CERN-TH.4761/87

QUANTUM APPROACH TO BEAMSTRAHLUNG

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A B S T R A C T

Bremsstrahlung energy losses by electrons and positrons in very high energy linear colliders are discussed in terms of Feynman diagrams. At least in some cases, it is possible to separate different contributions in the quantum radiation of the impinging particles. In view of this, we here focus on the radiation before beam crossing. This is in addition to the radiation during beam crossing.

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^{*)} Work supported in part by the United States Department of Energy under Grant DE-FG02-84ER40158.

The SLC, with beam energy in the 50 GeV range, should provide us much information about the dynamics of beam-beam interactions in high-energy linear colliders, in addition to the very exciting physics. It is clear that, because of the losses due to the intense synchrotron radiation in circular machines, linear colliders¹⁾ have to be preferred when one wishes to study electron-positron annihilation much above the energy range covered by SLC and LEP. Indeed several options have already been envisaged, and they are expected to lead to designs and later projects which could allow one to investigate this physics in the TeV and multi-TeV ranges²⁾, where many new phenomena should be found³⁾.

Since annihilation cross-sections decrease as the inverse of the square of the centre-of-mass energy, large luminosities have to be realized. This imposes very small transverse dimensions for the bunches and very high densities. The interpenetration of the electron and positron bunches will therefore involve strong disturbances which may strongly affect the effective annihilation conditions. It is important to study whether they can be acceptable within the constraints of machine operation conditions and of experimental conditions.

One of these effects is referred to as beamstrahlung. It corresponds to the bremsstrahlung energy loss by the electrons and positrons when the bunches interpenetrate each other. This leads to a spread in the centre-of-mass energy of the actually observed annihilation processes. If this spread is too severe, the effective energy and also the luminosity of the linear collider may become significantly less than the design values. For the SLC, this has been studied and found small^{4),5)}. In this case it can be argued that standard calculation methods are applicable. Because of the required high luminosity and small transverse size of the bunches, beamstrahlung is expected to be of greater importance in TeV and multi-TeV electron-positron linear colliders.

A complete treatment of beamstrahlung requires the simultaneous consideration of all the electrons and all the positrons in the bunches. This is a many-body problem and is very difficult. The process considered here is instead the interaction of a single electron (or positron) with the positron (or electron) bunch^{4),5)}. More precisely, the process is the radiation of a single photon:

$$e^{\pm} + \text{bunch} \rightarrow e^{\pm} + \gamma + \text{bunch} \quad (1)$$

Stated in this way, this purely QED process is most naturally treated by the method of Feynman diagrams in the rest frame of the bunch. Relative motion of the positrons (or electrons) in the bunch is neglected. The bunch in its rest frame acts merely as an external electrostatic potential, and the Feynman diagram is shown in Fig. 1 for the case of the electron.

From this Feynman diagram, the matrix element is, after summing over the number of interactions:

$$M_{\mu}(\vec{k}_i, \vec{k}_f, \vec{k}_\gamma) = -ie \int d^3x e^{-i\vec{k}_\gamma \cdot \vec{x}} \bar{\psi}_f(x) \gamma_{\mu} \psi_i(x) \quad (2)$$

where ψ_i and ψ_f are the initial and final electron wave functions in the presence of the bunch. They can be found by studying the high-energy behaviour of the Dirac equation. Both the lowest-order approximation⁶⁾ and the higher-order corrections⁷⁾ are available in the literature. For the purpose of a full beamstrahlung calculation, the lowest-order approximation is not quite accurate enough because of bending.

Before attempting to evaluate this matrix element, it is necessary to understand the various important parameters associated with the radiative process (1). It is useful to compare SLC with a linear collider at 100 times the energy, i.e., 5 TeV per beam^{8),9)}, which is here referred to as "super". Three characteristic lengths are particularly relevant. The first one is the correlation length $\bar{L}_c$. As the positron or the electron passes through the bunch it collides with, it is deflected and actually focused by its electromagnetic interaction with the bunch as a whole. Its radiation can remain coherent only as long as the deflection angle is less than $1/\gamma = m/E$, where m and E are respectively the electron mass and energy. This follows the classical argument. One indeed defines the correlation length $\bar{L}_c$ as the distance travelled for a deflection in angle of $1/\gamma$.

The second one is the bunch length $\bar{L}_b$, over which the effect of the bunch is directly felt. In the laboratory system it is the bunch longitudinal size σ_z , whereas σ_r will denote the transverse size.

