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LIGHT-CONE LATTICES AND THE EXACT SOLUTION
OF CHIRAL FERMION AND SIGMA MODELS

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ABSTRACT

A rich set of integrable two-dimensional quantum field theories are obtained from integrable lattice vertex models with q states per bound ($q \geq 2$) in the scaling limit by a generalization of the light-cone lattice approach.

Chiral fermion models with any simple Lie group of symmetry arise in this way (for finite q) as well as bosonic models like the principal chiral model (for $q = \infty$).

The Hamiltonian, momentum and colour-conserved currents are constructed on the lattice and the bare equations of motion are derived. The renormalized mass spectrum is given explicitly for the set of models considered here.

All these integrable vertex models yield conformal invariant theories if one takes the scaling limit in an appropriate, different way. It is argued that the values one obtains for the central charges are the same as those provided by the Sugawara construction (in the continuum) for all simple Lie algebras.

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1. - INTRODUCTION AND SUMMARY

The construction of exact solutions of 2D integrable statistical models has made impressive progress in the last years [1]. Eigenvalues and eigenvectors of a rich set of theories have been constructed by the Bethe Ansatz (BA) and its generalizations.

Usually, the physics of a statistical model at criticality (infinite correlation length) can be described by a continuum quantum field theory. Therefore, integrable lattice models in their scaling limit seem to be very appropriate starting points of a programme for the building of exactly solvable quantum field theories (QFT's).

As a first step in this programme, the massive Thirring model (MTM) fields were constructed from an exactly solvable lattice theory, the six-vertex model, in Ref. [2]. The light-cone lattice approach used there is actually appropriate for systematically building various integrable fermionic and bosonic QFT's from vertex models (Fig. 1), taking the scaling limit in the precise way illustrated in the present paper.

Lattice vertex models are usually treated using the row-to-row transfer matrix (Fig. 2). However, a simple relation exists between the light-cone (diagonal-to-diagonal) transfer matrices and the inhomogeneous row-to-row transfer matrices. By inhomogeneous we mean when the spectral argument θ of the vertex weights depends on the column of the lattice as

$$\theta \rightarrow \frac{1}{2}(\theta_0 - \theta_n) \quad , \quad 1 \leq n \leq N$$

Setting $\theta_n = \theta$ ($-\theta$) for odd (even) columns leads to a light-cone transfer matrix when one sets $\theta_0 = \theta$ or $\theta_0 = -\theta$, as is easily seen in Figs. 2-4. (An analytic proof is given in Section 2). More precisely

$$\begin{aligned} \tau(\theta; \{\theta_n = (-)^{n+1}\theta\}) &= U_L(\theta) \\ \tau(-\theta; \{\theta_n = (-)^{n+1}\theta\}) &= U_R(\theta)^\dagger \end{aligned} \quad (1.1)$$

Here U_L and U_R respectively produce a right or left unit translation in the diagonal direction over the lattice. U_R and U_L are depicted in Fig. 4.

The lattice Hamiltonian and momentum are defined as

$$H \pm P = \frac{2i}{a} \log U_{R,L}(\theta) \quad (1.2)$$

where a is the lattice spacing. All known integrable vertex models possess

gapless regimes where one can define a scaling limit $\alpha \rightarrow 0$ such that a relativistic theory emerges. That is, we choose $\theta = \theta(\alpha)$ such that a (non-empty) set of massive states remains in the spectrum of H and P in the $\alpha \rightarrow 0$ limit. We want to stress that several inequivalent QFT's may be obtained from a single lattice model by choosing different functions $\theta = \theta(\alpha)$. Specific examples are discussed in Section 2. In the Table we summarize the mass spectrum μm_λ for the fundamental models symmetric under all the different simple Lie algebras. The mass unit μ relates in the scaling limit to the lattice spacing α and the parameter θ as:

$$\mu = \frac{1}{\pi\alpha} e^{-\kappa\theta} \quad (1.3)$$

where $\alpha \rightarrow 0$ and $\theta \rightarrow \infty$ such that μ is fixed (the values of κ are listed in the Table). In addition to the mass spectrum, the exact S-matrix can be derived from the Bethe Ansatz equations which determine the eigenstates of $\tau(\theta_0; \{\theta\})$ (which are also eigenstates of U_R and U_L).

All rational R-matrices [i.e., matrices $R(\theta)$ depending rationally on θ] lead to gapless vertex models [see Eq. (2.3) or Fig. 1], and therefore a relativistic QFT can be constructed in the scaling limit (1.3). Rational R-matrices invariant under any simple Lie group G are known [3,4]. For large values of the spectral parameter θ they behave as

$$R(\theta) = P \left\{ 1 + \frac{i}{\theta} (\pi + \lambda) + O\left(\frac{1}{\theta^2}\right) \right\} \quad (1.4)$$

where λ is a numerical constant, P is the exchange operator, $P_{ab,cd} = \delta_{ad}\delta_{bc}$,

$$\pi = 2 \sum_{\alpha \in \dim G} T^\alpha \otimes T^\alpha \quad (1.5)$$

and the T^α 's are the generators of G in the fundamental representation, normalized according to $\text{tr} T^\alpha T^\beta = \frac{1}{2} \delta^{\alpha\beta}$. It must be realized that (1.4) is just the semiclassical expansion of the R-matrix. One can then introduce the lattice operator (Section 2)

$$T_n^\alpha = \mathbb{1} \otimes \dots \otimes \overset{\text{n-th site}}{\underbrace{T^\alpha}} \otimes \dots \otimes \mathbb{1} \quad (1.6)$$

