CERN-TH.4902/87

SPECTRUM DEGENERACY AND NEW SYMMETRIES FOR GENERALIZED EULER FORM ACTIONS

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A B S T R A C T

Higher-dimensional gravity theories based on dimensionally continued Euler forms lead to a degenerate spectrum. It contains fewer excitations than the one of a generic theory with arbitrary coefficients of the higher derivative terms. This can be traced back to a residual topological invariance of the action. The symmetry of the theory is enhanced. As a consequence the stability group of the vacuum can be larger than in the generic case. At the four-dimensional level this can even lead to infinite dimensional (Kac-Moody type) gauge algebras. A simple six-dimensional example leads to infinite towers of massless gravitons and scalars, but no spin-one bosons (gauge bosons) are present. We describe how modes decouple when the action approaches the Euler form. Concerning classical stability the Euler form corresponds to a singular point in the space of six-dimensional gravity actions.

CERN-TH.4902/87

November 1987

1. - INTRODUCTION

It is commonly accepted that any quantum theory of gravity, such as string theory, leads to higher derivative terms in the effective gravitational action. On the other hand, it is well known [1,2] that theories based on field equations with more than two derivatives suffer from tachyonic and ghost modes [3,4]. To circumvent this problem, it has been suggested [5] that all higher order corrections of the Einstein-Hilbert action should appear as dimensionally continued Euler forms which, when expanded around flat space, do not introduce any propagator corrections. They only modify the graviton-graviton interaction vertices. In Ref. [6] (henceforth referred to as I) we discussed in detail that this property is in general not sufficient to obtain a physically meaningful four-dimensional effective theory without unstable modes [7-10]. In I a classical stability analysis has been performed for the most general pure gravity action in $d = 4+D$ dimensions with at most four derivatives of the metric and for a ground state of the direct product form $\mathcal{M}^4 \times S^D$. The possible higher derivative terms are R^2 , $R_{\mu\nu} R^{\mu\nu}$ and $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$. The Euler form is obtained for a special choice of the coefficients of these invariants. For $D = 9$, for instance, it turned out that the Euler form action (together with the ordinary $\sqrt{g}R$ term and a higher-dimensional cosmological constant) did not meet the requirement of "low momentum stability", i.e., not all residues and mass terms of the lowest poles had the correct sign. Hence even if the corrections appear as Euler forms, a stable vacuum requires additional fields and/or a more complicated ground state. We have also argued that the effective action for gravity must not necessarily consist of dimensionally continued Euler forms. In fact, higher poles often indicate the breakdown of an approximation scheme (in the number of derivatives) rather than an inconsistency of the theory. Nevertheless, Euler forms are interesting points in parameter space. Approaching these forms the higher poles move to infinity and low momentum approximations can become reliable in the range of the compactification scale.

In this paper we want to show that besides introducing no higher poles dimensionally continued Euler forms lead to remarkable properties of the spectrum of particles emerging from spontaneous compactification. We shall see that Euler form actions with a given number of derivatives (e.g., $R^2 - 4R_{\mu\nu}^2 + R_{\mu\nu\rho\sigma}^2$ in the four derivative case) are singular points in the space of all actions (with arbitrary coefficients of R^2 , $R_{\mu\nu}^2$ and $R_{\mu\nu\rho\sigma}^2$). For Euler forms, many particles present in the general case are no longer propagating modes and decouple from the spectrum. Moreover, additional fields beyond the Yang-Mills gauge bosons and the graviton become massless [10]. This degeneracy of the spectrum can be traced back to the fact that Euler forms are topological invariants in their respective "critical" dimension. As a consequence, even in dimensions larger than the critical one, certain (topologically trivial) deformations of the metric do not change the action. This leads

to an enhanced symmetry of pure Euler form actions and renders some components of the higher dimensional metric non-dynamical. As a consequence of this enlarged symmetry, the linearized theory can no longer be coupled consistently to an arbitrary (conserved) energy-momentum tensor. Consistency of the linearized field equations implies constraints on the energy-momentum tensor which are much stronger than the standard covariant conservation requirement.

Another special feature of Euler form theories is the possibility of a stability group of the vacuum which is larger than $P_4 \times G$. (P_4 is the four-dimensional Poincaré group and G is the isometry group of internal space. Usually $P_4 \times G$ is the largest subgroup of general co-ordinate (and Lorentz) transformations $Gen(d)$ which leaves the ground state metric invariant [11,12].) In one of our examples the ground state symmetry corresponds to an infinite dimensional Kac-Moody type algebra. Infinite dimensional symmetries can be gauged by infinite towers of massless gravitons or vector gauge fields. Also different representations of G can become combined in a large representation of the new symmetry group.

We shall demonstrate the special properties of Euler form actions in a six-dimensional theory based on the four-derivative Euler form. Neither an Einstein-Hilbert term nor a cosmological constant will be included. The model has $\mathcal{M}^4 \times S^2$ as a solution of the field equations and we consider it as the (candidate) ground state. This example exhibits the new features of Euler form actions in a rather extreme form. Occasionally, we also consider the seven-dimensional theory with the same action and the ground state $\mathcal{M}^4 \times S^3$. Actions of this type recently have been investigated by Arik and Dereli [13] and by Ishikawa [10].

This paper is organized as follows. In Section 2 we introduce the model and derive its spectrum. We show how the constraints for the energy momentum tensor emerge. In Section 3 we explain the special features of the Euler form spectrum as a consequence of the topological properties of the action, i.e., of the fact that in four dimensions it is a topological invariant. In Section 4, the invariance group of the ground state is discussed. This gives further insight into the origin of the mass degeneracy found in Section 2. Finally, we describe in Section 5 the mechanisms by which states can decouple from the spectrum of propagating particles when the coefficients in a general action approach those of the Euler form.

2. - SPECTRUM AND SOURCE CONSTRAINTS

We discuss in this section the spectrum of six- and seven-dimensional gravity theories based on the dimensionally continued four-derivative Euler form E_4 , expanded around a ground state $\mathcal{M}^4 \times S^2(S^3)$. (We denote by E_{2N} the dimensionally continued Euler form with $2N$ derivatives. It is a topological invariant in its critical dimension $d = 2N$.) To stress the salient features of the model we consider it as a limiting case of the most general pure gravity action in $d = 4+D$ dimensions with at most four derivatives^{*)}:

$$S = -\frac{1}{N} \int d^d x \sqrt{|g|} \left(\alpha R^2 + \beta R_{\hat{\mu}\hat{\nu}} R^{\hat{\mu}\hat{\nu}} + \gamma R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} R^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} + \delta R + \varepsilon \right). \quad (2.1)$$

The normalization constant N is taken to be the volume of the D -dimensional internal manifold for the ground state. The parameters α , β and γ are dimensionless, whereas δ and ε have dimension $(\text{mass})^2$ and $(\text{mass})^4$, respectively. The Euler form model to be considered later is obtained from (2.1) for $(\alpha, \beta, \gamma) = \pm(1, -4, 1)$ and $\delta = \varepsilon = 0$.