The third one is the virtual electron length $\bar{L}_e$, that is the distance over which coherent radiation by the impinging electron is allowed before it enters

the bunch. An electron can radiate a photon losing this way a fraction x of its energy. While the massive bunch will not transfer energy, it will transfer momentum and the radiation process is thus allowed quantum mechanically over a distance which clearly increases as γ . It corresponds to the maximum value of $(\Delta_L)^{-1}$, where Δ_L will be later defined as the longitudinal momentum discrepancy.

We summarize for comparison in Table 1 the key parameters of two linear colliders, the SLC and super. These parameters can be used to calculate the three characteristic lengths which have been introduced. They are listed in Table 2, together with the values of the disruption factor and the deflection angle θ_d . The bremsstrahlung radiation angle $1/\gamma$ is denoted as θ_b . All lengths are given in the laboratory system and are accordingly written with a bar.

As already mentioned, the Feynman diagram of Fig. 1 is most conveniently treated in the frame where the bunch is at rest. We list in Table 3 the three lengths in this bunch frame. It is simply a matter of multiplying by γ .

The values given in these Tables 2 or 3 show remarkable differences as one changes in such a dramatic way the energy scale. For the SLC, the virtual electron length is smaller than the coherent length, which is smaller than the bunch length. One can indeed write

$$\bar{L}_e \ll \bar{L}_c \ll \bar{L}_b \quad (3a)$$

While for the super the winner is the virtual electron length. One then has

$$\bar{L}_c \ll \bar{L}_b \ll \bar{L}_e \quad (3a)$$

Radiation coherence not extending in practice over more than a correlation length, we can separate the radiation which occurs before bunch crossing, over the virtual distance $\bar{L}_e$, from radiation during beam crossing since $\bar{L}_e$ turns out to be so much larger than $\bar{L}_c$. For the "super", we can treat this radiation before beam crossing on its own as a separate process, and study its effect on electron energy loss, whereas such a consideration would be out of place with the SLC parameters. This is the purpose of this paper. It is a quantum process which becomes independent at very high energy and can thus justify a separate treatment. This is done because, for the calculation of this radiation before

bunch crossing, the lowest-order high-energy approximation⁶⁾ is sufficient. Thus ϕ_i and ϕ_f differ from plane waves only in the phase factors

$$\begin{aligned} e^{i\phi_i(x,y,z)} &= e^{-ie \int_{-\infty}^z dz' \phi(x,y,z')} \\ e^{i\phi_f(x,y,z)} &= e^{ie \int_z^{\infty} dz' \phi(x,y,z')} \end{aligned} \quad (4)$$

where ϕ is the potential felt by the electron as it crosses the bunch. The transverse component of $\vec{k}_f$ is neglected and the coordinate z is along the axis of the bunch.

For definiteness we shall restrict ourselves to the case of a uniform bunch in the form of a circular cylinder. Let the front sharp boundary of the bunch be placed at $z = 0$. The potential over which one integrates in (4) is taken as having a discontinuity at $z = 0$. The phase factors can therefore be considered as constant up to $z = 0$ while they rapidly oscillate thereafter.

The phase factor of ϕ_f varies rapidly as function of x and y . The integral in (4) changes indeed by a factor of $N\alpha \sim 10^6$, using the super parameter (Table 1), when going from the axis to the edge of the bunch. Yet in our separating out a radiation before bunch crossing, the detailed properties of the bunch, which enter in the calculation of ϕ_f , should not be relevant. The bunch acts as a black box for the incoming electron and its geometry should be the only relevant factor. We therefore first proceed in a somewhat heuristic but simple way which emphasizes this point, considering the radiation of a free electron, thus neglecting altogether the phase factors in (4), but limiting the volume of integration according to the bunch geometry and allowing this way for some non-conservation of momentum through radiation. This simple approach will be later compared to a calculation which takes the phase factor in (4) into account while integrating over the electron momentum after bunch crossing. The two results are identical.

We therefore calculate the radiation from an electron of energy E_i over a volume extending from $-\infty$ up to the bunch edge ($z = 0$) and transversally over the bunch dimensions $r < R$. The discrepancy in momentum is absorbed by the bunch. It reads

$$\vec{\Delta} = \vec{k}_i - \vec{k}_f - \vec{k}_\gamma$$

The photon carries a fraction x of the initial energy of the electron $k = xE_i$, while the electron retains $E_f = (1-x)E_i$. We are particularly interested in radiation processes involving a sizeable value of x and not so much in the x^{-1} infra-red component.