As is shown in Section 2, T_n^α obey local equations of motion on the lattice:

$$\begin{aligned} U_R T_{2n-2}^\alpha U_R^\dagger &= U_L T_{2n}^\alpha U_L^\dagger = \\ &= T_{2n}^\alpha + \frac{2i}{\theta} f^{\alpha\beta\gamma} T_{2n-1}^\beta T_{2n}^\gamma + O(1/\theta^2) \end{aligned} \quad (1.7a)$$

$$\begin{aligned}
 U_R T_{2n-1}^\alpha U_R^\dagger &= U_L T_{2n+1}^\alpha U_L^\dagger = \\
 &= T_{2n-1}^\alpha + \frac{2i}{\theta} f^{\alpha\beta\gamma} T_{2n-1}^\beta T_{2n}^\gamma + O(1/\theta^2)
 \end{aligned} \tag{1.7b}$$

A closed form of the terms of order $O(1/\theta^2)$ is known for $G = \text{SU}(n)$ [Eq. (2.24)]. A bare scaling limit $\theta \rightarrow \infty$ and then $a \rightarrow 0$ can be defined such that Eq. (1.6) yields the continuous zero-divergence and zero-curvature conditions on the vector current

$$J_R^\alpha(x) = \frac{1}{g a \theta} T_{2n}^\alpha, \quad J_L^\alpha(x) = \frac{1}{g a \theta} T_{2n-1}^\alpha; \quad x \approx n a \tag{1.8}$$

Therefore we have a lattice version of the $\hat{G}$ -algebra currents $J_\mu^\alpha(x)$ in the context of an exact discretization of field theoretical models.

Gapless lattice models exhibit conformal invariant behaviour for distances much larger than the lattice spacing a . Therefore, conformal field theories arise in the limit

$$a \rightarrow 0, \quad \frac{d\theta}{da} = 0 \tag{1.9}$$

Notice that this scaling limit differs from (1.3). The finite size corrections to the free energy and excitation energies yield the conformal properties of the model, that is, the central charge c and the conformal dimensions $\Delta, \bar{\Delta}$. The value of c is exactly known for all fundamental models associated to simply-laced Lie algebras [5,6] and for the spin-S $\text{SU}(2)$ symmetric model [7]. These results indicate the general formula for the central charge

$$c = \frac{x \dim G}{x + \tilde{h}} \tag{1.10}$$

Here $\dim G$ is the dimension of the Lie algebra, $\tilde{h}$ its dual Coxeter number^{*} and x stands for the number of fundamental R-matrices fused to produce the model. We have $x = 1$ for fundamental models and $x = 2S$ for the S-spin $\text{SU}(2)$ symmetric model. The Sugawara construction of the Virasoro algebra given in Ref. [8] also fulfils Eq. (1.10). We believe that the gapless integrable theories associated to a Lie

^{*}) It is given by $\tilde{h} = C_2 / \|\alpha\|$, where C_2 is the quadratic Casimir operator in the adjoint representation and $\|\alpha\|$ the length of the longest root of the algebra.

algebra $\hat{G}$ provide, through their long-distance behaviour, an alternative construction of the conformal algebra. It is interesting to notice that models like the six-vertex, where $c = 1$, explicitly provide an infinite number of primary fields in the long-distance regime.

Conformal field theories appear therefore in the limit (1.9) of lattice integrable models. These integrable models clearly have a much richer structure than the continuum field theories since, for example, they yield massive QFT in the limit (1.3).

In Section 3, we review the Co-ordinate Bethe Ansatz (CBA) for a general class of two-dimensional Chiral Fermionic Models (CFM) defined by the Lagrangian

$$\mathcal{L} = i \bar{\Psi} \not{\partial} \Psi - \frac{1}{4} (\bar{\Psi}_a \gamma_\mu \Psi_c) V_{ab,cd} (\bar{\Psi}_b \gamma^\mu \Psi_d) \quad (1.11)$$

Here ψ_a is a Dirac field. The CBA solution shows that this model describes massless free particles bearing non-zero chirality plus massive excitations. Actually, the CBA only works when the interaction matrix V yields a two-body S-matrix (3.12) obeying the YB equation (3.19). Then the massive sector of (1.11) coincides with the scaling limit (1.2)-(1.3) of the lattice model built from this S-matrix as vertex weights [Eq. (3.21)]. Together with the lattice current construction (1.6)-(1.8), this provides an explicit connection between the CBA approach in the continuum and the light-cone transfer matrix approach in the lattice.

We also check in Section 3 that the one-loop perturbative β -function for the CFM's is correctly reproduced by the BA for all fundamental models associated to simple Lie algebras. This is a highly non-trivial check in view of the completely different mathematical structure of the continuum Feynman diagrams and the lattice or CBA. Actually, the CBA approach has serious drawbacks and may sometimes lead to wrong results. This is illustrated in Section 3 for the massless Thirring model and in Section 4 for the multiflavour generalization of (1.11), that is:

$$\mathcal{L} = i \bar{\Psi}_r \not{\partial} \Psi_r - \frac{1}{2} g (\bar{\Psi}_r \gamma_\mu T^\alpha \Psi_r) (\bar{\Psi}_{r'} \gamma^\mu T^\alpha \Psi_{r'}) \quad (1.12)$$

where each ψ_r ($1 \leq r \leq N_F$) transforms under an irreducible representation ρ of the group G , and T_α are the G -generators in that representation. In Eq. (1.12), we used

$$V_{ab,cd} = g \Pi_{a,b,c,d}$$

where Π is given by Eq. (1.5).

In Section 4 we also briefly analyze the connection between the $N_f \rightarrow \infty$ limit of the Multiflavour Chiral Gross-Neveu (MCGN) model and the $SU(n)$ Principal Chiral σ -model. For $n = 2$, the latter can also be (partially) obtained in the $S \rightarrow \infty$ limit of the spin-S $SU(2)$ light-cone vertex model.