The field equations derived from (2.1) have classical solutions of the form $\mathcal{M}^4 \times S^D$ where $\mathcal{M}^4$ is four-dimensional Minkowski space and S^D is a sphere whose radius L_0 is given by [14]

$$L_0^{-2} \equiv \gamma = \frac{\delta}{2\xi}, \quad (2.2)$$

with

$$\xi \equiv D(D-1)\alpha + (D-1)\beta + 2\gamma. \quad (2.3)$$

For the four-dimensional space to be flat, the d -dimensional cosmological constant ε has to be chosen as

$$\varepsilon = \frac{1}{4} D(D-1) \delta^2 \xi^{-1}. \quad (2.4)$$

Then one finds for the Planck mass

$$M_P^2 = 16\pi [\beta(D-1) + 2\gamma] \delta \xi^{-1}. \quad (2.5)$$

*) Our conventions are those of I. In particular, indices $\hat{\mu}, \hat{\nu}, \dots$ run from 0 to $d-1$, Minkowski space indices $\mu, \nu, \dots$ run from 0 to 3, and internal indices $\alpha, \beta, \dots$ from 4 to $d-1$. Occasionally we use $\partial^2 = \partial_\mu \partial^\mu$ and $D^2 = D_\alpha D^\alpha$.

To extract the particle spectrum with respect to this ground state (for a detailed discussion we refer to I) we expand the metric $g_{\hat{\mu}\hat{\nu}}$ around the background

$$\bar{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \bar{g}_{\alpha\beta}(x^\alpha) \end{pmatrix} \quad (2.6)$$

($\bar{g}_{\alpha\beta}$ is the metric on S^D) according to

$$g_{\hat{\mu}\hat{\nu}} = \bar{g}_{\hat{\mu}\hat{\nu}} + h_{\hat{\mu}\hat{\nu}} . \quad (2.7)$$

We insert this ansatz into the action (2.1) and keep the quadratic terms in $h_{\hat{\mu}\hat{\nu}}$ only. This results in a quadratic action of the form

$$S^{(2)} = -\frac{1}{N} \int d^d x \sqrt{|g|} \left\{ h_{\hat{\mu}\hat{\nu}} \Omega^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} h_{\hat{\rho}\hat{\sigma}} + h_{\hat{\mu}\hat{\nu}} Q^{\hat{\mu}\hat{\nu}} \right\} \quad (2.8)$$

with a complicated fourth-order differential operator $\Omega^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}$. To eliminate gauge modes a coupling of the fluctuations to an external source $Q^{\hat{\mu}\hat{\nu}}$ has been introduced. It must be covariantly conserved:

$$D_{\hat{\mu}} Q^{\hat{\mu}\hat{\nu}} = 0 . \quad (2.9)$$

The propagators of the physical excitations are found [7] by solving the linear field equations of (2.8) and inserting their solution $h_{\hat{\mu}\hat{\nu}}(Q)$ into $S^{(2)}$:

$$S_{\text{phys}}^{(2)} = -\frac{1}{2N} \int d^d x \sqrt{|g|} h_{\hat{\mu}\hat{\nu}}(Q) Q^{\hat{\mu}\hat{\nu}} . \quad (2.10)$$

For general values α, β, γ we found one tower of spin-2 particles (with the graviton at the lowest level), two towers of spin-1 particles (with the Yang-Mills fields at the lowest level of one of them) and five towers of spin-0 particles (four towers for $D = 2$). Except for the graviton and the gauge bosons all excitations are massive. The equations of motion did not place any restrictions on $Q_{\hat{\mu}\hat{\nu}}$ except for covariant conservation.

Let us now look at what happens when (α, β, γ) approach their Euler form values $\pm(1, -4, 1)$. [The more general case $\pm(\alpha, -4\alpha, \alpha)$ obtains by a trivial multiplicative scaling of the action.] We first concentrate on $D = 2$. From (2.3) one has $\zeta = 0$. According to (2.2) a non-vanishing radius L_0 requires $\delta = 0$. Then, for a finite ratio δ/ζ , Eq. (2.4) implies that ϵ has to vanish too. Hence, for $D = 2$ and the Euler form values for α, β and γ , $\mathcal{M}^4 \times S^2$ is a solution of the field equations only if neither an Einstein-Hilbert term nor a cosmological constant is present (we may call this a pure Euler form action). To be precise, our Euler form model is defined by

$$(\alpha, \beta, \gamma) = -(1, -4, 1), \quad \varepsilon = \delta = 0, \quad D=2, \quad (2.11)$$

because only this choice of the overall sign leads to a positive value of M_p^2 . Note that in this model the compactification scale $L_0^{-1} \equiv y^{\frac{1}{2}}$ is not determined by the equations of motion since these do not contain any dimensionful parameter. Combining (2.2) and (2.5) we nevertheless can relate it to the Planck mass:

$$M_p^2 = 64 \pi L_0^{-2}. \quad (2.12)$$

We describe the detailed steps of the mode analysis for the pure Euler form action in the Appendix. One obtains the following results for the various towers of four-dimensional fields with given spin.

(A) Tensor fields. The equation of motion for the spin-2 fields reads:

$$k^2 H_{TT}^{\mu\nu}{}_{lm}(k) = (2y)^{-1} q_{TT}^{\mu\nu}{}_{lm}(k), \quad (2.13a)$$

and from (2.10) one obtains for the action

$$S_{\text{phys}}^{(2)}(H_{TT}^{\mu\nu}) = -\frac{1}{4yN} \int \frac{d^4k}{(2\pi)^4} \sum_{l=0}^{\infty} \sum_m |q_{TT}^{\mu\nu}{}_{lm}(k)|^2 / k^2. \quad (2.13b)$$

This means that there exists a massless spin-2 excitation for any value of l and m . We here encounter the remarkable situation that even at the higher levels $l = 1, 2, 3, \dots$, where the internal Laplacian $\bar{D}^2$ has non-zero eigenvalues when acting on scalar harmonics, the fields are massless. The only distinction between these infinitely many gravitons are their $SO(3)$ quantum numbers. We observe that the residues of the poles have the correct sign for the Euclidean action to be positive definite, i.e., for these modes the requirements of classical stability are fulfilled.

(B) Vector fields. (i) The field $V_{T^1}^{\mu}$ may be varied independently. It enters $S^{(2)}$ only via its source term. As a consequence, the variation yields [cf. Eq. (A.6)]

$$q_{T^1}^{\mu}{}_{lm} = 0, \quad l \geq 1. \quad (2.14a)$$

This is not a genuine field equation for $V_{T^1}^{\mu}$, but rather a constraint on its source. Since $V_{T^1}^{\mu}$ does not appear on the left-hand side of the equation of motion, a

consistent coupling to matter requires the corresponding component of $Q_{\mu\nu}$ to be zero. To understand this constraint dynamically, consider a field equation of the general form $ck^2(\text{field}) = (\text{source})$ and assume that c tends to zero. This is the situation of α , β and γ approaching their Euler form values. Then, for any given non-zero source function, the field excitation diverges like $1/c$. Thus the limit $c \rightarrow 0$ exists only if the relevant component of the energy-momentum tensor vanishes identically. In the generic model the fields $V_{T^1, \ell=1, m}^\mu$ represent the Yang-Mills gauge bosons which do not appear as propagating modes in the present case. This does not contradict the fact that they may be considered Goldstone bosons related to the symmetry breaking by the ground state since, loosely speaking, Goldstone's theorem only predicts the existence of flat directions of the potential but it does not guarantee a non-vanishing kinetic term.

