Clearly, (17) satisfies Eq. (4) with $\lambda = (m-1)/m$. Let us now define the lattice currents

$$T_n^A = \mathbb{1} \otimes \dots \otimes T^A \otimes \dots \otimes \mathbb{1} \quad \text{n-th link} \quad (19)$$

If n is even (odd), the n -th link is right- (left-) oriented.

Since the light-cone evolution is determined by U_R and U_L , it is natural to consider the one-step evolved operators $U_R T_n^A U_R^\dagger$ and $U_L T_n^A U_L^\dagger$. To evaluate them, it is sufficient to observe that

$$R(\theta)(T^A \otimes \mathbb{1})R(\theta)^\dagger = \frac{1}{1+\theta^2} \left\{ \theta^2 \mathbb{1} \otimes T^A + T^A \otimes \mathbb{1} + 2i\theta f^{ABC} T^B \otimes T^C \right\} \quad (20a)$$

$$R(\theta)(1 \otimes T^A)R(\theta)^\dagger = \frac{1}{1+\theta^2} \left\{ \theta^2 T^A \otimes 1 + 1 \otimes T^A - 2i\theta f^{ABC} T^B \otimes T^C \right\} \quad (20b)$$

where f^{ABC} are the $SU(m)$ structure constants, $[T^A, T^B] = if^{ABC} T^C$. From the definition (9) of $U_{R,L}$, it then follows that

$$U_R T_{2n-2}^A U_R^\dagger = U_L T_{2n}^A U_L^\dagger = \frac{1}{1+\theta^2} \left\{ \theta^2 T_{2n}^A + T_{2n-1}^A + 2i\theta f^{ABC} T_{2n-1}^B T_{2n}^C \right\} \quad (21a)$$

$$U_R T_{2n-1}^A U_R^\dagger = U_L T_{2n+1}^A U_L^\dagger = \frac{1}{1+\theta^2} \left\{ \theta^2 T_{2n+1}^A + T_{2n}^A - 2i\theta f^{ABC} T_{2n-1}^B T_{2n}^C \right\} \quad (21b)$$

These are the bare equations for the currents on the lattice. In order to obtain the corresponding continuous equations, we must take a (bare) scaling limit. As was already the case for the MTM, bare and renormalized scaling limits [this latter is defined by Eq. (16)] are quite different.

The bare scaling limit consists here of two steps:

- 1) Define $T_n^A = g\theta t_n^A$ and let $\theta \rightarrow \infty$ with the parameter g fixed (g will be the bare coupling constant). Equation (21a) yields in this limit

$$U_R t_{2n-2}^A U_R^\dagger = U_L t_{2n}^A U_L^\dagger = t_{2n}^A + 2ig f^{ABC} t_{2n-1}^B t_{2n}^C \quad (22)$$

with a similar result for (21b).

- 2) Let the lattice spacing a tend to zero and set [cf. Eq. (10)]

$$U_{R,L} \simeq 1 - \frac{i}{2} a (H \pm P) + O(a^2) \quad (23)$$

$$t_{2n}^A = a J_R^A(x), \quad t_{2n-1}^A = a J_L^A(x) \quad ; \quad x \simeq na \quad (24)$$

We then find from (21) and (22) the continuous field equations

$$\partial_\mu J^{\mu A} = 0 \quad (25a)$$

$$\partial_\mu J_\nu^A - \partial_\nu J_\mu^A + ig f^{ABC} [J_\mu^B, J_\nu^C] = 0 \quad (25b)$$

which are zero-divergence and (non-Abelian) zero-curl equations for the currents $J_\mu^A = \frac{1}{2}(J_R^A + J_L^A, J_R^A - J_L^A)$. These equations characterize the currents in the $SU(m)$ Chiral Gross-Neveu (CGN) model. In this case, $J_\mu^A \approx \bar{\psi} \gamma_\mu T^A \psi$.

Let us now generalize Eqs. (25) to any simple Lie group G and representation ρ of it where Eq. (4) holds (this happens for all representations of $SU(m)$ and many representations of other groups [5,10]). Indeed, the details of the $O(\theta^{-2})$ term are irrelevant in the "bare" scaling limit, and the non-Abelian quadratic term in Eqs. (20)-(22) and (25) follows directly from the presence of the invariant operator Π in Eq. (4). Also in the trigonometric case one can evidently derive an anisotropic version of Eqs. (25) provided a semiclassical expansion like (4) is valid. Equations (25) with f^{ABC} as structure constants of some simple Lie algebra $\dot{G}$ hold for a very large class of integrable QFTs. Actually, in all of them the generator of non-local conserved currents provides a representation of the YB algebra on the physical particle states [11]. Principal chiral models, $O(N)$ sigma models, the Gross-Neveu model and many others belong to the class defined by Eqs. (25).