In summary, we shall show in this paper how to generate integrable (massive) QFT's and conformal QFT's out of integrable vertex models by performing appropriate scaling limits. The scope of the light-cone lattice approach for integrable QFT's seems to cover a very general set of models including all previously-known examples.

2. - LIGHT-CONE VERSUS ROW-TO-ROW TRANSFER MATRICES AND FIELD-THEORETIC MODELS

The row-to-row transfer matrix is one of the fundamental objects for an integrable vertex model. Let us consider an inhomogeneous vertex model where both horizontal and vertical links take values in the same q -dimensional vector space (see Fig. 1). The row-to-row transfer matrix reads:

$$\tau(\theta_0; \{\theta_i\}) = \sum_{a_1, \dots, a_N=1}^q t_{a_1 a_2}(\frac{\theta_0 - \theta_1}{2}) t_{a_2 a_3}(\frac{\theta_0 - \theta_2}{2}) \dots t_{a_N a_1}(\frac{\theta_0 - \theta_N}{2}) \quad (2.1)$$

It is graphically depicted in Fig. 2. θ_0 is the spectral parameter and θ_i ($1 \leq i \leq N$, with N an even number) are the inhomogeneity parameters.

The vertex model is said to be integrable if the weights $t_{ab}(\theta)$ obey the Yang-Baxter (YB) algebra

$$R(\theta - \theta') [t(\theta) \otimes t(\theta')] = [t(\theta') \otimes t(\theta)] R(\theta - \theta') \quad (2.2)$$

where

$$R_{a\alpha}^{b\beta}(\theta) = [t_{ab}(\theta)]_{\alpha\beta} \quad (2.3)$$

It follows from (2.1) and (2.2) that

$$[\tau(\theta_0; \{\theta_i\}), \tau(\theta'_0; \{\theta_i\})] = 0, \quad \forall \theta_0, \theta'_0 \quad (2.4)$$

We choose here a regular solution of the YB equations (2.2) such that

$$[t_{ab}(0)]_{\alpha\beta} = \delta_{a\beta} \delta_{b\alpha} \quad (2.5)$$

Let us now make the following choice of inhomogeneities

$$\theta_k = (-)^{k+1} \theta \quad ; \quad k = 1, 2, \dots, N \quad (2.6)$$

In this case the sum over a_j ($1 \leq j \leq q$) in Eq. (2.1) can be trivially performed, using Eq. (2.5), whenever $\theta_0 = \theta$ or $\theta_0 = -\theta$:

$$\begin{aligned} \tau(\theta, \theta)_{\alpha, \beta} &\equiv \tau(\theta; \{\theta_k = (-)^{k+1} \theta\})_{\alpha_1 \dots \alpha_N, \beta_1 \dots \beta_N} = \\ &= [t_{\alpha_1 \beta_2}(\theta)]_{\alpha_2 \beta_2} [t_{\alpha_3 \beta_5}(\theta)]_{\alpha_4 \beta_4} \dots [t_{\alpha_{2N-1} \beta_1}(\theta)]_{\alpha_N \beta_N} \end{aligned} \quad (2.7)$$

Using Eq. (2.3), this reads

$$\tau(\theta, \theta)_{\alpha, \beta} = R_{\alpha_1 \alpha_2}^{\beta_2 \beta_3}(\theta) \dots R_{\alpha_N \alpha_1}^{\beta_{N-1} \beta_N}(\theta) \quad (2.8)$$

But this is precisely a diagonal-to-diagonal, or light-cone, transfer matrix. In the notation of Ref. [2],

$$\tau(\theta, \theta) = U_+(\theta) V^\dagger = U_L(\theta) \quad (2.9)$$

Similarly, if we consider

$$\tau(-\theta, \theta) \equiv \tau(-\theta; \{\theta_k = (-)^{k+1} \theta\}) \quad (2.10)$$

we find

$$\tau(-\theta, \theta) = V^\dagger U_+(-\theta) = [U_+(\theta) V]^\dagger = U_R(\theta)^\dagger \quad (2.11)$$

since $R(\theta)^\dagger = R(-\theta) = R(\theta)^{-1}$ (a property which can always be achieved by proper normalizations).

Therefore the light-cone transfer matrices $U_{R,L}(\theta)$ are particular cases of an inhomogeneous row-to-row transfer matrix with alternating inhomogeneities given by Eq. (2.6). The commutativity property (2.4) yields therefore

$$\begin{aligned} [\tau(\theta_0, \theta), U_L(\theta)] &= 0 = [\tau(\theta_0, \theta), U_R(\theta)] \\ [U_R(\theta), U_L(\theta)] &= 0 \end{aligned} \quad (2.12)$$

where $\tau(\theta_0, \theta) \equiv \tau(\theta_0; \{\theta_k = (-)^{k+1} \theta\})$. One can consider the infinite sequence of commuting operators

$$c_k^\pm = \frac{1}{k!} \left. \frac{\partial^k \tau(\theta_0, \theta)}{\partial \theta_0^k} \right|_{\theta_0 = \pm \theta}, \quad k=1, 2, \dots \quad (2.13)$$

They all commute with $U_L(\theta)$ and $U_R(\theta)$:

$$[c_k^\pm, U_L(\theta)] = [c_k^\pm, U_R(\theta)] = 0 \quad (2.14)$$

As explained in Ref. [2], the operators $U_R(\theta)$ and $U_L(\theta)$ are precisely the light-cone evolution operators in a discretized Minkowski space-time for the MTM. We shall generalize this interpretation to large class of QFTs. We define the Hamiltonian and momentum operators for the lattice QFT as

$$H_{R,L} \equiv \frac{1}{2}(H \pm P) = \frac{i}{a} \log U_{R,L}(\theta) \quad (2.15)$$

where a is the lattice spacing. The continuum limit is to be performed letting $\theta = \theta(a)$ such that the physical mass gap is finite. Let us consider the spectrum of $U_R(\theta)$ and $U_L(\theta)$ in order to find $\theta(a)$ and the mass spectrum.