(ii) The situation is similar for the coupled set $\{V_{T^2}^\mu, [H_T^\mu]; \ell \geq 1\}$. Using (A.7g) to eliminate H_T^μ and its analogue for the source one finds for the contribution to the action

$$S^{(2)}(V_{T^2}^\mu) = \frac{2}{N} \int \frac{d^4 k}{(2\pi)^4} \sum_{\ell m} \{k^2 - \ell(\ell+1)\gamma\} / k^2 V_{T^2 \ell m}^{\mu*} q_{T^2 \ell m}^\mu$$

which implies

$$q_{T^2 \ell m}^\mu = 0, \quad \ell \geq 1. \quad (2.14b)$$

Apart from the subtlety due to the mixing of $V_{T^2}^\mu$ and H_T^μ , the constraints (2.14) reflect the fact that for vanishing sources $S^{(2)}$ becomes independent of $h_{\alpha\mu}$, cf. Eq. (A.2) for the action E_4 .

(C) Scalar fields. (i) Using (A.7b), the set $\{S_2, [\phi_1]; \ell \geq 2\}$ contributes as

$$S^{(2)}(S_2) = -\frac{1}{N} \int \frac{d^4 k}{(2\pi)^4} \sum_{\ell m} \{1 - [\ell(\ell+1) - 2]\gamma/k^2\} S_{2 \ell m}^* Q_{2 \ell m}.$$

We again conclude

$$Q_{2 \ell m}(k) = 0, \quad \ell \geq 2. \quad (2.15a)$$

(ii) For the 0-level scalars $\{S_4, \sigma, [\eta]; \ell=0\}$ we first observe that S_4 enters only via its source term implying

$$Q_{4, \ell=0, m}(k) = 0. \quad (2.15b)$$

Eliminating η by (A.7f) and substituting the equation of motion (which is of the usual form in this case) into (2.10) yields

$$S_{\text{phys}}^{(2)}(\sigma | l=0) = \frac{1}{12 N y} \int \frac{d^4 k}{(2\pi)^4} |q_{\mu, l=0}^k|^2 / k^2, \quad (2.16)$$

i.e., we find a single massless spin-0 particle. The sign of the residue is opposite to the one of the graviton propagator. Hence this particle is a ghost. We note that scalar modes with $l = 0$ are in general related to the internal volume degree of freedom often discussed in Kaluza-Klein cosmology. In the present Euler form case the scalar singlet becomes a sort of dilaton with "wrong" sign of the kinetic term.

(iii) For the set $\{S_4, \sigma, [\eta, \phi_2]; l=1\}$ it is convenient to introduce the linear combinations $G_{\pm} = S_4 \pm (k^2/2y)\sigma$. After having eliminated the dependent fields, the relevant part of the action (A.6) reads

$$S^{(2)}(S_4, \sigma | l=1) = -\frac{y}{3N} \int \frac{d^4 k}{(2\pi)^4} \left\{ 8y^2 |G_+|^2 / k^2 + G_+ [Q_4' + q_{\mu}^k / k^2] + G_- [Q_4' - q_{\mu}^k / k^2] \right\}, \quad (2.17)$$

where $Q_4' \equiv 3(y^{-1} - 2/k^2)Q_4$. Obviously G_- has no propagator and we have to require $Q_4' - q_{\mu}^k / k^2 = 0$ or

$$q_{\mu, l=1}^k(k) = 3(k^2/y - 2)Q_{4, l=1}(k). \quad (2.15c)$$

The other linear combination is propagating; using (2.15c) we can express its action in terms of q_{μ}^k alone:

$$S_{\text{phys}}^{(2)}(\sigma | l=1) = \frac{1}{24 N y} \int \frac{d^4 k}{(2\pi)^4} |q_{\mu, l=1}^k|^2 / k^2. \quad (2.18)$$

One again finds a massless scalar ghost.

(iv) In the generic case the system $\{S_3, S_4, \sigma, [\eta, \phi_2]; l \geq 2\}$ contains three towers of scalars. However, similar as above, here one can introduce linear combinations of S_3 , S_4 and σ such that two of them have no kinetic term and hence lead to source constraints. The third linear combination gives rise to a tower of propagating scalars. A lengthy calculation yields the constraints

$$Q_{3, l=2}(k) = Q_{4, l=2}(k) = 0, \quad l \geq 2. \quad (2.15d)$$

and the action for the propagating modes

$$S_{\text{phys}}^{(2)}(s_3, s_4, \sigma) = \frac{1}{24 N_Y} \int \frac{d^4 k}{(2\pi)^4} \sum_{\ell=2}^{\infty} \sum_m |q_{\ell m}^k + \dots|^2 / k^2. \quad (2.19)$$

The dots denote terms containing Q_3 and Q_4 which vanish when the constraints are imposed. As in the graviton case, the particles are massless at all levels. The sign of the residues corresponds to a ghost. Note that all propagating scalars found here are contained in the Minkowski space trace $\sigma \equiv h_{\mu}^{\mu}$, cf. Eqs. (2.16), (2.18) and (2.19).

To conclude, we found that the spectrum of the Euler form model (2.11) consists of one tower of massless spin-2 particles and one tower of massless scalars^{*)}. There are no massive states at all. If the overall sign of the action is chosen such that the tensor fields have positive Euclidean action, all scalars have negative (Euclidean) kinetic terms. [Without interactions between the tensor tower and the scalar tower both sectors of the model are separately classically stable. Possible interactions between the scalars and the tensors would render the ground state unstable. Since in this paper we are not primarily interested in classical stability, but rather in symmetry aspects and mode decoupling, we will not pursue this problem here further.] Furthermore, the linearized theory can be consistently coupled only to energy momentum tensors respecting the source constraints (2.14) and (2.15). [Going beyond the linear approximation it is possible that these constraints become weaker since the field equations could equate interaction terms to source components which to lowest order have been constrained to be zero. The respective fields are still non-propagating, of course.] As compared to the generic theory, the Euler form model has two vector towers and three scalar towers less in its spectrum.

Before closing this section, let us briefly look at the Euler form model for $D = 3$. The parameter ζ still vanishes and from (2.2), (2.4) and (2.5) we see that a classical solution with $M_P^2 > 0$ is again possible for $\delta = \varepsilon = 0$. The simplification of the quadratic action is less dramatic than for $D = 2$. For instance, the coefficients G_3 and G_4 in Eq. (A.2) are non-zero and render the tensor particles massive (except for the graviton). Furthermore, one has $F_2 \neq 0$, but $F_3 = F_4 = 0$, i.e., there is a kinetic term for $h_{\alpha\mu}$, but no mass term. A careful analysis using the gauge constraints indeed reveals a whole tower of massless vector particles with the ordinary Yang-Mills fields sitting at its lowest level. This has to be compared to the two massive (except for the gauge fields) vector towers of the generic

*) The massless spin-2 tower also has been found in Ref. [10]. As far as the scalar sector is concerned, our results differ, however.

theory. The analysis of the scalar sector is more involved than for $D = 2$ and we do not go into the details here.

3. - NEW SYMMETRIES FOR dimensionally CONTINUED EULER FORM ACTIONS

In this section we show that the six-dimensional gravity action based on the Euler form E_4 has a symmetry much larger than the usual six-dimensional general covariance. This symmetry generalizes to other dimensions and arbitrary Euler forms. It is connected with topological properties of dimensionally continued Euler forms. Using this symmetry we show how the spectrum degeneracy of the Euler form theory and its source constraints can be understood from simple geometrical arguments.

