CERN-TH.4907/87

QUANTUM CALCULATION OF BEAMSTRAHLUNG: THE DIRAC CASE

M. Jacob
CERN - Geneva

and

Tai Tsun Wu^{*)}
CERN - Geneva
and
Gordon McKay Laboratory, Harvard University

ABSTRACT

Bremsstrahlung energy losses by electrons and positrons in very high energy linear colliders are calculated in terms of Feynman graphs. A follow-up to two previous papers, "Quantum Approach to Beamstrahlung" and "Quantum Calculation of Beamstrahlung: The Spinless Case", this paper deals with the calculation of radiative energy losses taking spin effects into account. The calculation applies to the special conditions set by machines in the TeV and multi-TeV ranges, when the disruption parameter remains small.

*) Work supported in part by the US Department of Energy under grant No. DE-FG02-84ER40158.

1. - INTRODUCTION

Electron-positron linear colliders in the TeV and multi-TeV energy range are promising future machines [1,2]. However, they have to operate at very high luminosity in order to compensate for very small cross-sections. The luminosity is, up to a geometrical numerical factor, given by

$$L = \frac{N^2 f}{\pi R^2} \quad (1)$$

where N is the number of particles per bunch, f the repetition frequency and R the bunch radius. Since increasing N and f increases the power cost, one is led to reduce R as much as possible. Bunch radii of the order of 10^{-9}m are usually considered [1,2]. Electrons (positrons) crossing a very high density positron (electron) bunch thus suffer strong acceleration and radiate a sizeable fraction of their incident energy. This is usually referred to as beamstrahlung [2,3]. Conditions are such that a quantum treatment is required.

In two previous papers [4,5], we analyzed this intense beamstrahlung, using the distorted wave Born approximation. The method was presented in Ref. [4], "Quantum Approach to Beamstrahlung" and radiation before bunch-crossing was separately calculated, this being justified at very high energy. In Ref. [5], "Quantum Calculation of Beamstrahlung: The Spinless Case", our calculation procedure was applied to radiation during bunch-crossing, while neglecting complications due to spin. A very simple expression was derived for the fractional energy loss, namely

$$\delta = K \frac{\alpha}{\pi} \frac{L_b}{\ell_c} \quad (2)$$

where L_b is the bunch length and ℓ_c is a coherent radiation length special to this particular regime. It was introduced through the study of the phase dependence of the radiation process [5]. The quantity K in (2) is a purely numerical factor of order 1. While the value found for K was sensitive to our neglecting spin complications, the form of Eq. (2) is quite general. It is accurate provided that the disruption parameter D remains small ($D \ll 1$), and that $L_e \gg L_c$ where L_e and L_c are respectively the virtual electron length and the classical coherent radiation length [4]. Both conditions are met with the "super" parameters [2,4].

In the present paper we extend our calculation procedure to the Dirac case, showing explicitly the parallel with, and departure from, the spinless case. We refer the reader to Ref. [5] for a detailed presentation of the calculation procedure which is here merely outlined as follows.

We consider the radiation of an electron in the field created by a positron bunch, an approach to beamstrahlung which is justified provided that the disruption parameter is small. The radiation amplitude is treated in the distorted wave Born approximation. It is written as [4]

$$M = \varepsilon^\mu M_\mu \quad (3)$$

$$M_\mu = -ie \int d^3\vec{x} e^{-i\vec{k}_\gamma \cdot \vec{x}} \bar{\psi}_f(\vec{x}) \gamma_\mu \psi_i(\vec{x}) \quad (4)$$

where ε and k_γ are the polarization vector and the momentum of the radiated photon, and $\psi_f(\psi_i)$ are the final (initial) electron wave functions in the presence of the bunch.

The main steps in the calculation of the radiation rate are the following:

- i) Calculate ψ_i and ψ_f in the presence of the bunch in the high-energy and small D approximations.
- ii) Study the stationary phase conditions which control radiation.
- iii) Compute the radiation matrix elements under such stationary phase conditions.
- iv) Compute the space integral in (4). This is done separately for the two photon polarization components taken in the classical bending plane and normal to it respectively, in analogy with the classical treatment of synchrotron radiation [6].
- v) Compute the radiation rate $I(X)$, where X is the fraction of the electron energy taken away by the photon.
- vi) Compute the fractional energy loss

$$\delta = \int_0^1 X \times I(X) dX$$

In Ref. [4], we emphasize the rôle of three characteristic lengths, namely the bunch length L_b , the coherent radiation length L_c and the virtual electron length L_e , with

$$L_c^3 = L_b^2 L_e \quad (5)$$

and in Ref. [5], we showed how δ can be simply expressed in terms of them (2).

This paper is organized as follows. Section 2 deals with point (i). We show that for both the Dirac case and the Klein-Gordon case which was previously studied [5], the phase conditions [point (ii)] are the same. There are, as expected, non-trivial effects specific to the Dirac case which affect the spinor wave function. In Section 3, we compute the different radiation amplitudes [points (iii)] and

(iv)], picking up first the non-helicity flip terms. Borrowing from Ref. [5], we then complete the calculation of the radiation rate in that case [points (v) and (vi)]. The non-flip contribution is by far the dominant one. The flip contribution, recently considered by Bell and Bell [3], is studied in Section 4. The result on beamstrahlung is summarized in Section 5 and general conclusions are presented in Section 6.

In order to spell out through an analytic calculation features special to this new machine regime, we retain the idealized bunch properties used in Ref. [5], namely a uniform cylindrical bunch with sharp boundaries. Typical values of the relevant parameters have been given in Refs. [4] and [5] and are summarized in the Table.

2. - THE ELECTRON WAVE FUNCTION

As in Ref. [5], we work in the bunch rest frame. The incident electron has an energy E and gains, in the bunch, a potential energy $eV(\vec{x})$ (Eq. (10) of Ref. [5]). In order to stress parallels and differences between the spinless and spin- $\frac{1}{2}$ cases, we treat them both together. However, to avoid confusion, equations referring explicitly to the Klein-Gordon case are denoted with a prime.