From (2) the radiation amplitude is written as

$$A = \int_{r=0}^R dx dy \int_{-\infty}^0 dz e^{i \vec{\Delta} \cdot \vec{x}} M \quad (5)$$

with

$$M = -ie \bar{u}(k_f) \delta_\mu u(k_i) \varepsilon^\mu \quad (6)$$

where ε is the polarization vector of the radiated photon. We have omitted the phase factors (4) in (6). With the integral (5) cut off at the origin, the dependence of this phase on z is irrelevant. Here the electron is free.

The calculation of the volume integral over the x dependent phase factor gives

$$A = - \frac{2i\pi}{\Delta_L} \frac{1}{\Delta_T} R J_1(\Delta_T R) M \quad (7)$$

where Δ_L is the discrepancy in longitudinal momentum

$$\Delta_L = k_i - k_{fL} - k_{\gamma L} \quad (8)$$

and Δ_T is the modulus of the discrepancy in transverse momentum

$$\Delta_T = |\vec{\Delta}_T| \quad \vec{\Delta}_T = -(\vec{k}_{fT} + \vec{k}_{\gamma T}) \quad (9)$$

We define $\vec{\tau}$ as

$$\vec{\tau} = \vec{k}_{\gamma T} - \vec{k}_{fT} \quad (10)$$

While $\vec{\tau}$ can be large in modulus, $\vec{\Delta}_T$ will be kept small since it measures quantum variation over a transverse dimension R which, though small, is relatively large as compared to the electron Compton wavelength.

The photon radiation rate is given by:

$$\frac{1}{2E_i} \int |A|^2 2\pi \delta(E_i - E_f - k) \frac{d^3 \vec{k}_f d^3 \vec{k}_\gamma}{(2\pi)^6 4E_f k} \quad (11)$$

where $|A|^2$ implies a summation and averaging over polarization states. A particularly important quantity in A is Δ_L^{-1} and an easy calculation gives

$$\Delta_L^{-1} = 2E_i \left(\frac{m^2 x}{1-x} + \frac{|\vec{\Delta}_T + \frac{\vec{\tau}}{c}|^2}{4(1-x)} + \frac{|\vec{\Delta}_T - \frac{\vec{\tau}}{c}|^2}{4x} \right)^{-1} \quad (12)$$

showing its expected energy dependence.

We next have to compute $|\mathcal{M}|^2$. It is given by

$$\frac{e^2}{2} \bar{u} \left\{ (\gamma \cdot k_f) \gamma_\mu (\gamma \cdot k_i) \gamma_\nu \right\} \varepsilon^{\mu\nu} \quad (13)$$

with

$$\varepsilon^{\mu\nu} = g^{ij} + \frac{k_\gamma^i k_\gamma^j}{|\vec{k}_\gamma|^2} \quad (14)$$

using the radiation gauge.

For reasons previously given we drop terms proportional to Δ_T in the final result since we are not particularly interested in the $x \rightarrow 0$ limit. Using (13) and (14) one gets:

$$|\mathcal{M}|^2 = \frac{e^2}{2} \left\{ \frac{4m^2 x^2 + \frac{\tau^2}{c^2}}{x^2(1-x)} (1 + (1-x)^2) \right\} \quad (15)$$

Combining (7) and (15) with (11) we can then obtain the cross-section for radiation of a photon with fractional momentum x .

The integral over Δ_T , when we neglect Δ_T terms next to mx terms, can be carried out separately, by using Ref. 10), namely

$$\int_0^\infty \frac{J_1^2(R \Delta_T)}{\Delta_T} d\Delta_T = \frac{1}{2} \quad (16)$$

We normalized the cross-section for radiation to the surface area πR^2 offered by the bunch in order to obtain a radiation probability density as a function of x . It reads:

$$P(x) = \frac{\alpha}{2\pi} \frac{1 + (1-x)^2}{x} \ln \left(1 + \frac{\tau_{max}^2}{4 m^2 x^2} \right) \quad (17)$$

This is our main result. The probability density is proportional to α . This is lowest-order perturbation theory for the radiation of an electron which receives a violent perturbation through bunch crossing, breaking any coherence between radiation before and during crossing. The x term shows a well-known spin factor and the expected infra-red behaviour. Since we eventually took the $\Delta_T = 0$

limit, (17) is identical to the result obtained assuming an infinitely wide bunch. It remains to determine $\tau_{\max}^2$ as it appears in (17). Using (12), with $\vec{\Delta}_T = 0$, we have

$$\Delta_L = \frac{1}{2E_i} \frac{4m^2x^2 + \tau^2}{4x(1-x)} \quad (18)$$

The maximum value of Δ_L determines the maximum value of τ^2 . It is related to the minimum length beyond which our approximation breaks down. For a uniform bunch the order of magnitude of this length is clearly defined by L_c . For the super parameters (Table 3), since indeed $L_c \ll L_e$, we can separate out a calculation over L_e , performed down to L_c , and corresponding to radiation before bunch crossing.