Euler forms are topological invariants in their respective critical dimension. This means that a gravity action in $2N$ dimensions based on the Euler form E_{2N} is invariant under all continuous deformations of the geometry (which leave the topology of the manifold unchanged). Infinitesimally, this corresponds to arbitrary variations of the metric $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$ with the only constraint on $\delta g_{\mu\nu}$ that it is consistent with the topology of the manifold with metric $g_{\mu\nu}$. We denote the symmetry group of such continuous changes of the metric by $\text{Def}(2N)$, the group of deformations in $2N$ dimensions.

Consider now the Euler form E_4 in six dimensions and a given metric which describes a direct product of a two-dimensional flat space ($\mathcal{M}^2$ or $\mathbb{R}^2$) with an arbitrary four-dimensional space $\mathcal{W}^4$ parametrized by $g_{\eta\phi}^{(\mathcal{W})}$. The action based on E_4 is left invariant by an arbitrary continuous deformation of the four-dimensional metric $g_{\eta\phi}^{(\mathcal{W})} \rightarrow g_{\eta\phi}^{(\mathcal{W})} + \delta g_{\eta\phi}^{(\mathcal{W})}$. [Only $g_{\eta\phi}^{(\mathcal{W})}$ contributes to E_4 in this case.] We will denote the symmetry group of the six-dimensional action by $\text{Def}(4|2)^*$. The symmetry $\text{Def}(4|2)$ means that the action is independent of the fields $g_{\eta\phi}^{(\mathcal{W})}$ parametrizing the metric on $\mathcal{W}^4$. Hence these fields do not correspond to dynamical degrees of freedom.

We are interested in the action of $\text{Def}(4|2)$ transformations on fluctuations $h_{\hat{\mu}\hat{\nu}}$ around the ground state $\mathcal{M}^4 \times S^2$. We consider metric configurations $g_{\hat{\mu}\hat{\nu}} = g_{\hat{\mu}\hat{\nu}} + h_{\hat{\mu}\hat{\nu}}$

*) This defines the action of $\text{Def}(4|2)$ only for metric configurations corresponding to direct product spaces $\mathcal{W}^4 \times \mathcal{M}^2$ or $\mathcal{W}^4 \times \mathbb{R}^2$. The possibility of a (perhaps non-linear and non-local) extension to neighbouring configurations is left open. We note that $\text{Gen}(d)$ is a subgroup of $\text{Def}(d)$, but neither $\text{Gen}(6)$ is contained in $\text{Def}(4|2)$ nor $\text{Def}(4|2)$ is contained in $\text{Gen}(6)$. New symmetry transformations can be obtained by combining $\text{Def}(4|2)$ and $\text{Gen}(6)$ transformations.

where two of the Minkowski space indices (μ_1, μ_2) correspond to a flat space $\mathcal{M}^2(\mathbb{R}^2)$. The other two Minkowski co-ordinates (μ_3, μ_4) together with the curved internal co-ordinates form the curved space $\mathcal{W}^4$ on which the action of Def(4|2) is defined. All fluctuations corresponding to this type of configuration can be obtained from $\mathcal{M}^4 \times S^2$ by Def(4|2) transformations. They cannot correspond to propagating degrees of freedom. We can give an explicit representation of the Def(4|2) transformations at the linearized level. Any variation of the metric changes the action by an amount

$$\delta S^{(2)} = -\frac{1}{N} \int d^4x \sqrt{|g|} Q^{\hat{\mu}\hat{\nu}} \delta g_{\hat{\mu}\hat{\nu}} . \quad (3.1)$$

We found that $Q^{\hat{\mu}\hat{\nu}}$ has to fulfill several constraint equations. Hence if we vary a component of the metric whose source is constrained to be zero, the action is invariant. From the constraints (2.14) and (2.15) one finds for the Def(4|2) transformations which do not correspond to general co-ordinate transformations

$$\begin{aligned} \delta Y_{\tau}^{\mu}{}_{i\ell m}(k) &= \varepsilon_{\tau}^{\mu}{}_{i\ell m}(k), \quad i=1,2, \quad \ell \geq 1, \\ \delta S_{4,\ell=0,m}(k) &= \varepsilon_{4,\ell=0,m}(k), \\ \delta \sigma_{\ell=1,m}(k) &= \varepsilon_{4,\ell=1,m}(k), \\ \delta S_{4,\ell=1,m}(k) &= -\frac{3}{8}(k^2/\gamma - 2)\varepsilon_{4,\ell=1,m}(k) \\ \delta S_{i\ell m}(k) &= \varepsilon_{i\ell m}(k), \quad i=2,3,4, \quad \ell \geq 2. \end{aligned} \quad (3.2)$$

The ε 's are arbitrary infinitesimal parameter functions. They parametrize at the linearized level an infinite set of symmetries of the six-dimensional action, all local in the four-dimensional space $\mathcal{M}^4$.

Our considerations can be trivially generalized to other dimensionalities. In particular, the seven-dimensional model based on E_4 has a Def(4|3) symmetry group consisting of all $\mathcal{W}^4$ -deformations if the total space is $\mathcal{W}^4 \times \mathcal{M}^3(\mathbb{R}^3)$ with a flat $\mathcal{M}^3(\mathbb{R}^3)$ factor. A very simple example is the Einstein-Hilbert action in two dimensions with symmetry group Def(2) $\equiv$ Def(2|0). Those deformations of a metric on a two-dimensional space which are not elements of Gen(2) can be identified with the conformal transformations.

Next we want to show how Def(4|2) invariance forbids certain kinetic and mass terms in the dimensionally reduced four-dimensional action. This is the deeper reason for the existence of massless towers and the absence of certain towers which would be present in the generic case. Our strategy is as follows. The different possible kinetic and mass terms of the reduced four-dimensional action are considered as an approximation quadratic in $h_{\mu\nu}$ of the corresponding four-dimensional invariants [with respect to local $\text{Gen}(4) \times \text{SO}(3)$]. If such an invariant can be made non-zero for some $h_{\mu\nu}$ which is a pure Def(4|2) deformation of the ground state, we know that its coefficient must be zero and the corresponding invariant cannot be present in the reduced four-dimensional action. Otherwise we would change the action by a Def(4|2) transformation in contradiction to Def(4|2) being a symmetry. We now discuss the different towers separately.