Since U_R and U_L are expressed in terms of the row-to-row transfer matrix [Eqs. (2.9) and (2.11)], it is enough to analyze the spectrum of $\tau(\theta_0; \{\theta_k\})$. The eigenvectors and eigenvalues can be exactly computed by the BA and its nested generalizations [1,9]. The ground-state eigenvector just corresponds (in the scaling limit) to the vacuum state of the QFT. The particle states follow from the lowest-lying excitations. Since there is a factor a^{-1} in the Hamiltonian [see Eq. (2.15)], only gapless vertex models may yield finite energy states in the scaling limit. Therefore, in order to calculate the energy and momentum in the scaling limit, it is enough to know the eigenvalues of $\tau(\theta_0; \{\theta_k\})$ close to the bottom of the spectrum.

Let us take the family of fundamental vertex models associated to simple Lie algebras [3,4] (other models will be considered later on). The low-lying excitations in these models are associated to holes with large rapidity ϕ . Moreover, the large ϕ behaviour of the eigenvalues is independent of the inhomogeneity parameters θ_j ($1 \leq j \leq N$); its explicit form was derived in Ref. [6]:

$$\begin{aligned} \lambda_\ell(\varphi) &\equiv - \lim_{N \rightarrow \infty} \log \left[\Lambda_{\max}(\theta_0; \{\theta_k\})^{-1} \Lambda_\ell(\varphi, \theta_0; \{\theta_k\}) \right] \\ &\underset{\varphi + \theta_0 \rightarrow \pm \infty}{\sim} \frac{i m_\ell}{\pi} e^{\mp \kappa(\varphi + \theta_0)} + O(e^{\mp 2\kappa(\varphi + \theta_0)}) \end{aligned} \quad (2.16)$$

Here $\Lambda_\ell(\phi, \theta_0; \{\theta_k\})$ is the contribution of a hole in the ℓ th branch ($1 \leq \ell \leq \text{rank } G$) to the BA eigenvalue of $\tau(\theta_0; \{\theta_k\})$. The dimensionless parameters κ and m_ℓ are given

in the Table for all these models associated to simple Lie algebras in their fundamental representations.

Combining Eq. (2.16) with (2.9), (2.11) and (2.15) yields the energy-momentum dispersion relations (for large values of θ)

$$e_\ell(\varphi) = \left(\frac{e^{-\kappa\theta}}{\pi a} \right) m_\ell \operatorname{ch} \kappa \varphi + O(e^{-2\kappa\theta}) \quad (2.17a)$$

$$p_\ell(\varphi) = \left(\frac{e^{\kappa\theta}}{\pi a} \right) m_\ell \operatorname{sh} \kappa \varphi + O(e^{-2\kappa\theta}) \quad (2.17b)$$

We then define the scaling limit according to

$$a \rightarrow 0, \quad \theta \rightarrow \infty, \quad \mu \equiv \frac{e^{-\kappa\theta}}{\pi a} \text{ fixed.} \quad (2.18)$$

μ is the renormalized, or physical, mass scale and the mass spectrum of these integrable QFT's is given by

$$M_\ell = \mu m_\ell \quad (2.19)$$

From the relativistic form of the dispersion law (2.17), we also recognize in $\kappa\phi$ the physical rapidity of the particles.

This is a very general way of constructing integrable QFT's. The operators H and P given by Eq. (2.15) are well defined on the lattice as well as all the higher conserved charges (2.13). In the continuum limit $a \rightarrow 0$, they provide the energy and momentum of a relativistic invariant QFT, as long as the spectrum of the original vertex model is gapless. This is usually the case for statistical weights $t_{ab}(\theta)$ which are rational or trigonometric functions of the spectral parameter θ . In addition to the particle spectrum, the S-matrix is exactly calculable from the BA equation by standard methods [1].

As was the case for the MTM, the evolution operators U_R and U_L of the light-cone lattice are much simpler than the Hamiltonian and momentum. This fact was exploited in Ref. [2] to obtain a lattice field equation for the bare fundamental fields of the MTM regularized on the lattice. An analogous local construction of lattice field operators would be very interesting to obtain in the general case. We present here a lattice construction for the currents of a large class of integrable

models that will be specified below. Let us first give an example of the construction in a particular model, in which the R-matrix $R(\theta)$ [symmetric under the action of $SU(n)$ in the fundamental representation, so that $q = n$ in this case] reads:

$$R(\theta) = \frac{1 + i\theta P}{1 + i\theta} \quad (2.20)$$

Here $P_{ab,cd} = \delta_{ad}\delta_{bc}$ ($1 \leq a,b,c,d \leq n$) can be written in terms of the $SU(n)$ Hermitian generators T^α as

$$P = \frac{1}{n} + 2\pi \quad ; \quad \pi \equiv 2 \sum_{\alpha=1}^{n^2-1} T^\alpha \otimes T^\alpha \quad (2.21)$$

The T^α are such that

$$[T^\alpha, T^\beta] = if^{\alpha\beta\gamma} T^\gamma \quad ; \quad \text{tr} T^\alpha T^\beta = \frac{1}{2} \delta^{\alpha\beta}$$

where $f^{\alpha\beta\gamma}$ are totally antisymmetric $SU(n)$ structure constants.