For the spin- $\frac{1}{2}$ case, the wave function satisfies the Dirac equation

$$\left(E - eV + i \vec{\alpha} \cdot \vec{\nabla} - m\beta \right) \psi(\vec{x}) = 0 \quad (6)$$

whereas, in the spinless case [5], it satisfies the Klein-Gordon equation

$$\left((E - eV)^2 + \vec{\nabla}^2 - m^2 \right) \psi(\vec{x}) = 0 \quad (6')$$

In Eq. (6), ψ is a spinor while in (6') it is a scalar.

In both cases, we write

$$\psi(\vec{x}) = e^{i\phi(\vec{x})} A(\vec{x}) \quad (7)$$

with

$$\phi(\vec{x}) = \vec{k} \cdot \vec{x} - \eta(\vec{x}) \quad (8)$$

where k is the momentum of the initial (final) electron.

We are at very high energy with relatively weak focusing (small disruption

parameter D), and the most rapidly varying part of $\phi(\vec{x})$ is therefore to be found with the first term in $\phi(\vec{x})$. It therefore makes sense to write (6) and (6') as equations for $\phi(\vec{x})$ and for $A(\vec{x})$ separately. On the basis of physical optics, it is natural to write the equation for $\phi(\vec{x})$ as

$$(E - eV)^2 - |\vec{\nabla}\phi|^2 - m^2 = 0 \quad (9)$$

for both cases.

Using Eq. (9), the equations for $A(\vec{x})$ are

$$(E - eV - \vec{\alpha} \cdot \vec{\nabla}\phi + i \vec{\alpha} \cdot \vec{\nabla} - m\beta) A(\vec{x}) = 0 \quad (10)$$

and

$$(2 \vec{\nabla}\phi \cdot \vec{\nabla} + \vec{\nabla}^2\phi - i \vec{\nabla}^2) A(\vec{x}) = 0 \quad (10')$$

We solve the common equation for ϕ (9) to second order in $1/k$, thus writing (8) as

$$\eta = \eta_0 + \frac{1}{k} \eta_1 \quad (11)$$

In Eq. (9), we treat separately the term linear in k and the one independent of k , neglecting terms of order k^{-1} . We thus use the assumption that D is reasonably small.

For the initial state, where the momentum is along the z axis, this gives

$$\begin{aligned} \frac{\partial \eta_0}{\partial z} &= eV \\ \frac{\partial \eta_1}{\partial z} &= \frac{1}{2} |\vec{\nabla}\eta_0|^2 - \frac{1}{2} e^2 V^2 \end{aligned} \quad (12)$$

Up to a constant term, we thus find

$$\eta_{0,i}(\vec{x}) = S_i(\vec{x}) \quad (13)$$

$$\eta_{1,i}(\vec{x}) = \frac{1}{2} \int_{-\infty}^z \left(\left(\frac{\partial S_i(\vec{x}')}{\partial x} \right)^2 + \left(\frac{\partial S_i(\vec{x}')}{\partial y} \right)^2 \right) dz' \quad (14)$$

where $\vec{x}'$ stands for x, y, z' and where

$$S_i(\vec{x}) = \int_{-\infty}^z e V(\vec{x}') dz' \quad (15)$$

Similarly for the final state, where the momentum of the electron $\vec{k}_f$ is used to define the xz plane [5], we have

$$\eta_{of}(\vec{x}) = -S_f(\vec{x}) \quad (16)$$

$$\eta_{if}(\vec{x}) = -\frac{1}{2} \int_z^\infty \left(\left(\frac{\partial S_f(\vec{x}')}{\partial x} \right)^2 + \left(\frac{\partial S_f(\vec{x}')}{\partial y} \right)^2 + 2k_{fx} \frac{\partial S_f(\vec{x}')}{\partial x} \right) dz' \quad (17)$$

where $\vec{x}'$ stands for x, y, z' and

$$S_f(\vec{x}) = \int_z^\infty e V(\vec{x}') dz' \quad (18)$$

The last term in (17) is calculated using (18). This gives

$$\eta_{if}(\vec{x}) = -\frac{1}{2} \int_z^\infty \left(\left(\frac{\partial S_f(\vec{x}')}{\partial x} \right)^2 + \left(\frac{\partial S_f(\vec{x}')}{\partial y} \right)^2 - 2k_{fx}(z-z') \frac{\partial}{\partial x} e V(\vec{x}') \right) dz' \quad (19)$$

The presence of the bunch changes the phase of the electron wave function. In both cases (Dirac and Klein-Gordon), we can write the phase ϕ defined by (8) and (9) as

$$\phi_i = k_i z - S_i - \frac{1}{2k_i} \int_{-\infty}^z \left(\left(\frac{\partial S_i}{\partial x} \right)^2 + \left(\frac{\partial S_i}{\partial y} \right)^2 \right) dz' \quad (20)$$

$$\begin{aligned} \phi_f = \vec{k}_f \cdot \vec{x} + S_f + \frac{1}{2k_f} \int_z^\infty & \left(\left(\frac{\partial S_f}{\partial x} \right)^2 + \left(\frac{\partial S_f}{\partial y} \right)^2 \right. \\ & \left. - 2k_{fx}(z-z') \frac{\partial}{\partial x} e V \right) dz' \end{aligned} \quad (21)$$

These are Eqs. (7) of Ref. [5], which were not derived there [7,8]. We remark that the ratio of the third to second terms on the right-hand side of both (20) and (21) is of order D . Our approach should therefore be looked at as an expansion in D , rather than as an expansion in $1/k$.

Once $\eta(\vec{x})$ is known [Eqs. (13)-(18)], one has to solve the equation for $A(\vec{x})$.