(A) Tensor fields. Let us consider a fluctuation $h_{\mu\nu}(x^0; x^\alpha)$ describing a (massless) graviton propagating along the z -axis of Minkowski space. Then $h_{\mu\nu}$ contains a factor of $\varepsilon_{\mu\nu} \exp[ik(z-t)]$ where the polarization tensor $\varepsilon_{\mu\nu}$ has non-zero components only for μ and ν lying in the directions perpendicular to t and z . Thus to describe a graviton one needs four different Minkowski space indices. Hence it is clear that the total space with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $g_{\mu\alpha} = 0$, $g_{\alpha\beta} = g_{\alpha\beta}$ is not a direct product of $\mathcal{M}^2$ with some other space since the internal dimensions already "use up" two of the four dimensions of $\mathcal{N}^4$. (Excitations leading to a space $\mathcal{N}^4 \times \mathcal{M}^2$ must not involve more than two directions of Minkowski space.) Thus we conclude that a massless tensor particle is not forbidden by Def(4|2) invariance. Now let us look at a possible mass term for the spin-2 fields. It would be of the Pauli-Fierz form

$$\mathcal{L}_{\text{PF}} = -\frac{1}{2} m_T^2 \left(h_{\mu\nu} h^{\mu\nu} - h_\mu^\mu h_\nu^\nu \right). \quad (3.3)$$

(Instead of using ℓ, m -indices we consider $h_{\mu\nu}$ to be x^α -dependent.) The bracket in (3.3) can be made non-vanishing by a fluctuation with $h_{\mu\nu}(x^0; x^\alpha) \equiv h_{\mu\nu}(z, t; x^\alpha) \neq 0$ and $\mu, \nu = z$ or t . Identifying $\mathcal{N}^4$ with $(z, t; x^\alpha)$ -space and $\mathcal{M}^2$ with the (x, y) -subspace of Minkowski space we find that the bracket changes under deformations of $\mathcal{N}^4$. Since we know that these do not alter the action, we have to conclude that $m_T^2 = 0$. This means that all fields $h_{\ell m}^{\mu\nu}(x^0)$ are massless. On the other hand, for $D = 3$ a mass term is no longer forbidden since internal space now occupies three of the $\mathcal{N}^4$ co-ordinates.

(B) Vector fields. The kinetic term of a vector field $V_\mu(x^0; x^\alpha)$ is of the form $(\partial_\mu V_\nu - \partial_\nu V_\mu)^2$. It can be made non-zero by a fluctuation which involves only two Minkowski space co-ordinates: $V_\mu \equiv V_\mu(z, t; x^\alpha) \neq 0$ for $\mu = z$ or t . Again identifying $\mathcal{N}^4$ with $(z, t; x^\alpha)$ -space one concludes that this term is forbidden by

Def(4|2) invariance. In seven dimensions, Def(4|3) is not "strong" enough to rule out a kinetic term because V_μ would affect at least five co-ordinates. A mass term $m^2 V_\mu V^\mu$ is forbidden, however, because it is non-zero even for a configuration like $V_\mu \equiv V_\mu(t; x^\alpha) \neq 0$ only for $\mu = t$ involving only a single space-time co-ordinate.

(C) Scalar fields. Propagating excitations of the scalars S_2, S_3 and S_4 parametrizing $h_{\alpha\beta}(x^\rho; x^\gamma)$ would contain Minkowski space factors like $\exp[ik(z-t)]$ involving two co-ordinates. Hence the total space again decomposes into the flat (x,y) -part and the curved $(z,t; x^\alpha)$ -manifold which could be arbitrarily deformed without changing the action. A possible kinetic term would be changed by a $\mathcal{M}^4$ deformation and hence is forbidden. Furthermore, from the gauge fixing conditions (A.7) one infers that if S_2, S_3 and S_4 are non-dynamical, ϕ_1 and ϕ_2 are non-dynamical too, and that η always can be expressed in terms of $\sigma \equiv h_\mu^\mu$. This is the only topologically allowed propagating scalar. A mass term for σ is ruled out by Def(4|2) symmetry. In seven dimensions Def(4|3) does not forbid a kinetic term for $S_1, \dots, S_4$.

To conclude, the spectrum consistent with Def(4|2) and Def(4|3) invariance, respectively, is precisely the one found by the explicit calculation: a massless tensor and a massless scalar tower for $D = 2$ and a massive tensor tower, a massless vector tower and possibly several scalar towers for $D = 3$.

4. - THE STABILITY GROUP OF THE GROUND STATE

For the generic theory the global symmetry of the ground state is $P_4 \times G$ where P_4 is the four-dimensional Poincaré group and $G = SO(3)$ or $SO(4)$ is the isometry group of $g_{\alpha\beta}$. This is the only subgroup of $\text{Gen}(6)$ which leaves the ground state metric invariant. For the Euler form theory the symmetry group of the six-dimensional action is much bigger and we now have to ask which subgroup of the full six-dimensional symmetry [which includes Def(4|2)] leaves the ground state metric invariant. This stability group of the ground state can be used for a classification of the four-dimensional spectrum. We will construct elements of this stability group by a combination of $\text{Gen}(6)$ transformations with appropriate Def(4|2) transformations not belonging to $\text{Gen}(6)$.

A general co-ordinate transformation with parameters $\xi^{\hat{\mu}} \equiv \xi^{\hat{\mu}}(x^\nu; x^\alpha)$ acts on the metric as

$$\delta g_{\hat{\mu}\hat{\nu}} = -\partial_{\hat{\mu}} \xi^{\hat{\sigma}} g_{\hat{\sigma}\hat{\nu}} - \partial_{\hat{\nu}} \xi^{\hat{\sigma}} g_{\hat{\mu}\hat{\sigma}} - \xi^{\hat{\sigma}} \partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}} \quad (4.1)$$

We first specialize for a global, i.e., x^μ -independent, Poincaré transformation with parameters

$$\xi^\mu = \alpha^\mu(x^\alpha) + \omega^\mu{}_\nu(x^\alpha) x^\nu, \quad \xi^\alpha = 0. \quad (4.2)$$

Its effect on the ground state metric (2.6) is given by

$$\begin{aligned} \delta \tilde{g}_{\mu\nu} &= \delta \tilde{g}_{\alpha\beta} = 0, \\ \delta \tilde{g}_{\alpha\mu} &= -\partial_\alpha \alpha_\mu(x^\gamma) - \partial_\alpha \omega_{\mu\nu}(x^\gamma) x^\nu. \end{aligned} \quad (4.3)$$

This is a symmetry of the vacuum only if the parameters α_μ and $\omega_{\mu\nu}$ are independent of x^γ . In the present case it is easy to see that the change in $g_{\alpha\mu}$ can be compensated for by a subsequent Def(4|2) transformation. For any given translation parameter $\alpha_\mu(x^\gamma)$ in a basis with only one value of μ non-zero, the metric deformation $\delta_{\alpha\mu}(x^\gamma)$ involves three co-ordinates and hence is an element of Def(4|2). The same is true for $\delta_{\alpha\mu\nu}(x^\gamma)x^\nu$ since we may choose $\omega_{\mu\nu}$ as a rotation in a fixed co-ordinate plane of Minkowski space involving only two co-ordinates. Therefore the vacuum is invariant under global, but x^α -dependent, Poincaré transformations combined with compensating Def(4|2) deformations. The group of these composite transformations will be denoted by $P_4(S^2)$. Its algebra is a Kac-Moody type extension of the Poincaré algebra [11,12] since via harmonic analysis the x^α -dependence is translated into the infinite grading of the generators labelled by the indices l and m . We expect $P_4(S^2)$ to be gauged by the infinitely many gravitons found in the spectrum yielding a graded Gen(4) symmetry. Note that $P_4(S^2)$ is in our case not a subgroup of Gen(6) as in the case of the infinite dimensional algebras studied by Dolan and Duff [11] and by Aulakh and Sahdev [12]. This clearly shows that the enlarged stability group results from a bigger local symmetry of the action rather than from a less effective breaking of Gen(6) by the vacuum. In seven dimensions the above argument cannot be repeated since the term $\delta_{\alpha\mu\nu}(x^\gamma)x^\nu$ involves at least five co-ordinates now. This is reflected by the fact that in this theory the tensor fields at the higher levels are massive.