Let us now compare a little more closely the "bare" scaling limit presented above with the "renormalized" limit Eq. (16). The splitting of the former into two independent steps, with no interrelation between the $\theta \rightarrow \infty$ and $a \rightarrow 0$ limits, is needed to introduce the notion of a "bare" coupling constant. Suppose now that the QFT corresponding to the renormalized scaling limit is strictly renormalizable, with a non-trivial coupling constant flow. In this case, g is really a function of a , the inverse UV cut-off. Taking into account that g can actually be reabsorbed with the rescaling $\theta \rightarrow \theta/g$ and that the renormalized limit (16) requires $\theta \rightarrow \infty$ as $\ln a^{-1}$ (i.e., the semiclassical behaviour at vanishing lattice spacing), we conclude that the massive QFT describing the renormalized scaling is asymptotically free, with $g \sim \ln a^{-1}$.

As a matter of fact, it is possible to reproduce the same eigenvalue-eigenvector structure and S-matrix of a light-cone rational model with the chiral fermionic theory

$$\mathcal{L} = i\bar{\psi} \not{\partial} \psi - \frac{1}{4} \bar{\psi}_{\alpha'} \gamma_\mu \psi_\alpha V(g)_{\alpha\beta}^{\alpha'\beta'} \bar{\psi}_{\beta'} \gamma_\mu \psi_\beta \quad (26)$$

where $V(g)$ is a Hermitian interaction matrix related to the R-matrix of the vertex model by a Cayley transform

$$R(\bar{g}')P = \frac{2i + V(g)}{2i - V(g)} \quad (27)$$

One finds that the spectrum of (26) decouples into a massive part, carrying the quantum numbers but no $U(1) \times U(1)$ charge, and a massless chiral sector, carrying $U(1) \times U(1)$ but neutral with respect to G . The zero chirality sector of this model is identical to the scaling limit of the light-cone vertex model with the integrable R-matrix associated to G and ρ . Details of these equivalences can be found in Ref. [12], together with a critical analysis of the limitations of the co-ordinate BA (CBA).

If the R-matrix has the semiclassical expansion (4), then $V(g)$ has the semiclassical form

$$V(g) \simeq g(\pi + \lambda) + O(g^2) \quad (28)$$

and the interaction in (26) takes the typical form of (non-Abelian) current-current coupling

$$\mathcal{L} = i\bar{\psi}\not{\partial}\psi - \frac{1}{4}g[J_\mu^A J_A^\mu + \lambda J_\mu J^\mu] ; J_\mu^A = \bar{\psi}\gamma_\mu T^A \psi, J_\mu = \bar{\psi}\gamma_\mu \psi \quad (29)$$

This is a generalization to arbitrary simple Lie groups and representations [such that Eq. (4) holds] of the $SU(N)$ CGN model. As for classical integrable models, these were already considered in Ref. [13], except for the $U(1)$ Thirring term in Eq. (29), which is of quantum origin. The model Eq. (29), with finite g and no $O(g^2)$ term (and possibly a unique value for λ), is by itself quantum-mechanically exactly integrable for all classical Lie groups in their fundamental representations [14].

The fermionic chiral theories (26) and (29) are asymptotically free, in the sense that the one-loop coefficient of the β -function is negative (for any representation of the Lie algebra). This coefficient follows from the two ladder diagrams (the only ones logarithmically divergent in two space-time dimensions), so that its value is just $(2\pi)^{-1}$ times the quadratic Casimir of the adjoint representation (we use a normalization such that the shortest roots are of unit length). The examination of κ in the Table shows that the dynamically-generated mass in the continuum limit (16) of the light-cone vertex models (recall $\theta = 1/g$) scales exactly according to the one-loop renormalization group, that is, κ^{-1} coincides with the first perturbative coefficient of the β -function. This result provides a non-trivial check of the universality for the BA solutions of the large class of models at hand.