In the Klein-Gordon case, we use (10'). For the initial wave function, $\phi = kz - \eta$, and

$$k \frac{\partial A}{\partial z} - \vec{\nabla} \eta \cdot \vec{\nabla} A - \frac{1}{2} (\vec{\nabla}^2 \eta) A - \frac{1}{2} \vec{\nabla}^2 A = 0 \quad (22')$$

We expand A in powers of the $1/k$ and solve the equation for each term

$$A = A_0 + \frac{1}{k} A_1 \quad (23)$$

To leading order in k in (22')

$$\frac{\partial A_{0i}}{\partial z} = 0 \quad A_{0i} \equiv 1 \quad (24')$$

and, neglecting terms of order $1/k$,

$$\frac{\partial A_{1i}}{\partial z} = \frac{1}{2} \vec{\nabla}^2 \eta_0$$

or

$$A_{1i}(\vec{x}) = \frac{1}{2} e V(\vec{x}) + \frac{1}{2} \int_{-\infty}^z (\vec{\nabla}_\perp^2 S_i) dz' \quad (25')$$

For the final wave function, $\phi_f = kz - k_{fx} \times \eta$.

Proceeding in the same way, we find

$$A_{0f} \equiv 1$$

$$A_{1f}(\vec{x}) = \frac{1}{2} e V(\vec{x}) + \frac{1}{2} \int_z^\infty (\vec{\nabla}_\perp^2 S_f) dz' \quad (26')$$

Since we work to order $1/k$, there is no significant change in the normalization of the wave function. Provided that D remains small, the presence of the bunch only leads to changes in phase for the initial and final wave functions which are given by (20) and (21). This is not the case for the Dirac equation, where A_{1i} and A_{1f} include contributions which have to be kept.

We use (10) where $A(\vec{x})$ is a spinor, which is again written as

$$A = A_0 + \frac{1}{k} A_1 \quad (23)$$

It is normalized to $\bar{A}A = 1$ and, in the asymptotic limit, it can therefore be written explicitly as

$$A = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \chi \\ \frac{\vec{\sigma} \cdot \vec{k}}{E+m} \chi \end{pmatrix} \quad (27)$$

which we write, to order $1/k$, as

$$A \simeq \sqrt{\frac{k}{2m}} \left\{ \begin{pmatrix} \chi \\ \hat{\sigma} \cdot \hat{k} \chi \end{pmatrix} + \frac{m}{2k} \begin{pmatrix} \chi \\ -\hat{\sigma} \cdot \hat{k} \chi \end{pmatrix} \right\} \quad (28)$$

The asymptotic final spinors read

$$\begin{aligned} A_{0f}(\infty) &= \sqrt{\frac{k}{2m}} \begin{pmatrix} \chi_f \\ \sigma_3 \chi_f \end{pmatrix} \\ A_{1f}(\infty) &= \sqrt{\frac{k}{2m}} \begin{pmatrix} \frac{m}{2} \chi_f \\ (\sigma_1 k_x - \sigma_3 \frac{m}{2}) \chi_f \end{pmatrix} \end{aligned} \quad (29)$$

We separate the different terms in the Dirac equation (10)

$$\left(E - eV - \alpha_3 \frac{\partial \phi}{\partial z} + i \alpha_3 \frac{\partial}{\partial z} - \vec{\alpha}_\perp \cdot \vec{\nabla}_\perp \phi + i \vec{\alpha}_\perp \cdot \vec{\nabla}_\perp - m\beta \right) A = 0$$

Operating on the left with $1 \pm \alpha_3$, we can separate it into two equations [8]

$$\left(E - eV - \frac{\partial \phi}{\partial z} + i \frac{\partial}{\partial z} \right) A^{(+)} + \left(-\vec{\alpha}_\perp \cdot \vec{\nabla}_\perp \phi + i \vec{\alpha}_\perp \cdot \vec{\nabla}_\perp - m\beta \right) A^{(+)} = 0 \quad (30)$$

$$\left(E - eV + \frac{\partial \phi}{\partial z} - i \frac{\partial}{\partial z} \right) A^{(-)} + \left(-\vec{\alpha}_\perp \cdot \vec{\nabla}_\perp \phi + i \vec{\alpha}_\perp \cdot \vec{\nabla}_\perp - m\beta \right) A^{(-)} = 0 \quad (31)$$

where

$$\begin{aligned} A^{(+)} &= \frac{1 + \alpha_3}{2} A \\ A^{(-)} &= \frac{1 - \alpha_3}{2} A \end{aligned} \quad (32)$$

From Eqs. (9), we have indeed

$$\frac{\partial \phi}{\partial z} = E - eV + O\left(\frac{1}{k}\right)$$

and therefore there is a term linear in $E(k)$ in front of $A^{(-)}$ for (31), while none in front of $A^{(+)}$ for (30). This is the reason for solving separately two equations instead of one.

From (31), we obtain

$$A_0^{(-)} = 0 \quad A^{(-)}(\vec{x}) = \frac{1}{k} A_1^{(-)}(\vec{x}) \quad (33)$$

and, as a result, keeping only leading terms

$$A_1^{(-)} = \frac{1}{2} \left(\vec{\alpha}_\perp \cdot \vec{\nabla}_\perp \phi - i \vec{\alpha}_\perp \vec{\nabla}_\perp + m\beta \right) A_0^{(+)} \quad (34)$$

We discuss first the final-state wave function. In that case, to solve (30) we write, to order $1/k^2$,

$$\frac{\partial \phi}{\partial z} = E - \frac{m^2}{2k} - \frac{k_{fx}^2}{2k} - eV - \frac{1}{k} \frac{\partial T}{\partial z}$$

with, according to (21)

$$T = \frac{1}{2} \int_z^\infty \left(-2k_{fx}(z-z') \frac{\partial eV}{\partial x} + \left(\frac{\partial S_f}{\partial x} \right)^2 + \left(\frac{\partial S_f}{\partial y} \right)^2 \right) dz' \quad (35)$$

This gives, to order $1/k$, and with the help of (33),

$$\begin{aligned} & \left(\frac{m^2}{2k} + \frac{k_{fx}^2}{2k} - \frac{1}{k} \frac{\partial T}{\partial z} + i \frac{\partial}{\partial z} \right) \left(A_0^{(+)} + \frac{1}{k} A_1^{(+)} \right) + \\ & \left(-\vec{\alpha}_\perp \left(\vec{k}_\perp + \vec{\nabla}_\perp S \right) + i \vec{\alpha}_\perp \vec{\nabla}_\perp - m\beta \right) \frac{1}{k} A_1^{(-)} = 0 \end{aligned} \quad (36)$$

where, from (35)

$$\begin{aligned}\frac{\partial T}{\partial \mathfrak{z}} &= -\hbar_{fx} \int_{\mathfrak{z}}^{\infty} \frac{\partial eV}{\partial x} d\mathfrak{z}' - \frac{1}{2} \left(\left(\frac{\partial S_f}{\partial x} \right)^2 + \left(\frac{\partial S_f}{\partial y} \right)^2 \right) \\ &= -\hbar_{fx} \frac{\partial S_f}{\partial x} - \frac{1}{2} \left(\left(\frac{\partial S_f}{\partial x} \right)^2 + \left(\frac{\partial S_f}{\partial y} \right)^2 \right)\end{aligned}$$