Next we consider global Yang-Mills type transformations $\xi^\mu = 0$, $\xi^\alpha = \xi^\alpha(x^\gamma)$. The ground state metric changes by

$$\delta \dot{g}_{\mu\nu} = \delta g_{\mu\alpha} = 0 ,$$

$$\delta \dot{g}_{\alpha\beta} = -\partial_\alpha \xi^\gamma \dot{g}_{\beta\gamma} - \partial_\beta \xi^\gamma \dot{g}_{\alpha\gamma} - \xi^\gamma \partial_\gamma \dot{g}_{\alpha\beta} \equiv \mathcal{L}_\xi \dot{g}_{\alpha\beta} \quad (4.4)$$

Usually the vacuum is invariant only if ξ^γ is a Killing vector of $\dot{g}_{\alpha\beta}$. In our case it is obvious that for any ξ^γ the variation $\delta \dot{g}_{\alpha\beta}$ can be compensated for by a suitable Def(4|2) transformation acting on internal space only. It is this group (we shall denote it by Y) consisting of ξ^α -gauge transformations followed by compensating Def(4|2) transformations which replaces the isometry group G in the generic theory. For D = 2 this symmetry is realized only globally since there are no vector fields to gauge it. For D = 3 we can assume it is made local by the tower of massless vectors. (The above argument even holds for D = 4, but $\mathcal{M}^4 \times S^4$ is not a solution of the field equations for $\delta = \varepsilon = 0$.) We now study, for D = 2, the action of Y on the tensor fields $H_{TT}^{\mu\nu}$ and scalars σ_{lm} . This will provide us with additional insight for their mass degeneracy. An arbitrary fluctuation transforms under a $\xi^\mu = 0$, $\xi^\alpha = \xi^\alpha(x^\gamma)$ -transformation according to

$$\begin{aligned} \delta h_{\mu\nu} &= -\xi^\alpha \partial_\alpha h_{\mu\nu} , \\ \delta h_{\alpha\mu} &= -\partial_\alpha \xi^\gamma h_{\mu\gamma} - \xi^\gamma \partial_\gamma h_{\alpha\mu} , \\ \delta h_{\alpha\beta} &= \mathcal{L}_\xi h_{\alpha\beta} . \end{aligned} \quad (4.5)$$

Employing the same arguments as above it can be seen that by using suitable Def(4|2) operations all components of $h_{\mu\nu}$ except for $h_{TT}^{\mu\nu}$ and the trace $\sigma \equiv h_\mu^\mu$ can be deformed back to zero. The remaining fields simply transform as scalars under Y:

$$\begin{aligned} \delta_Y h_{TT}^{\mu\nu}(x^S; x^X) &= -\xi^\alpha(x^X) \partial_\alpha h_{TT}^{\mu\nu}(x^S; x^X) , \\ \delta_Y \sigma(x^S; x^X) &= -\xi^\alpha(x^X) \partial_\alpha \sigma(x^S; x^X) . \end{aligned} \quad (4.6)$$

Harmonic analysis of the transformations (4.6) yields an infinite dimensional, irreducible matrix representation of Y acting in the space of the $H_{TT}^{\mu\nu}$ and σ_{lm} fields. The irreducibility follows from the fact that by applying Y any function $f(x^Y)$ can be transformed into any other function of x^Y . Since for the harmonic

analysis of a general function $f(x^Y)$ and its image the harmonics $T_{\ell m}(x^Y)$ are needed for all ℓ and m , the representation must be irreducible^{*}). Furthermore, the mass operator $M^2 = P_\mu P^\mu$ (where the generator P_μ belongs to the $\ell = 0$ level of the graded Poincaré algebra) commutes with the transformations of Y . Hence all the fields $H_{TT}^{\mu\nu}$ and $\sigma_{\ell m}$ of the two infinite dimensional multiplets must have the same mass. As compared to the generic theory we have here the additional possibility of symmetry transformations connecting different $SO(3)$ representations. This is another way to look at the spectrum degeneracy of the Euler form model.

5. - DECOUPLING MECHANISMS

In this section we consider the Euler form theory as being defined by a special point in (α, β, γ) -parameter space and ask what happens to the modes present in the generic theory but not in the Euler form theory when the parameters α , β and γ approach their Euler form values. Considering a contribution to $S^{(2)}$ of the form $(A+Bk^2) \cdot (\text{field})^2$, there exist essentially three different possibilities for the behaviour of A and B in this limit:

- (i) A approaches a non-zero constant, B vanishes,
- (ii) A and B vanish, A/B becomes a finite constant,
- (iii) A and B vanish, A/B diverges.

Let us give some examples for the various cases.

- (i) After rescaling the field the physical mass of the excitation is $M^2 = -A/B$.

Hence decoupling occurs because the respective particle acquires an infinite mass. A typical example is the heavy poles arising from the higher derivatives in the action. For the second pole in the graviton propagator, say, one finds

$$M^2 = \frac{2\beta + 4\gamma}{\beta + 4\gamma} \gamma. \quad (5.1)$$

A rather general feature is that one can have both $M^2 \rightarrow +\infty$ and $M^2 \rightarrow -\infty$ depending on the direction in parameter space from which the Euler form is approached. This is a singular point. Many modes change the sign of their mass square by going through infinity, i.e., ordinary particles become tachyons for parameters crossing the Euler form point and vice versa.

^{*}) Strictly speaking this is not quite correct since x^α -independent functions are invariant under Y . However, the representation is not fully reducible, which is all we need for our argument, since a function with vanishing ($\ell=0$)-component can acquire such a component by a Y transformation.

(ii) For A/B becoming constant, the physical mass does not diverge. A typical example is the vector fields. Upon redefining the field variable so as to get a conventionally normalized kinetic term, the coupling of the mode gets changed. Consider for instance a term like $B F_{\mu\nu} F^{\mu\nu}$ for a gauge boson A_μ . Under the redefinition $A_\mu \rightarrow B^{-\frac{1}{2}} A_\mu$ the Yang-Mills coupling g scales as $g \rightarrow B^{-\frac{1}{2}} g$ and hence it diverges when we approach the Euler form. For the explicit form of the spin-1 propagators we refer to I. One finds that, depending on the direction in parameter space, going away from the Euler form point can yield both signs for the residue corresponding to ordinary particles and ghosts, respectively. It is therefore the borderline between classical stability and instability (even for low momenta).

(iii) Like in (i), this possibility amounts to the physical particle mass becoming infinite. It might be realized in some of the scalar towers.

These observations indicate that for certain modes the Euler form point is exactly on the borderline between classical stability and instability. If case (ii) is realized, it is obvious that the stability properties for the Euler form model are highly sensitive to small perturbations of the parameters. No conclusions on the stability of "neighbouring" theories should be drawn in this case.

6. - CONCLUSION

In this paper we tried to show that higher-dimensional theories based on effective actions for gravity in the form of dimensionally continued Euler forms have very special properties which go beyond the well-known fact that no propagator corrections are introduced in flat space. As an example we considered the pure four-derivative Euler form E_4 in six dimensions. This theory shows the special features in their most extreme form, but our general conclusions apply equally well to any action which is a sum of Euler forms of various dimensionalities [10,13,17]. If the rank of the lowest form is n , one always has a symmetry $\text{Def}(n|d-n)$. Similarly, our choice of the background, $\mathcal{M}^4 \times S^2$, is inessential for the general features.