We define the $SU(n)$ currents on the lattice by attaching a T^α generator to each link of a unit horizontal slice of the diagonal lattice (see Fig. 5)

$$T_n^\alpha = \mathbb{1} \otimes \dots \otimes \overbrace{T^\alpha}^{\text{nth link}} \otimes \dots \otimes \mathbb{1} \quad , \quad 1 \leq n \leq N \quad (2.22)$$

We must apply $U_{L,R}(\theta)$ and $U_{L,R}(\theta)^\dagger$ to T_n^α from left and right respectively, in order to derive the unit space and time evolutions of T_n^α on the lattice. These calculations reduce to the following local algebra (compare Figs. 4 and 5):

$$R(\theta)(T^\alpha \otimes \mathbb{1})R(\theta)^\dagger = \frac{1}{1+\theta^2} \left\{ \theta^2 (\mathbb{1} \otimes T^\alpha) + T^\alpha \otimes \mathbb{1} + 2i\theta f^{\alpha\beta\gamma} T^\beta \otimes T^\gamma \right\} \quad (2.23a)$$

$$R(\theta)(\mathbb{1} \otimes T^\alpha)R(\theta)^\dagger = \frac{1}{1+\theta^2} \left\{ \theta^2 (T^\alpha \otimes \mathbb{1}) + \mathbb{1} \otimes T^\alpha - 2i\theta f^{\alpha\beta\gamma} T^\beta \otimes T^\gamma \right\} \quad (2.23b)$$

Therefore

$$\begin{aligned} U_R T_{2n-2}^\alpha U_R^\dagger &= U_L T_{2n}^\alpha U_L^\dagger = \\ &= \frac{1}{1+\theta^2} \left(\theta^2 T_{2n}^\alpha + T_{2n-1}^\alpha + 2i\theta f^{\alpha\beta\gamma} T_{2n-1}^\beta T_{2n}^\gamma \right) \end{aligned} \quad (2.24a)$$

$$\begin{aligned}
 U_R T_{2n-1}^\alpha U_R^\dagger &= U_L T_{2n+1}^\alpha U_L^\dagger = \\
 &= \frac{1}{1+\theta^2} \left(\theta^2 T_{2n-1}^\alpha + T_{2n}^\alpha - 2i\theta f^{\alpha\beta\gamma} T_{2n-1}^\beta T_{2n}^\gamma \right) \quad (2.24b)
 \end{aligned}$$

These are the bare equations for the currents on the lattice. In order to obtain the corresponding continuous equations, we must take a (bare) scaling limit. As was already the case for the MTM, bare and renormalized scaling limits are different.

The bare scaling limit consists here of two steps:

- 1) Define $T_n^\alpha = g \theta t_n^\alpha$ and let $\theta \rightarrow \infty$ with the parameter g fixed (g will be the bare coupling constant). Equations (2.23)-(2.24) yield in this limit:

$$\begin{aligned}
 U_R t_{2n-2}^\alpha U_R^\dagger &= U_L t_{2n}^\alpha U_L^\dagger = t_{2n}^\alpha + 2ig f^{\alpha\beta\gamma} t_{2n-1}^\beta t_{2n}^\gamma \\
 U_R t_{2n-1}^\alpha U_R^\dagger &= U_L t_{2n+1}^\alpha U_L^\dagger = t_{2n+1}^\alpha - 2ig f^{\alpha\beta\gamma} t_{2n-1}^\beta t_{2n}^\gamma \quad (2.25)
 \end{aligned}$$

- 2) Let the lattice spacing a tend to zero. Then we can set

$$\begin{aligned}
 U_A &\simeq 1 - ia H_A, \quad A = R, L \\
 t_{2n}^\alpha &= a J_R^\alpha(x), \quad t_{2n+1}^\alpha = a J_L^\alpha(x); \quad x \simeq na \quad (2.26)
 \end{aligned}$$

We find for $a \rightarrow 0$ from Eqs. (2.25) and (2.26)

$$\begin{aligned}
 \partial_\mu J^{\mu\alpha} &= \partial_0 J_0^\alpha - \partial_1 J_1^\alpha = 0 \\
 \partial_\mu J_\nu^\alpha - \partial_\nu J_\mu^\alpha + i f^{\alpha\beta\gamma} [J_\mu^\beta, J_\nu^\gamma] &= 0 \quad (2.27)
 \end{aligned}$$

where $J_0 = \frac{1}{2}(J_R + J_L)$, $J_1 = \frac{1}{2}(J_R - J_L)$. Therefore the lattice operators T_n^α provide an integrable discretization of the zero divergence and zero curvature equations (2.27). These equations characterize the currents in the $SU(n)$ CGN model. In addition, H and P provided by Eq. (2.15), using the R -matrix (2.20), are the discretized (integrable) version of the CGN Hamiltonian and momentum in the zero chirality sector (see Section 3).