We therefore obtain, to zeroth order in k ,

$$\frac{\partial A_0^{(+)}}{\partial \mathfrak{z}} = 0$$

hence

$$A_{0f}^{(+)} = A_{0f}^{(+)}(\infty) = \sqrt{\frac{\hbar_f}{2m}} \begin{pmatrix} \chi_f \\ \sigma_3 \chi_f \end{pmatrix} \quad (37)$$

For the $1/k$ order terms of (36), we get an equation for $A_1^{(+)}$ using (33) and (34). It reads

$$\begin{aligned}-i \frac{\partial}{\partial \mathfrak{z}} A_1^{(+)} &= \left\{ \frac{1}{2} \left(m^2 + |\vec{k}_\perp + \vec{\nabla}_\perp S_f|^2 \right) \right. \\ &\quad \left. + \frac{1}{2} \left(-\vec{\alpha}_\perp (\vec{k}_\perp + \vec{\nabla}_\perp S_f) + i \vec{\alpha}_\perp \vec{\nabla}_\perp - m\beta \right) \left(\vec{\alpha}_\perp (\vec{k}_\perp + \vec{\nabla}_\perp S_f) - i \vec{\alpha}_\perp \vec{\nabla}_\perp + m\beta \right) \right\} A_0^{(+)}\end{aligned} \quad (38)$$

and from the explicit form of $A_0^{(+)}$ (37), this simplifies to give

$$\begin{aligned}\frac{\partial}{\partial \mathfrak{z}} A_{1f}^{(+)} &= -\frac{1}{2} \left(\vec{\nabla}_\perp^2 S_f \right) \sqrt{\frac{\hbar_f}{2m}} \begin{pmatrix} \chi_f \\ \sigma_3 \chi_f \end{pmatrix} \\ A_{1f}^{(+)}(\vec{x}) &= A_{1f}^{(+)}(\infty) + \frac{1}{2} \int_{\mathfrak{z}}^{\infty} \left(\vec{\nabla}_\perp^2 S_f(\vec{x}') \right) \sqrt{\frac{\hbar_f}{2m}} \begin{pmatrix} \chi_f \\ \sigma_3 \chi_f \end{pmatrix} d\mathfrak{z}'\end{aligned} \quad (39)$$

We can now combine the different pieces of the final spinor wave function $A_f(\vec{x})$, namely

$$\begin{aligned}A_f(\vec{x}) &= \sqrt{\frac{\hbar_f}{2m}} \left\{ \begin{pmatrix} \chi_f \\ \sigma_3 \chi_f \end{pmatrix} + \frac{1}{\hbar_f} \begin{pmatrix} \frac{m}{2} \chi_f \\ (\sigma_1 \hbar_{fx} - \sigma_3 \frac{m}{2}) \chi_f \end{pmatrix} \right. \\ &\quad \left. + \frac{1}{2\hbar_f} \left(\vec{\alpha}_\perp \vec{\nabla}_\perp S_f(\vec{x}) + \int_{\mathfrak{z}}^{\infty} \vec{\nabla}_\perp^2 S_f(\vec{x}') d\mathfrak{z}' \right) \begin{pmatrix} \chi_f \\ \sigma_3 \chi_f \end{pmatrix} \right\} e^{i\phi_f(\vec{x})}\end{aligned} \quad (40)$$

We thus find explicitly how the spin orientation is affected by the bending. The

relevant term $\vec{\alpha}_\perp \vec{\nabla}_\perp S_i$, while of order $1/k$, cannot be neglected!

We now turn to the initial spinor wave function. The asymptotic spinor reads

$$\begin{aligned} A_{0i}(-\infty) &= \sqrt{\frac{k_i}{2m}} \begin{pmatrix} \chi_i \\ \sigma_3 \chi_i \end{pmatrix} \\ A_{1i}(-\infty) &= \sqrt{\frac{k_i}{2m}} \begin{pmatrix} \frac{m}{2} \chi_i \\ -\frac{m}{2} \sigma_3 \chi_i \end{pmatrix} \end{aligned} \quad (41)$$

As with (37) and (33), we now have

$$\begin{aligned} A_{0i}^{(+)} &= A_{0i}^{(+)}(-\infty) = \sqrt{\frac{k_i}{2m}} \begin{pmatrix} \chi_i \\ \sigma_3 \chi_i \end{pmatrix} \\ A_{0i}^{(-)} &= 0 \end{aligned} \quad (42)$$

Equation (34) now reads

$$A_{1i}^{(-)} = \frac{1}{2} \left(-\vec{\alpha}_\perp \cdot \vec{\nabla}_\perp S_i + m\beta \right) A_{0i}^{(+)} \quad (43)$$

and Eq. (30) with, to order $1/k^2$,

$$\frac{\partial \phi}{\partial z} = E - \frac{m^2}{2k} - eV - \frac{1}{2k} \left(\left(\frac{\partial S_i}{\partial x} \right)^2 + \left(\frac{\partial S_i}{\partial y} \right)^2 \right)$$

gives

$$\frac{\partial}{\partial z} A_{1i}^{(+)}(\vec{x}) = \frac{1}{2} \left(\vec{\nabla}_\perp^2 S_i(\vec{x}) \right) \sqrt{\frac{k_i}{2m}} \begin{pmatrix} \chi_i \\ \sigma_3 \chi_i \end{pmatrix} \quad (44)$$

and

$$A_{1i}^{(+)}(\vec{x}) = A_{1i}^{(+)}(-\infty) + \frac{1}{2} \int_{-\infty}^z \vec{\nabla}_\perp^2 S_i(\vec{x}') dz' \sqrt{\frac{k_i}{2m}} \begin{pmatrix} \chi_i \\ \sigma_3 \chi_i \end{pmatrix} \quad (45)$$