The spectrum of Euler form theories contains far less states than the generic one. When approaching the Euler form, particles can decouple either by an infinite mass or, as in the case of massless particles, by an infinitely strong coupling to other fields. The degeneracy of the spectrum is a consequence of the residual topological invariance of the various Euler forms beyond their critical dimensions. The additional local symmetry of the action can even lead to an infinite dimensional stability group of the vacuum. If the corresponding gauge fields are

present, an infinite dimensional Kac-Moody type gauge algebra can emerge. At least at the linearized level a consistent matter coupling imposes on the energy-momentum tensor a set of constraint equations to make it compatible with the symmetry of the gravity action.

We finally should note an observation which could have some relation with possible solutions of the cosmological constant problem: the four-dimensional Einstein-Hilbert action has an additional symmetry $\text{Def}(2|2)$ since the curvature scalar is the two-dimensional Euler form. This symmetry is destroyed by a non-vanishing cosmological constant. A vanishing cosmological constant is singled out as a special point in parameter space with higher symmetry.

ACKNOWLEDGEMENTS

We would like to thank M. Arik and H. Römer for stimulating discussions.

APPENDIX

In this appendix we sketch the mode decomposition for the six-dimensional pure Euler form model. We follow the analysis of I, where the reader is referred to for additional details. The steps leading to the quadratic action (2.10) could be performed without fixing a gauge. It is computationally advantageous, however, to impose the co-ordinate condition

$$D_{\hat{\mu}} h^{\hat{\mu}\hat{\nu}} = 0. \quad (\text{A.1})$$

Then the quadratic action reads:

$$\begin{aligned} S^{(2)} = & -\frac{1}{2N} \int d^4x \sqrt{|g|} \\ & \cdot \left\{ h^{\alpha\beta} [A_1(\partial^4 + 2\partial^2 \mathring{D}^2 + \mathring{D}^4) + A_2 \partial^2 + A_3 \mathring{D}^2 + A_4] h_{\alpha\beta} \right. \\ & + h_{\alpha}^{\alpha} [B_1(\partial^4 + 2\partial^2 \mathring{D}^2 + \mathring{D}^4) + B_2 \partial^2 + B_3 \mathring{D}^2 + B_4] h_{\beta}^{\beta} \\ & + h_{\alpha}^{\alpha} [C_1(\partial^4 + 2\partial^2 \mathring{D}^2 + \mathring{D}^4) + C_2 \partial^2 + C_3 \mathring{D}^2 + C_4] h_{\mu}^{\mu} \\ & + h^{\alpha\mu} [F_1(\partial^4 + 2\partial^2 \mathring{D}^2 + \mathring{D}^4) + F_2 \partial^2 + F_3 \mathring{D}^2 + F_4] h_{\alpha\mu} \\ & + h^{\mu\nu} [G_1(\partial^4 + 2\partial^2 \mathring{D}^2 + \mathring{D}^4) + G_2 \partial^2 + G_3 \mathring{D}^2 + G_4] h_{\mu\nu} \\ & + h_{\mu}^{\mu} [H_1(\partial^4 + 2\partial^2 \mathring{D}^2 + \mathring{D}^4) + H_2 \partial^2 + H_3 \mathring{D}^2 + H_4] h_{\nu}^{\nu} \\ & + 4\gamma\gamma [h_{\alpha\mu} \partial^{\mu} \partial^{\nu} h_{\nu}^{\alpha} + h_{\mu\nu} \partial^{\mu} \partial^{\nu} h_{\nu}^{\mu} \\ & \quad - h_{\alpha}^{\alpha} \partial_{\mu} \partial_{\nu} h^{\mu\nu} - h_{\mu}^{\mu} \partial_{\nu} \partial_{\alpha} h^{\alpha\nu}] \\ & \left. + 2 [h_{\mu\nu} Q^{\mu\nu} + 2 h_{\mu\alpha} Q^{\mu\alpha} + h_{\alpha\beta} Q^{\alpha\beta}] \right\} \quad (\text{A.2}) \end{aligned}$$

Explicit expressions for the coefficients $A_i, B_i, \dots$ can be found in I. They depend on $\alpha, \beta, \dots$ as well as on D . For general parameter values, all of them are non-zero. For the pure Euler form (2.11) the action (A.2) simplifies considerably. The coefficients are ($\gamma = L_0^{-2}$):

$$\begin{aligned}
A_1 &= A_2 = 0, \quad A_3 = 2\gamma, \quad A_4 = 4\gamma^2; \\
B_1 &= B_2 = 0, \quad B_3 = -2\gamma, \quad B_4 = -2\gamma^2; \\
C_i &= 0, \quad i=1, \dots, 4; \\
F_i &= 0, \quad i=1, \dots, 4; \\
G_1 &= G_3 = G_4 = 0, \quad G_2 = 2\gamma; \\
H_1 &= H_3 = H_4 = 0, \quad H_2 = -2\gamma.
\end{aligned} \tag{A.3}$$

As expected, all fourth derivatives have vanishing coefficients. Moreover, all kinetic terms (containing ∂^2) vanish except the one for $h_{\mu\nu}$. Therefore we suspect that upon harmonic analysis the field equations for $h_{\alpha\beta}$ and $h_{\alpha\mu}$ are purely algebraic and hence do not lead to propagating modes. In particular this applies to the gauge bosons usually contained in $h_{\alpha\mu}^{(*)}$. We hence expect the spectrum of the Euler form theory to contain much fewer states than the generic one, but, strictly speaking, no conclusion can be drawn on the basis of (A.2) with (A.3) alone since the fields $h_{\hat{\mu}\hat{\nu}}$ are subject to the gauge condition $D_{\hat{\mu}} h^{\hat{\mu}\hat{\nu}} = 0$.

Due to the algebraic complexity of the problem we did not explicitly determine the spectrum for some of the coupled scalars of the generic model. For the Euler form action, however, everything can be solved analytically. One first expands the fluctuations $h_{\hat{\mu}\hat{\nu}}$ with respect to their dependence on the internal co-ordinates x^Y in terms of tensor harmonics on S^2 [15,16] of the respective rank. The S^D -scalar $h_{\mu\nu}$ can be expanded in terms of the scalar harmonics $T_{\ell m}(x^Y)$, the S^D -vector $h_{\alpha\mu}$ in terms of the transverse and longitudinal vector harmonics $T_{\ell m}^\alpha(x^Y)$ and $L_{\ell m}^\alpha(x^Y)$ and the S^D -second rank tensor $h_{\alpha\beta}$ in terms of the transverse-traceless tensor harmonics $T_{\ell m}^{\alpha\beta}(x^Y)$, the longitudinal-transverse harmonics $L_{\ell m}^{\alpha\beta}(x^Y)$, the longitudinal-longitudinal harmonics $L_{\ell m}^{\alpha\beta}(x^Y)$ and the trace part $g^{\alpha\beta}(x^Y) T_{\ell m}(x^Y)$. The case $D = 2$ is special because there is no transverse-traceless tensor in two dimensions. A detailed description of these harmonics including their eigenvalues and degeneracies with respect to the internal Laplacian $\hat{D}^2 \equiv D_\alpha D^\alpha$ can be found in Ref. [15]. The eigenvalues are labelled by the index ℓ and m is a degeneracy index. We parametrize the fluctuations as follows:

*) The helicity-one part of $h_{\mu\nu}$ still could be propagating, however.