Let us now generalize the current construction for all rational R-matrices admitting the large θ (semiclassical) expansion [10]

$$R(\theta) = P \left\{ 1 + \frac{1}{i\theta} (\Pi + \lambda) + O(1/\theta^2) \right\} \quad (2.28)$$

where λ is a numerical constant, $\Pi = \sum_{\alpha\beta} K^{\alpha\beta} t_\alpha \otimes t_\beta$, t_α are the generators of the simple Lie algebra $\dot{G}$ in some representation ρ and $K^{\alpha\beta}$ is proportional to the inverse of the Killing form. Solutions of the YB equations of this form exist for all representations ρ when $\dot{G} = A_n$ and for some ρ , including the fundamental representation, for all the other simple Lie algebras. As in the $SU(n)$ case, we define the lattice current operator by Eq. (2.22). Relations analogous to Eqs. (2.23)-(2.24) hold in the general case (2.28), but the details of the terms of order θ^{-2} depend on the specific Lie algebra $\dot{G}$ and representation ρ chosen. However, in the bare scaling limit, $\theta \rightarrow \infty$, and only the leading terms precisely provided by Eq. (2.28) survive. Therefore Eqs. (2.25)-(2.27) hold in the general case where $f_{\beta\gamma}^\alpha$ now stand for the structure constants of $\dot{G}$. We achieve in this way an integrable discretization for the currents in the integrable models associated to a rational R matrix symmetric under $\dot{G}$ in the representation ρ . For finite dimensional ρ , this corresponds in the continuum to the zero-chirality sector of the chiral invariant model [11]

$$\mathcal{L} = i\bar{\Psi}\gamma\Psi - \frac{1}{4}g K^{\alpha\beta} (\bar{\Psi}\gamma_\mu t_\alpha \Psi)(\bar{\Psi}\gamma^\mu t_\beta \Psi) \quad (2.29)$$

where we can identify

$$J_\mu^\alpha = \bar{\Psi}\gamma_\mu t^\alpha \Psi \quad (2.30)$$

although no lattice version of ψ is available at present (except for the MTM [2]). Moreover, it is shown in Section 3 [Eqs. (3.18)-(3.23)] that the Hamiltonian and momentum derived from Eq. (2.29) by CBA coincide with the lattice operators (2.15) in the zero-chirality massive sector. It must be noticed that (2.29) can be rewritten as in (1.11) with $V = g\Pi$.

For infinite dimensional representations ρ the lattice models can be related to continuum bosonic models like the PC σ -model (see below).

The spin S ($S > \frac{1}{2}$) generalization of the six-vertex model (with $2S+1$ states per link rather than just 2) provides a family of gapless vertex models associated to the Lie algebra $\dot{G} = A_1$ [12]. These models are characterized, besides the

spectral parameter θ , by an anisotropy parameter γ ($0 \leq \gamma < \pi$), with $\gamma \rightarrow 0$ corresponding to the isotropic $SU(2)$ -invariant limit. It is convenient to represent π/γ as a continued fraction

$$p_0 = \frac{\pi}{\gamma} = b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \dots}}} \quad (2.31)$$

where $b_0, b_1, b_2, \dots$ are positive integers or zero. One also introduces the numbers p_i, q_i, y_i and n_i according to

$$\begin{aligned} p_i &= 1, \quad p_{i+1} = p_{i-1} - b_{i-1} p_i, \quad b_i \equiv \left[\frac{p_i}{p_{i+1}} \right]; \quad i \geq 1 \\ q_0 &= 0, \quad q_1 = b_0, \quad q_{i+1} = q_i + b_i; \quad i \geq 0 \\ y_{-1} &= 0, \quad y_0 = 1, \quad y_{i+1} = y_{i-1} + b_i y_i; \quad i \geq 0 \\ n_j &= y_{i-1} + (j - q_i) y_i, \quad q_i \leq j \leq q_{i+1}; \quad i \geq 0 \end{aligned} \quad (2.32)$$

The analysis of Ref. [12] holds when there exists numbers r and σ such that $q_r \leq \sigma \leq q_{r+1}$, $2S+1 = n_\sigma$. Then one finds a family of low-energy excitations associated to each type of string configuration (with string length n_i) present in the ground state [12]. There exists in each family a kink state with mass

$$M_j = \frac{m_j}{\alpha} \exp \left[-\frac{\pi \theta}{2 p_{j-1}} \right] \quad (2.33)$$

where the number m_j does not depend on θ and the index j , which labels the families, takes all even values from 0 up to r (r even) or $r+1$ (r odd). In addition, one finds breather states with masses

$$M_j^{(k)} = 2 M_j \sin \left[(k - q_{j-1} + 1) \frac{\pi p_j}{2 p_{j-1}} \right]; \quad q_{j-1} \leq k < q_j \quad (2.34)$$

Therefore one can take the scaling limit in $[(r+1)/2] + 1$ different ways. That is, one can keep fixed any one of the kink masses (2.33), say M_λ , as $\alpha \rightarrow 0$ and $\theta \rightarrow \infty$ [cf. Eq. (2.18)]. The families with mass scale smaller or larger than M_λ become, respectively, massless or infinitely heavy in the scaling limit. They decouple in both cases from the particles of the λ th family. The physical S-matrix can be found in Ref. [12].

The field theoretic models discussed up to here correspond to finite values of q , namely, a finite dimensional vector space for each link in the light-cone

lattice. This is clearly appropriate for fermionic fields. Since there exist representations of the YB algebra for $q = \infty$, also bosonic QFT's may be described in this framework.

The $S = \infty$ representation of the XXX magnet relates to the $SU(2)$ Principal Chiral σ -Model (PCM), as was developed in Ref. [13]. Let us recall that the physical particle states of this model transform under the $SU(2)_L \times SU(2)_R$ group. The counting of states in the BA equation [13] and our derivation (Section 4, see Ref. [14] for details) show that only the $SU(2)_L$ singlet sector of the model is described by the H and P associated through Eq. (2.15) to the infinite S limit of the R-matrix [15]

$$R_{12}(\theta) = \frac{\Gamma(2S+1+i\theta) \Gamma(J+1-i\theta)}{\Gamma(2S+1-i\theta) \Gamma(J+1+i\theta)} \quad (2.36)$$

Here the operator J is defined by

$$J(J+1) = 2S(S+1) + 2\vec{S}_1 \cdot \vec{S}_2 \quad (2.37)$$

$\vec{S}_1$ and $\vec{S}_2$ are spin-S operators [$\vec{S}_1^2 = \vec{S}_2^2 = S(S+1)$] acting on the horizontal and vertical spaces respectively.