Combining the different pieces of the initial spinor wave function $A_i(\vec{x})$, we thus obtain

$$A_i(\vec{x}) = \sqrt{\frac{\hbar c}{2m}} \left\{ \begin{pmatrix} \chi_i \\ \sigma_3 \chi_i \end{pmatrix} + \frac{1}{\hbar c} \begin{pmatrix} \frac{m}{2} \chi_i \\ -\frac{m}{2} \sigma_3 \chi_i \end{pmatrix} \right. \\ \left. + \frac{1}{2\hbar c} \left(-\vec{\alpha}_\perp \cdot \vec{\nabla}_\perp S_i(\vec{x}) + \int_{-\infty}^{\delta} \vec{\nabla}_\perp^2 S_i(\vec{x}') d\zeta' \right) \begin{pmatrix} \chi_i \\ \sigma_3 \chi_i \end{pmatrix} \right\} e^{i\phi_i(\vec{x})} \quad (46)$$

These calculations are spelled out in detail in order to show the relevance of the high-energy, low-disruption-parameter, approximation. This concludes point (i). The overall phase $\phi(\vec{x})$ is the same in both the Dirac and Klein-Gordon cases. It is the one which controls the coherence conditions as obtained in Ref. [5]. In particular, relations (13) and (30) of Ref. [5] give the phase variation of the radiation amplitude for the uniform cylindrical bunch. We can thus turn directly to point (iii).

3. - THE RADIATION AMPLITUDES AND THE ENERGY LOSS

In order to follow as closely as possible the classical approach to synchrotron radiation [6], we consider separately two polarization components, namely the one in the classical bending plane (xz) and the one perpendicular to it.

As in Ref. [5], they are written

$$\Sigma_{||} = \begin{pmatrix} 1 & 0 & -\frac{\hbar \gamma x}{X \hbar c} \end{pmatrix} \\ \Sigma_{\perp} = \begin{pmatrix} 0 & 1 & -\frac{\hbar \gamma y}{X \hbar c} \end{pmatrix} \quad (47)$$

The Feynman radiation matrix element is

$$2m A_f^{\dagger}(\vec{x}) \vec{\alpha} \cdot \vec{\Sigma} A_i(\vec{x}) e^{i(\phi_i(\vec{x}) - \phi_f(\vec{x}) - \vec{k}_r \cdot \vec{x})} \quad (48)$$

which we write explicitly using (40) and (46) as:

$$\begin{aligned}
 & k_i \sqrt{1-X} \left\{ \begin{pmatrix} x_f \\ \sigma_3 x_f \end{pmatrix}^+ \left(1 + \frac{1}{2k_f} \int_0^\infty (\vec{\nabla}_\perp^2 S_f) d\delta' \right) + \frac{1}{k_f} \left(\frac{m}{2} x_f \right. \right. \\
 & \quad \left. \left. + \frac{1}{2k_f} \begin{pmatrix} x_f \\ \sigma_3 x_f \end{pmatrix}^+ \vec{\alpha}_\perp \vec{\nabla}_\perp S_f \right) \right\} \vec{\alpha}_\perp^+ \left\{ \left(1 + \frac{1}{2k_i} \int_{-\infty}^0 (\vec{\nabla}_\perp^2 S_i) d\delta' \right) \begin{pmatrix} x_i \\ \sigma_3 x_i \end{pmatrix} \right. \\
 & \quad \left. + \frac{1}{k_i} \begin{pmatrix} \frac{m}{2} x_i \\ -\frac{m}{2} x_i \end{pmatrix} - \frac{1}{2k_i} \vec{\alpha}_\perp \vec{\nabla}_\perp S_i \begin{pmatrix} x_i \\ \sigma_3 x_i \end{pmatrix} \right\} e^{i(\phi_i(x') - \phi_f(x) - k_f \cdot \vec{x})}
 \end{aligned} \tag{49}$$

This can be expressed as a matrix element between two two-component spinors, namely

$$M = x_f^+ (U + i \vec{\sigma}_\perp \cdot \vec{V}) x_i \tag{50}$$

separating a non-flip and a spin-flip amplitude. In the small disruption parameter approximation, only V_1 and V_2 contribute to the helicity flip term.

We first set $m = 0$ in (49) and pick up the leading terms. This eliminates the helicity flip terms (proportional to m). The amplitude then reads:

$$\begin{aligned}
 M = 2k_i \sqrt{1-X} x_f^+ \left\{ \varepsilon_3 + \frac{1}{2k_f} \sigma_1 k_{fx} \vec{\sigma}_\perp \cdot \vec{\varepsilon}_\perp + \right. \\
 \left. \frac{1}{2k_f} \vec{\sigma}_\perp (\vec{\nabla}_\perp S_f) (\vec{\sigma}_\perp \cdot \vec{\varepsilon}_\perp) - \frac{1}{2k_i} (\vec{\sigma}_\perp \cdot \vec{\varepsilon}_\perp) \vec{\sigma}_\perp (\vec{\nabla}_\perp S_i) \right\} x_i.
 \end{aligned} \tag{51}$$