$$\begin{aligned}
 h^{\alpha\beta}(x) &= \int \frac{d^4 k}{(2\pi)^4} e^{i k_p x^p} \left\{ \sum_{l=2}^{\infty} \sum_m \left(S_{1lm}(k) T_{lm}^{\alpha\beta}(x^x) \right. \right. \\
 &\quad \left. \left. + S_{2lm}(k) L_{lm}^{\alpha\beta}(x^x) + S_{3lm} L_{lm}^{\alpha\beta}(x^x) \right) + g^{\alpha\beta}(x^x) \sum_{l=0}^{\infty} \sum_m S_{4lm}(k) T_{lm}(x^x) \right\}, \\
 h^{\alpha\mu}(x) &= \int \frac{d^4 k}{(2\pi)^4} e^{i k_p x^p} \sum_{l=1}^{\infty} \sum_m \left(V_{1lm}^{\mu}(k) T_{lm}^{\alpha}(x^x) \right. \\
 &\quad \left. + V_{2lm}^{\mu}(k) L_{lm}^{\alpha}(x^x) \right), \quad (A.4) \\
 h^{\mu\nu}(x) &= \int \frac{d^4 k}{(2\pi)^4} e^{i k_p x^p} \sum_{l=0}^{\infty} \sum_m H_{lm}^{\mu\nu}(k) T_{lm}(x^x).
 \end{aligned}$$

To include the $D = 3$ theory as well, we kept the $T_{lm}^{\alpha\beta}$ harmonics; for $D = 2$ one has to set $S_{1lm} \equiv 0$. In order to exploit (2.9) and (A.1) it is advantageous to further decompose V_{ilm}^{μ} and $H_{lm}^{\mu\nu}$ into pieces which are irreducible with respect to the four-dimensional Lorentz group:

$$\begin{aligned}
 V_{ilm}^{\mu} &= V_{Tilm}^{\mu} + i k^{\mu} \phi_{ilm}, \quad i=1,2, \\
 H_{lm}^{\mu\nu} &= H_{TTlm}^{\mu\nu} + i(k^{\mu} H_{Tlm}^{\nu} + k^{\nu} H_{Tlm}^{\mu}) \\
 &\quad + (k^{\mu} k^{\nu} - \frac{1}{4} \eta^{\mu\nu} k^2) \eta_{lm} + \frac{1}{4} \eta^{\mu\nu} \sigma_{lm}. \quad (A.5)
 \end{aligned}$$

Here V_{ilm}^{μ} , H_{ilm}^{μ} and $H_{TTlm}^{\mu\nu}$ are transverse vectors and a transverse-traceless tensor, respectively. For the sources $Q^{\hat{\mu}\hat{\nu}}$ one introduces similar expansions. The source functions belonging to S_{ilm} , V_{ilm}^{μ} , ϕ_{ilm} , $H_{TTlm}^{\mu\nu}$, H_{ilm}^{μ} , η_{lm} and σ_{lm} are denoted by Q_{ilm} , q_{ilm}^{μ} , φ_{ilm} , $q_{TTlm}^{\mu\nu}$, w_{ilm}^{μ} , w_{ilm} and $q_{\mu l m}^{\mu}$, respectively. Inserting these series into the quadratic action (A.2) with the coefficients (A.3) one obtains

$$\begin{aligned}
 S^{(2)} &= -\frac{1}{N} \int \frac{d^4 k}{(2\pi)^4} \sum_{lm} \left\{ [l(l+1)-2] (|S_2|^2 + |S_3|^2) \right. \\
 &\quad - 2\gamma^2 l(l+1) |S_4|^2 - \gamma k^2 |H_{TT}^{\mu\nu}|^2 + \frac{3}{8} k^2 (\gamma |\sigma|^2 + k^4 |\eta|^2) \\
 &\quad - k^2 S_4^* (3k^2 \gamma + \sigma) - \frac{3}{4} k^4 \sigma^* \gamma - 2\gamma k^4 (|\phi_1|^2 + |\phi_2|^2) \\
 &\quad \left. + \sum_{i=1}^3 S_i^* Q_i + 2 S_4^* Q_4 \right\}
 \end{aligned}$$

$$\begin{aligned}
 & -2 \sum_{i=1}^2 \left(V_i^{\mu*} q_{i\mu} + k^2 \phi_i^* \varphi_i \right) + H_{\tau\tau}^* q_{\tau\tau}^{\mu\nu} \\
 & + 2k^2 H_{\tau\mu}^* W^\mu + \frac{3}{4} k^4 \gamma^* w + \frac{1}{4} \sigma^* q_\mu^\mu \} \quad (A.6)
 \end{aligned}$$

For simplicity we have omitted the indices l and m . Before deriving field equations from (A.6) one has to remember that the fields obey the gauge condition $D_\mu h^{\mu\hat{\nu}} = 0$ and hence cannot be varied independently. Similarly, because of the continuity equation $D_\mu Q^{\mu\hat{\nu}} = 0$ not all components of the source are independent. In terms of the Lorentz irreducible fields the gauge condition becomes

$$k^2 \phi_{l,l=1,m} = 0, \quad (A.7a)$$

$$k^2 \phi_{1lm} = [\{l(l+1)-2\}\gamma/2]^{1/2} S_{2lm}, \quad l \geq 2, \quad (A.7b)$$

$$k^2 \phi_{2,l=1,m} = (2\gamma)^{1/2} S_{4,l=1,m}, \quad (A.7c)$$

$$k^2 \phi_{2lm} = [\{l(l+1)-2\}\gamma/2]^{1/2} S_{3lm} + [l(l+1)\gamma]^{1/2} S_{4lm}, \quad l \geq 2, \quad (A.7d)$$

$$k^2 H_{\tau}^{\mu}{}_{l=0,m} = 0, \quad (A.7e)$$

$$(3k^2 \gamma + \sigma)_{l=0,m} = 0, \quad (A.7f)$$

$$k^2 H_{\tau}^{\mu}{}_{lm} = [l(l+1)\gamma]^{1/2} V_{\tau}^{\mu}{}_{2lm}, \quad l \geq 1, \quad (A.7g)$$

$$[l(l+1)\gamma]^{1/2} \phi_{2lm} = -\frac{3}{4} k^2 \gamma_{lm} - \frac{1}{4} \sigma_{lm}, \quad l \geq 1. \quad (A.7h)$$

Up to a trivial relabelling the conservation law for the sources has the same form. Note that the towers of the various S^D -harmonics start for different values of l and hence the levels $l = 0$ and $l = 1$ have to be discussed separately. We group the fields into sets whose members are connected by (A.7) and then eliminate the dependent ones along with their sources. As in I we shall use the sets

$$\{H_{\tau\tau}^{\mu\nu}\}, \{V_{\tau 1}^{\mu}\}, \{S_1\}, \{V_{\tau 2}^{\mu}, [H_{\tau}^{\mu}]; l \geq 1\},$$

$$\{S_2, [\phi_1]; l \geq 2\}, \{S_4, \sigma, [\eta]; l=0\},$$

$$\{S_4, \sigma, [\eta, \phi_2]; l=1\} \text{ and } \{S_3, S_4, \sigma, [\eta, \phi_2]; l \geq 2\}.$$

The fields in the square brackets have to be eliminated via (A.7) and the result is inserted into (A.6). The field equations can be obtained from the variation of $S^{(2)}$ with respect to the remaining independent fields.

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