In other words, the Hamiltonian of the quantum PCM does not follow from the vertex constructions (2.15) and (2.36), even in the scaling limit. Only at the classical level can an equivalence be established between the respective classical analogues [13]. Although (2.15) and (2.36) do not provide at $S = \infty$ the full PCM Hamiltonian, they correctly reproduce its restriction to the $SU(2)_L$ -singlets, and this is enough to calculate all particle masses as well as the invariant S-matrix amplitudes. The same considerations apply to the anisotropic $SU(2)$ PC field [16], from which the $O(3)$ non-linear σ -model can be obtained.

The lattice current construction, Eqs. (2.22) + (2.27), also applies to the PCM. For large θ the R-matrix (2.36) admits a semiclassical expansion of the type (2.28). Therefore the whole construction holds. It must be noted that we have once again only one conserved and curvature-less matrix current: either the one associated to $SU(2)_R$ or that associated to $SU(2)_L$.

In conclusion, the light-cone transfer matrices U_R and U_L associated to each integrable gapless vertex model yield integrable and massive QFT's in the continuum limit. Since the scaling limit can sometimes be performed in several inequivalent ways, one can construct different QFT's from a unique given vertex model.

Depending on q , the number of allowed states per link, one finds fermionic theories without internal degrees of freedom ($q = 2$, from the six-vertex to the MTM), fermionic models with internal symmetries ($q > 2$) and bosonic models ($q = \infty$).

3. - CO-ORDINATE BETHE ANSATZ (CBA)

We derive in this section the CBA for a general class of 2D relativistic Chiral-invariant Fermionic Models (CFM). A CBA also exists [1] for an important non-chiral case: the MTM. The MTM is special in a sense, since there exists a complete light-cone lattice regularization based on the six-vertex model. The $U(1)$ invariance of this R-matrix is identified with bare particle conservation and leads to a proper identification of the canonical lattice fields ψ_n and $\psi_n^\dagger$. In the general case, the symmetries of the R-matrix are not obviously linked to a canonical field structure. This implies that the scaling limit of the light-cone vertex models in general describes only a sector (the zero chirality sector, to be precise) of a continuum QFT which can be locally written in terms of bare canonical operators. The CBA exactly provides the means for the proper identification of such continuum QFT's.

Consider a CFM described by the Lagrangian

$$\mathcal{L} = (\bar{\psi} \not{\partial} \psi + \psi_{a+}^\dagger \psi_{b-}^\dagger V_{ab,cd} \psi_{c+} \psi_{d-}) \quad (3.1)$$

where V is a constant Hermitian matrix acting in the internal q -dimensional spaces and determining the interaction between right-movers (ψ_+) and left-movers (ψ_-). We choose here as Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so that $\gamma^5 \psi_\pm = \pm \psi_\pm$. Equation (3.1) coincides with Eq. (1.11) upon using $\gamma^\mu \otimes \gamma_\mu = \mathbf{1} \otimes \mathbf{1} - \gamma^5 \otimes \gamma^5$. For generic internal space and generic V , Eq. (3.1) essentially describes all relativistic chiral-invariant (and left-right symmetric) fermionic QFT. A further generalization is possible by breaking the left-right symmetry using different left and right internal spaces. Such a model is considered in Ref. [17] in connection with the Wess-Zumino-Witten (WZW) σ -model [18].

The Hamiltonian of the model (3.1) reads

$$H = \int_{-L/2}^{L/2} dx \left[-i \psi^\dagger \gamma^5 \partial_x \psi + \psi_{a+}^\dagger \psi_{b-}^\dagger V_{ab,cd} \psi_{d-} \psi_{c+} \right] \quad (3.2)$$

and commutes with the $U(1) \times U(1)$ charges of the theory

$$Q = N_+ + N_- \quad ; \quad Q_5 = N_+ - N_-$$

$$N_{\pm} = \int_{-L/2}^{L/2} dx \psi_{a\pm}^\dagger \psi_{a\pm} \quad (3.3)$$

so that $\dot{Q} = \dot{Q}_5 = 0$. While Q requires at least the VEV subtraction to be finite in the physical states, Q_5 does not, since the necessarily unbroken chiral symmetry implies $\langle N_+ - N_- \rangle = 0$ in the vacuum.

The generators of the symmetries exhibited by V also commute with H . For example, V may be invariant under the action of a Lie group G in a given representation ρ . That is, under the change $\psi \rightarrow \rho(g)\psi$, $g \in G$, H will be invariant if

$$[\rho(g) \otimes \rho(g)]^\dagger V [\rho(g) \otimes \rho(g)] = V \quad , \quad \forall g \in G \quad (3.4)$$

For instance, the most general V invariant under $SU(n)$ in the fundamental representation is ($q = n$ here)

$$V_{ab,cd} = g \delta_{ad} \delta_{bc} + (g' - \frac{g}{n}) \delta_{ac} \delta_{bd} \quad , \quad 1 \leq a, b, c, d \leq n \quad (3.5)$$

where g and g' are arbitrary couplings. This interaction can be written as

$$-\psi_{a+}^\dagger \psi_{b-}^\dagger V_{ab,cd} \psi_{c+} \psi_{d-} = \frac{1}{2} g \sum_{\alpha=1}^{n^2-1} J_\mu^\alpha J^{\mu\alpha} + \frac{1}{2} g' J_\mu J^\mu \quad (3.6)$$

where

$$J_\mu^\alpha = \bar{\psi} \gamma_\mu T^\alpha \psi \quad , \quad J_\mu = \bar{\psi} \gamma_\mu \psi$$

$$T^\alpha = T^{\alpha\dagger} \quad , \quad \text{tr} T^\alpha = 0 \quad , \quad \text{tr} T^\alpha T^\beta = \frac{1}{2} \delta^{\alpha\beta} \quad (3.7)$$

J_μ^α and J_μ are, respectively, the "colour" $SU(N)$ current and the $U(1)$ current. This model [alternatively called Chiral Gross-neveu model or $SU(n)$ Thirring model] was exactly diagonalized by the CBA in Refs. [19] and [20].