In this case, only the third component of V (50) appears. The amplitude is written as

$$M = x_f^+ (U + i \sigma_3 V) x_i \tag{52}$$

and we obtain for the two polarization components (47)

$$\begin{aligned}
 U_\perp &= -k_{xy} \sqrt{1-X} \left(\frac{2}{X} + \frac{1-\tilde{\gamma}}{2(1-X)} - \frac{1+\tilde{\gamma}}{2} \right) \\
 V_\perp &= \Delta_{Tx} \sqrt{1-X} \left(\frac{k_{fx}}{\Delta_{Tx}(1-X)} + \frac{1}{2} \frac{1-\tilde{\gamma}}{1-X} + \frac{1}{2} (1+\tilde{\gamma}) \right) \\
 U_\parallel &= -\Delta_{Tx} \sqrt{1-X} \left(\frac{2k_{fx}}{X \Delta_{Tx}} - \frac{k_{fx}}{(1-X) \Delta_{Tx}} - \frac{1-\tilde{\gamma}}{2(1-X)} + \frac{1+\tilde{\gamma}}{2} \right) \\
 V_\parallel &= \frac{k_{xy}}{2} \sqrt{1-X} \left(\frac{1-\tilde{\gamma}}{1-X} + 1 + \tilde{\gamma} \right)
 \end{aligned} \tag{53}$$

when the calculation of all quantities is performed for a uniform cylindrical bunch and with the coherence conditions obtained in Ref. [5]. We have introduced [5]

$$\tilde{z} = \frac{2z}{L_L} \quad \vec{\Delta}_T = -\vec{k}_{\gamma T} - \vec{k}_{\perp T}$$

All terms have the same phase, factored out in (49), and which has thus not been written explicitly in (53).

We now pick up leading terms in (53), neglecting k_{yy} in front of k_{yx} (k_{fx}) and $(\tilde{z}-\tilde{z}_1)$, in front of 1, where z_1 is the radiation point picked up by the stationary phase conditions. This procedure and these approximation have been discussed in Ref. [5]. This gives:

$$\begin{aligned} U_{\perp} &= k_{\gamma y} \frac{k_{fx}}{\Delta_T} \frac{2-x}{x\sqrt{1-x}} \\ V_{\perp} &= -\Delta_T (\tilde{z}-\tilde{z}_1) \frac{x}{2\sqrt{1-x}} \\ U_{\parallel} &= -\frac{\Delta_T (2-x)}{2\sqrt{1-x}} (\tilde{z}-\tilde{z}_1) \\ V_{\parallel} &= k_{\gamma y} \frac{k_{fx}}{\Delta_T} \frac{1}{\sqrt{1-x}} \end{aligned} \quad (54)$$

The value of $\tilde{z}_1$ is given by relation (28) of Ref. [5], namely

$$\tilde{z}_1 = -1 - \frac{2}{x} \frac{k_{\gamma x} \Delta_{Tx} - k_{\gamma T}^2}{\Delta_T^2}$$

The discussion of the stationary phase is identical to that presented in detail in Ref. [5]. The amplitudes $U_{\perp}$ and $V_{\parallel}$, when integrated over space (4), can be expressed in terms of Airy functions, and the amplitudes $U_{\parallel}$ and $V_{\perp}$ in terms of derivatives of Airy functions. Denoting by script quantities the corresponding integrals, the sum and average over electron polarization retains only $\frac{1}{2} \text{Tr} \{ (u_{-i\sigma_3}^* v^*) (u_{+i\sigma_3} v) \}$, and the radiation rate is given by

$$R = |u_{\perp}|^2 + |v_{\perp}|^2 + |u_{\parallel}|^2 + |v_{\parallel}|^2 \quad (55)$$

The term involving the Airy integral (transverse photon polarization in the spinless case) now combines $U_{\perp}$ and $V_{\parallel}$. While we had an overall factor $(2k_{yy}(k_{fx}/\chi\Delta_T))^2$, we now have instead an overall factor (55):

$$\left(k_{\gamma y} \frac{k_{fx}}{\Delta_T} \right)^2 \left(\frac{(2-x)^2}{x^2(1-x)} + \frac{1}{1-x} \right)$$

This gives for that contribution an overall ratio between the Dirac and the Klein-Gordon cases which is simply given by $\frac{1}{2}((1+(1-x)^2)/(1-x))$.

The term involving the derivatives of the Airy integral (photon polarization in the classical bending plane for the spinless case) now combines $U_{\parallel}$ and $v_{\perp}$ (54). The overall factor $4\Delta_T^2/L_b^2$ of the spinless case is now replaced by an overall factor

$$\frac{4\Delta_T^2}{L_b^2} \frac{1}{4} \frac{1}{1-x} (x^2 + (2-x)^2)$$

and we therefore find again the same ratio $\frac{1}{2}((1+(1-x)^2)/(1-x))$ between the Dirac and Klein-Gordon cases.

For the non-helicity flip contribution, combining the non-flip U term and the part of the spin-flip amplitude V_3 , we can therefore take the rate computed in the spinless case and merely multiply it by that factor. In place of relation (58) of Ref. [5], we now have

$$I(x) = 1.94 \frac{\kappa}{\pi} \left(\frac{L_b}{e_c} \right) \pi R^2 \frac{1 + (1-x)^2}{2 x^{2/3} (1-x)^{1/2}} \quad (56)$$

and, for the fractional energy loss during bunch-crossing

$$\delta = 0.97 \frac{\kappa}{\pi} \left(\frac{L_b}{e_c} \right) \int_0^1 x^{1/3} (1-x)^{-1/3} (1 + (1-x)^2) dx \quad (57)$$

The integral over x is the sum of two beta functions which are readily calculated. The numerical value of δ is that calculated in the spinless case (Eq. (60) of Ref. [5]) times a factor 16/9 or 1.78.

The overall fractional energy loss is therefore of the order of 17% with the "super" parameters [2]. We thus confirm the overall proportionality between the spinless case and the non-helicity flip contribution in the Dirac case, obtained by Blankenbecler and Drell [2]. We obtain the simple scaling expression

$$\delta = 1.38 \frac{\kappa}{\pi} \left(\frac{L_b}{e_c} \right), \quad (58)$$

accurate in that regime.

4. - THE HELICITY FLIP CONTRIBUTION

We now pick up terms linear in m in the radiation amplitude (49), which were dropped when writing (51). It is clear that such terms, taken to leading order in $1/k$, contribute to the flip amplitude discussed by Bell and Bell [3]. Besides the V_3 term in (50), we now also have V_1 and V_2 terms. They are

$$\begin{aligned} V_{1L} &= -im \frac{x}{\sqrt{1-x}} & V_{1H} &= 0 \\ V_{2L} &= 0 & V_{2H} &= im \frac{x}{\sqrt{1-x}} \end{aligned} \quad (59)$$

These spin-flip terms contribute to the helicity flip amplitude, whereas V_3 does not in the low D approximation.