For the radiation probability over L_e one natural choice is $\Delta_L^{\max} = L_c^{-1}$. With this choice, we get from (17)

$$P(x) = \frac{\alpha}{2\pi} \frac{1 + (1-x)^2}{x} \ln \frac{2(1-x)E_i}{x m^2 L_c} \quad (19)$$

We can then calculate from (19) the fractional energy loss δ , namely $\int_0^1 P(x) x dx$. It reads

$$\delta = \frac{\alpha}{2\pi} \left(\frac{1}{2} + \frac{4}{3} \ln \frac{2E_i}{m^2 L_c} \right) \quad (20)$$

With the parameter of the super (Table 3) we find a value of 1.7%.

A more detailed study of the integral in (2) suggests instead $(\Delta_L^{\max})^{-1} \sim (L_c^2 L_e)^{1/3}$. The value of δ for the super would then be 1.2%.

This particular contribution to beamstrahlung, so far negligible at lower energy, adds to the loss during bunch crossing. Note that it is independent of the size of the bunch and hence is expected to hold also for non-circular bunches, such as ribbon bunches since the width will be much larger than m^{-1} and thus justify taking the $\Delta_T = 0$ limit.

We now start again our calculation in a different way, writing instead of Eq. (5)

$$A = \int_{r=0}^{\infty} dx dy \int_{-\infty}^0 dz e^{i\vec{\Delta} \cdot \vec{x}} m e^{-i\varphi_f(x, y, z)} \quad (21)$$

The potential inside the bunch is well known

$$e\phi(r) = \frac{N\alpha r^2}{R^2 L_e} \quad (22)$$

and the second phase factor in (21) is therefore: $-iN\alpha r^2/R^2$.

The integral over x , y and z can be readily carried out. One gets:

$$\begin{aligned} A &= \frac{2\pi i}{\Delta_L} \int_0^\infty J_0(\Delta_T r) e^{-i \frac{N\alpha}{R^2} r^2} r dr M \\ &= -\frac{2\pi}{\Delta_L} \frac{R^2}{2N\alpha} e^{\frac{i}{4} \frac{R^2 \Delta_T^2}{N\alpha}} M \end{aligned} \quad (23)$$

We remark that Δ_T which appears in (23) is now the transverse momentum imposed by the crossing of the bunch, a large quantity compared with the transverse momenta Δ_T and τ previously defined in connection with radiation before bunch crossing (9), (10).

Indeed with the potential $\phi(r)$ of (22) one obtains:

$$\Delta_T = \frac{2N\alpha r}{R^2} \quad (24)$$

The calculation of the radiation probability proceeds as done previously, taking the limit of zero global transverse momentum before bunch crossing, and then performing the integration over Δ_T . The value of $|M|^2$ and of Δ_L are as previously obtained, (12) and (15). Instead of (16), we now have, from (23), a factor $(R/2N\alpha)^2$ times the phase space integral over Δ_T , the transverse momentum after bunch crossing, namely:

$$K = \left(\frac{R}{2N\alpha}\right)^2 \int_0^{\Delta_{Tmax}} \Delta_T d\Delta_T \quad (25)$$

The maximum value of Δ_T is obtained from (24), $\Delta_{Tmax} = 2N\alpha/R$, which used in (25) gives $K = \frac{1}{2}$. We thus obtain the same result.

We calculated the new Δ_T in a classical way. We may also calculate an effective impact parameter r_0 , matching the two rapidly varying phases on both sides of (23). One obtains:

$$r_0 = \frac{\Delta_T R^2}{2N\alpha} \quad (26)$$

The maximum value of Δ_T is obtained for $r_0 = R$ and one obtains again the value calculated through the classical expression (24).

Reaching the same answer for the mean energy loss following these two different approaches, we can thus single out this quantum radiation before bunch crossing which is, as it should be, independent of the number of particles in the bunch.

The technique used here to calculate beamstrahlung on the basis of Feynman diagram can be generalized to include radiation during bunch crossing, taking into account the explicit form of ϕ_i and ϕ_f beyond the approximation (4) and a phase varying with z for $z > 0$. This involves a more complex calculation which will be the object of another paper.