The finite dimensional nature of the vector space attached to each link leads naturally to a fermionic interpretation, which the equivalence just mentioned exactly confirms. Integrable R-matrices exist, however, for arbitrarily large irreps of certain Lie groups [for instance $SU(N)$]. For simplicity, consider $SU(2)$. The R-matrix for the irrep of spin S reads [15]

$$R_{12}(\theta) = \frac{\Gamma(2S+1+i\theta)\Gamma(J+1-i\theta)}{\Gamma(2S+1-i\theta)\Gamma(J+1+i\theta)} \quad (30)$$

where the operator $\mathbf{J}$ is defined by $\mathbf{J}(\mathbf{J}+1) = (\vec{S}_1 + \vec{S}_2)^2$, with $\vec{S}_1$ and $\vec{S}_2$ spin operators of weight S .

The $S \rightarrow \infty$ limit of Eq. (30) provides through the general light-cone construction (9)-(10), (19)-(20), the Hamiltonian, momentum and currents corresponding to the Principal Chiral Model (PCM), taking the appropriate scaling limit. Actually in this continuum model the Lagrangian reads

$$\mathcal{L}_{PCM} = \frac{1}{2g} \text{tr} \partial_\mu \Omega \partial^\mu \Omega^{-1}, \quad \Omega \in SU(2) \quad (31)$$

and one has left and right conserved currents $\Omega^{-1} \partial_\mu \Omega$ and $\Omega \partial_\mu \Omega^{-1}$ respectively. Only one of them, say the left-invariant one, is represented in the lattice by Eq. (19). Hence, only left-singlets are produced by this method. The construction of H and P in Ref. [16] is actually equivalent to ours, and a detailed calculation confirms the absence of non-left singlet states, as expected. This fact means that the H and P of Eq. (10) are not the full Hamiltonian and momentum operators even in the scaling limit for the PCM (31), but only its left-singlet restrictions.

In an approach which is in fact closely related, Polyakov and Wiegmann claimed [17] that the PCM (29) arises as the $N_F = \infty$ limit of the model (29) when the fermions have an additional flavour index a ($1 \leq a \leq N_F$), such that

$$J_\mu^A = \sum_{a=1}^{N_F} \bar{\Psi}_a \gamma_\mu T^A \Psi_a \quad (32)$$

A careful study of the Euclidean partition function $Z_{N_F}(\beta)$ of the multiflavour generalization of the model (29) in a finite imaginary time interval β , shows [12] that the precise connection between the models is

$$\lim_{N_F \rightarrow \infty} Z_{N_F}(\beta) = \text{Tr} [e^{\beta H_{PCM}} P_0] \quad (33)$$

where H_{PCM} is the full Hamiltonian of the PCM model (29) and P_0 the projector on left-singlet states.

In summary, any gapless integrable vertex model yields one or more (even an infinite number [12]) of integrable QFTs through the lattice light-cone approach exposed in Ref. [3] and here. Besides the mass spectrum, the S-matrix can always be computed from the BA equations of the model [1]. This considerably enlarges the set of known 2D integrable QFTs, and includes in particular all models previously solved by CBA. In some sense, the light-cone lattice approach supersedes completely the CBA.

Lie Algebra	Dynkin's Diagram	κ	m_k
A_n		$2\pi/n+1$	$\sin(\pi k/n+1), 1 \leq k \leq n$
B_n		$\pi/2n-1$	$\sin(\pi k/2n-1), 1 \leq k \leq n-1; m_n = \frac{1}{2}$
C_n		$\pi/n+1$	$\sin(\pi k/2(n+1)), 1 \leq k \leq n$
D_n		$\pi/n-1$	$\sin(\pi k/2(n-1)), 1 \leq k \leq n-2; m_{n-1} = \frac{1}{2}$
E_6		$\pi/6$	$m_1 = m_5 = \frac{m_6}{2} = \frac{\sqrt{3}}{2}; m_2 = m_4 = \frac{3+\sqrt{3}}{2}; m_3 = \frac{3+\sqrt{3}}{\sqrt{2}}$
E_7		$\pi/9$	(*)
E_8		$\pi/15$	(*)
F_4		$\pi/9$	(*)
G_2		$\pi/6$	(*)

(*) These values can be (very laboriously) extracted from Refs. [2b] and [4].

T A B L E

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Fig. 1

Fig. 2

Fig. 3