When polarization is not observed, there is no interference between the contribution of these flip terms and that calculated in Section 3. The former are proportional to m^2 . While they were a priori considered small, they should be compared to contributions to the non-flip rate, obtained from (54), which are proportional to k_{yy}^2 since, as discussed in Ref. [5], $\Delta_T(\tilde{z}-\tilde{z}_1) \sim k_{yy}$ and $k_{fx}/\Delta_T \sim 1$.

The helicity flip contribution will be negligible insofar as $k_{yy}^2 \gg m^2$. We found that (relation (43) of Ref. [5])

$$k_{xy} \sim \frac{2N\Delta}{R} \frac{\ell_c}{L_e} \quad (60)$$

With the "super" parameter, k_{yy} is therefore definitely large as compared with m ($k_{yy} \approx 40 m$). It was with this in mind that in the expression for Δ_L (relation (17) of Ref. [5]) we dropped $m^2 X^2$ terms in front of the others involving transverse momenta squared. In our treatment of the phase of the amplitude, and in particular in Eqs. (30) and (41) of Ref. [5], this same $m^2 X^2$ term should be added to $(k_{fx}^2/\Delta_T^2)k_{yy}^2$ as it appears. This looks simple, but keeping it leads to a much less transparent result, since the integration over Airy functions is no longer elementary.

It is clear that our final and very simple result for δ (58) is accurate insofar as $m \ll k_{yy}$ and, as repeatedly stated, as the disruption parameter remains small. Both these conditions are clearly satisfied with the "super" parameters [2,5], but neither are satisfied with the CLIC parameters [1]. The condition $m \ll k_{yy}$ can be written in a clearer way as $L_c \ll \lambda_c$, which, as can be easily checked, corresponds to the condition $C \ll 1$ in the notation of Blankenbecler and Drell [2]. More precisely, one has $C = \frac{1}{2}(L_c/\lambda_c)^3$. While the "super" regime corresponds to $L_c \ll \lambda_c$ ($L_e \gg L_c$), the CLIC regime gives $\lambda_c \approx L_c$.

Despite its relative smallness, we compute the flip rate using our calculation of the space integral. We take the expression for M_L (relation (44) of Ref. [5]) and merely substitute $mX/\sqrt{1-X}$ for $2k_{yy}(k_{fx}/X\Delta_T)$. Instead of relation (50) of Ref. [5], we now have

$$\begin{aligned} \int |M_F|^2 dk_{xy} &= \int_{-\infty}^{+\infty} (|V_{1L}|^2 + |V_{2L}|^2) dk_{xy} \\ &= 16 e^2 m^2 \left(\frac{x^2}{1-x} \right) \left(\frac{\pi R^2 L \ell}{N \alpha} \right)^2 \omega^{2/3} \int_0^\infty A_i^2(u) dk_{xy} \end{aligned} \quad (61)$$

with (Ref. [5])

$$\begin{aligned} u &= \frac{k_{fx}^2 k_{xy}^2}{\Delta_T^4} \left(\frac{L \ell \Delta_T^2}{6 x^2 (1-x) k_i} \right)^{2/3} \\ \omega &= \frac{6(1-x)}{x} \frac{k_i}{\Delta_T^2 L \ell} \end{aligned}$$

Equation (61) is transformed into

$$\int |M_F|^2 dk_{xy} = 16 m^2 e^2 x^2 \left(\frac{\pi R^2 L \ell}{N \alpha} \right)^2 \frac{k_i}{k_{fx} \ell} \int_0^\infty A_i^2(u^2) du \quad (62)$$

and the integral is rewritten as [9]

$$\int_0^\infty A_i^2(u^2) du = \frac{1}{18} \int_0^\infty K_{1/2}^2(u) du = \frac{\pi^2}{36} \quad (63)$$

The rate associated with the flip contribution is then obtained from relation (49) of Ref. [5], namely

$$\mathcal{I}_F(x) = \frac{\alpha}{36} \pi R^2 \left(\frac{m^2 L \ell}{k_i} \right) x^2 (1-x)^{-1} dx \quad (64)$$

to be added to (56).

In order to calculate this rate, and its contribution to δ , it should be realized that the apparent logarithmic divergence in (64) is due to the approximation of a small disruption parameter. In order to estimate the flip contribution to δ , the upper limit of integration over x should be replaced by $1-D$ (where $D = 0.11$ for the "super" parameters). One finds

$$\delta_F = \frac{\alpha}{36} \left(\frac{m^2 L \ell}{k_i} \right) \int_0^{1-D} \frac{x^3}{1-x} dx \quad (65)$$

The bracket contribution should of order L_b/ℓ_c (the non-flip contribution) times

$(m/k_{yy})^2 \sim (L_c/\ell_c)^2$. One can indeed rewrite (65) as

$$\delta_F = \frac{\alpha}{36} \left(\frac{L_c}{\ell_c} \right) \int_0^{1-\Delta} \frac{x^3}{1-x} dx \quad (66)$$

For the "super" parameter the flip contribution to δ is very small, namely 3×10^{-6} . It could of course become larger with different machine parameters, as discussed by Bell and Bell [3].

5. - BEAMSTRAHLUNG ENERGY LOSS

Neglecting the relatively small contribution from the helicity flip process discussed in Section 4, we can add the results of Section 3 to those of Ref. [4]. Therefore

$$\mathcal{J}(x) = I(x) + I_A(x) + I_B(x) \quad (67)$$

where $I(x)$ is given by (56), and I_A and I_B are the contributions after and before bunch-crossing resulting from the integration over space of the radiation amplitude (4), corresponding to the regions after the bunch $z > L_b/2$ and before the bunch $z < -L_b/2$ respectively, which do not interfere because of bending with the contribution originating during bunch-crossing. The overall relative energy loss is

$$\delta = \int_0^1 \mathcal{J}(x) \cdot x dx \quad (68)$$

Explicitly we have from (58) and Ref. [4]

$$I_B(x) = \frac{\alpha}{2\pi} \frac{1+(1-x)^2}{x} \ln \frac{2(1-x)E}{x m^2 \ell_c} \quad (69)$$

$$I(x) = \frac{\alpha}{\pi} 0.97 \frac{L_c}{\ell_c} \frac{1+(1-x)^2}{x^{2/3}(1-x)^{1/3}} \quad (70)$$

$$I_A(x) = \frac{\alpha}{2\pi} \frac{1+(1-x)^2}{x} \ln \frac{2(1-x)^2 E}{x m^2 \ell_c} \quad (71)$$

We should separate out the leading terms, including a large factor multiplying the overall α/π one. This gives

$$\delta = \frac{\alpha}{\pi} \left(1.38 \frac{L_e}{e_c} + \frac{4}{3} \ln \frac{4L_e}{e_c} \right) + O\left(\frac{\alpha}{\pi}\right) \quad (72)$$

We have to neglect terms of order α/π only, which have no reason to appear in view of the approximations made.