In a recent paper Blankenbecler and Drell⁹⁾ have calculated beamstrahlung at high energy. Only after a full calculation is completed could one compare their approach to the present one. It, however, seems that the quantum effect here spelled out is not included in their calculation and could perhaps be simply added to their result.

ACKNOWLEDGEMENT

For helpful discussions we are greatly indebted to Professor Jochen Bartels and Professor John Bell. We have also had useful conversations with Dr. Pisan Chen, Dr. Albert Hofmann and Dr. Wolfgang Schnell. One of us (T.T.W.) wishes to thank the Theory Division for its kind hospitality at CERN.

	SLC	Super
E (energy)	50 GeV×50 GeV	5 TeV×5 TeV
f (frequency of collision)	180 Hz	100 Hz
σ_r (radius of bunch)	1 μm	5 $\text{\AA}$
σ_z (length of bunch)	1 mm	1/3 μm
N (number of particle in each bunch)	5×10^{10}	3×10^8

TABLE 1

Machine parameters for SLC and super

	SLC	Super
$\gamma = E/m$	10^5	10^7
$D = r_e \sigma_z N / \sigma_r^2 \gamma$ (disruption factor)	1.4	0.11
$\bar{\theta}_b = 1/\gamma$ (bremsstrahlung angle)	10^{-5}	10^{-7}
$\bar{\theta}_d = r_e N / \sigma_r \gamma$ (deflection angle)	1.4×10^{-3}	1.7×10^{-4}
$\bar{L}_c = \sigma_z \bar{\theta}_b / \bar{\theta}_d$ (correlation length)	$7 \times 10^{-6} \text{m}$	$2 \times 10^{-10} \text{m}$
$\bar{L}_b = \sigma_z$ (bunch length)	10^{-3}m	$3 \times 10^{-7} \text{m}$
$\bar{L}_e = \gamma r_e / \alpha$ (virtual e length)	$4 \times 10^{-8} \text{m}$	$4 \times 10^{-6} \text{m}$

TABLE 2

Some important quantities related to beamstrahlung. Here r_e is the classical radius of the electron: α/m

	SLC	Super
$L_c = \gamma \bar{L}_c$	0.7	2×10^{-3}
$L_b = \gamma \bar{L}_b$	100	3
$L_e = \gamma \bar{L}_e$	4×10^{-3}	40

TABLE 3
The three key lengths for SLC
and super in the bunch frame.
All quantities are given in
metre.

REFERENCES

- 1) M. Tigner, *Nuovo Cimento* 37 (1965) 1228.
- 2) U. Amaldi, *Phys. Lett.* 61B (1976) 313;
B. Richter, *Nucl. Instr. and Methods* 136 (1976) 47.
- 3) Snowmass '82 DPF Summer Study: R. Donaldson, R. Cristafson and F. Paize, eds. and Proceedings of the following Snowmass meetings;
CERN-EFCA Workshop '84 CERN 84-10;
La Thuile Workshop '87 to be published.
- 4) A.A. Sokolov and I.M. Ternov, *Soviet Physics - JETP* 25 (1953) 698;
Synchrotron Radiation (Pergamon Press, 1968) Chapter 2.
- 5) T. Himel and J. Siegrist, SLAC-PUB-3572 (1985).
- 6) L.I. Schiff, *Phys. Rev.* 103 (1956) 443;
D.S. Saxon, *Phys. Rev.* 107 (1957) 871;
H. Cheng and T.T. Wu in High Energy Collisions, ed. C.N. Yang (Gordon and Breach, 1969) 329.
- 7) T.T. Wu, *Phys. Rev.* 108 (1957) 466;
D.S. Saxon and L.I. Schiff, *Nuovo Cimento* 6 (1957) 614;
V.P. Maslov and M.V. Fedoriuk, Semi-classical Approximation in Quantum Mechanics (D. Reidel Publishing Company, 1981) Chapter 14.
- 8) B. Richter, *Proceedings 1984 ICFA Seminar on Future Perspectives in High Energy Physics* (1984) 226.
- 9) R. Blankenbecler and S.D. Drell, SLAC - PUB-4186 (1987).
- 10) Bateman Manuscript Project, Higher Transcendental Functions, ed. A. Erdelyi, Vol. 2 (1953) 51, Eq. (30).

Fig.1: Feynman diagram for $e^- + \text{bunch} \rightarrow e^- + \gamma + \text{bunch}$