The following qualitative picture may be intuitively useful. For a given impact parameter (of order R), one can consider the beamstrahlung energy loss as a function of the angle of the radiated photon. One then finds two peaks, one at small angle and one at an angle corresponding to full bending, proportional to the impact parameter. Energy loss is small in intensity beyond these two peaks and is practically uniform in between. The two peaks correspond to δ_B and δ_A respectively (obtained from I_B and I_A) and the flat region in between corresponds to δ , as given by (58).

While the value of δ for a given luminosity should change according to the geometry of the bunch, the values of δ_A and δ_B are independent of the shape of the bunch and result from its presence.

6. - CONCLUSIONS

This is the third of a series of papers [4,5] which presents a distorted wave Born approximation approach to the problem of beamstrahlung. We itemize again our approximations.

We consider the regime of very high energy colliders ($E \sim \text{TeV}$) operating with large enough luminosity ($L \sim s$) and hence large transverse momentum values ($\Delta_T \gg m$), yet small compared with the incident energy ($E \gg \Delta_T$). The bending over the whole bunch is assumed to remain small ($D \ll 1$) so that the single electron approach makes sense, and so that one can limit oneself to leading terms in $1/k$, or rather D . In all practical cases, the bunch granularity turns out to be negligible ($\ell_c \ll N/L_b$).

Our approach makes close contact with previous ones [2,3], in particular with the recent and detailed analysis of Blankenbecler and Drell [2], but also with the classical treatment of synchrotron radiation [6], through our treatment of the coherence conditions. We give a complete quantum mechanical treatment of the beamstrahlung process for the uniform cylindrical bunch and reach a very simple scaling relation (72) for the radiative energy loss in the quantum case. We confirm that there is a close proportionality between the spinless case [5] and the Dirac case, up to a different shape for the radiation spectrum. The overall factor in the

fractional energy loss is a factor 16/9 in favour of the Dirac case. The flip contribution is relatively small and, with the "super" parameters considered, negligible.

We emphasize the rôle of three characteristic lengths in the problem, namely the bunch length L_b , the coherent radiation length L_c and the virtual electron length L_e , with a very special rôle for the length $\lambda_c = (L_c^2 L_e)^{1/3}$, which becomes the actual coherent radiation length in that regime.

ACKNOWLEDGEMENTS

We would like to acknowledge discussions with Jochen Bartels and John Bell, and an exchange of correspondence with Richard Blankenbecler and Sidney Drell. One of us (T.T.W.) wishes to thank the CERN Theoretical Physics Division for its hospitality.

	Super	CLIC
γ	10^7	2×10^6
$\bar{L}_b$	$3 \times 10^{-7} \text{ m}$	$5 \times 10^{-4} \text{ m}$
R	$5 \times 10^{-10} \text{ m}$	$6.5 \times 10^{-8} \text{ m}$
N	3×10^8	5.4×10^9

Table: "Super" and "CLIC" parameters used throughout. These parameters, taken from Refs. [1] and [2], are to be considered as provisional.

REFERENCES

- [1] M. Tigner, *Nuovo Cimento* 37 (1965) 1228;
B. Richter, *Nucl. Inst. and Methods* 136 (1976) 47;
U. Amaldi, *Phys. Lett.* 61B (1976) 313;
B. Richter, *Proceedings of the 1984 ICFA Seminar on "Future Perspectives in High-Energy Physics"*;
B. Richter, *IEEE trans. Nucl. Sci.* 32 (1985) 3928;
U. Amaldi, "Introduction to the Next Generation of Linear Colliders", CERN-EP/87-169. This paper contains the provisional CLIC parameters listed in the Table.
- [2] R. Blankenbecler and S. Drell, *Phys. Rev.* B36 (1987) 277. This paper contains the provisional "super" parameters listed in the Table.
- [3] T. Himel and J. Siegreist, *AIP Conference Proceedings* 130 (1985);
R.J. Noble, *SLAC-PUB-3871* (1986);
K. Yokoya, *Nucl. Inst. and Methods* A251 (1986) 1, and KEK preprint 85-9 (1985);
M. Bell and J.S. Bell, *CERN report PS 87-53* (1987);
P. Chen, *SLAC-PUB 4391* (1987).
- [4] M. Jacob and T.T. Wu, *Phys. Lett.* 197B (1987) 253.
- [5] M. Jacob and T.T. Wu, *CERN preprint TH.4848/87* (1987).
- [6] J.D. Jackson, "Classical Electrodynamics", Chapter 14 (John Wiley, 1975).
- [7] L.I. Schiff, *Phys. Rev.* 103 (1956) 443;
D.S. Saxon, *Phys. Rev.* 107 (1957) 871;
T.T. Wu, *Phys. Rev.* 108 (1957) 466;
D.S. Saxon and L.I. Schiff, *Nuovo Cimento* 6 (1957) 614;
V.P. Maslov and M.V. Fedoriuk, "Semiclassical Approximation in Quantum Mechanics", (D. Reidel Publ. Co., 1981).
- [8] H. Cheng and T.T. Wu, in "High Energy Collisions", C.N. Yang, ed., (Gordon and Breach, 1969), p. 329.
- [9] Bateman manuscript project, "Higher Transcendental Functions", A. Erdelyi, ed., (Vol. 2, 1953).