Let us now return to the general model (3.2). The CBA approach to the diagonalization of H consists of three steps:

- i) The canonical anticommutation rules on the fields ψ , $\psi^\dagger$ are given a Fock representation based on the unphysical reference state $|0\rangle$ killed by all ψ 's: $\psi_{a\pm}(x)|0\rangle = 0$, $\forall a, x$.

- ii) The second-quantized Hamiltonian H is reduced to a differential operator h acting on states with a fixed number of pseudoparticles N_+ and N_- which are both conserved. More explicitly, one sets

$$|F\rangle = \int_{-L/2}^{L/2} d^N x \sum_{a_1 \dots a_N} F_{a_1 \dots a_N}(x_1 \dots x_N) \prod_{j=1}^N \psi_{a_j \alpha_j}^+(x_j) |0\rangle \quad (3.8)$$

where $\alpha_j = \pm$ ($1 \leq j \leq N \equiv N_+ + N_-$) define any chirality configuration with prescribed N_+ and N_- . Then

$$H|F\rangle = |hF\rangle \quad (3.9)$$

with

$$h = -i \sum_{j=1}^N \alpha_j \frac{\partial}{\partial x_j} + \sum_{j < l} \frac{1}{2} (1 - \alpha_j \alpha_l) \delta(x_j - x_l) V_{jl} \quad (3.10)$$

where V_{jl} stands for the operator acting as V on the tensor product of the j -th and l -th internal subspaces. It is useful to introduce here the following notation: for any set of quantities $q = (q_1, \dots, q_N)$ associated to the N pseudoparticles, let $q_+ = (q_{j_1}, \dots, q_{j_{N_+}})$, $1 \leq j_1 < j_2 < \dots < j_{N_+} \leq N$, $\alpha_{j_r} = +$, be the restriction of q to the right-moving pseudoparticles; similarly, one defines q_- , which is clearly complementary to q_+ . The Hilbert space at fixed N_+, N_- is formed by the square-integrable wave functions $F_{a_1 \dots a_N}(x_1 \dots x_N)$ which are separately totally antisymmetric in the two sets of index pairs $(\underline{x}a)_+$ and $(\underline{x}a)_-$.

- iii) h is diagonalized on CBA states $|\underline{k}, \Phi\rangle$, whose wave functions are given by

$$\langle \underline{x}a | \underline{k}, \Phi \rangle = \left[\prod_{j=1}^N e^{ik_j x_j} \right]_{N_+, N_-} \sum_{Q \in S_N} \theta(x_Q) [S(Q) \Phi]_{a_1 \dots a_N} \quad (3.11)$$

In this expression, $[\dots]_{N_+, N_-}$ means complete antisymmetrization with respect to $\underline{x}_+$ and $\underline{x}_-$ separately; $\underline{k} = (k_1, \dots, k_N) = \underline{k}_+ \cup \underline{k}_-$ is an arbitrary set of pseudomomenta (with $\underline{k}_+$ and $\underline{k}_-$ separately given by all distinct entries); $\Phi = (\Phi_{a_1}, \dots, \Phi_{a_N})$ is any global internal state; the sum $\sum_{Q \in S_N}$ runs over all permutations of N objects, which form the symmetric group S_N ; $\theta(x_Q) \equiv \theta(x_{Q1} < \dots < x_{QN})$ is 1 if the argument is true and 0 otherwise; and, finally, the Q -dependent unitary S -matrix $S(Q)$ is obtained by taking the product, along any path of neighbouring transpositions connecting Q to the identity, of the unitary two-body S -matrices:

$$S_{je} = P_{je} \quad , \quad \alpha_j = \alpha_e \quad (3.12a)$$

$$S_{je} = \frac{i(\alpha_j - \alpha_e) - V_{je}}{i(\alpha_j - \alpha_e) + V_{je}} \quad , \quad \alpha_j \neq \alpha_e \quad (3.12b)$$

where $P_{j\ell}$ is the exchange operator between the j th and ℓ th internal subspaces. $S(Q)$ does not depend on the particular path chosen to reach Q , thanks to the (discrete) factorizability conditions satisfied by $S_{j\ell}$:

$$S_{je} S_{ej} = 1 \quad (3.13a)$$

$$S_{je} S_{kn} = S_{kn} S_{je} \quad , \quad j, e, k, n \text{ all distinct} \quad (3.13b)$$

$$S_{jk} S_{je} S_{ke} = S_{ke} S_{je} S_{jk} \quad (3.13c)$$

The eigenvalues of the Hamiltonian H and momentum operator P on $|\underline{k}, \Phi\rangle$ are the same as those of pure plane waves:

$$H|\underline{k}, \Phi\rangle = \left(\sum_{j=1}^N \alpha_j k_j \right) |\underline{k}, \Phi\rangle \quad (3.14a)$$

$$P|\underline{k}, \Phi\rangle = \left(\sum_{j=1}^N k_j \right) |\underline{k}, \Phi\rangle \quad (3.14b)$$

It is also easy to check that the CBA state $|\underline{k}, \Phi\rangle$ has the correct symmetry properties, since Eq. (3.12a) implies

$$S(Q_+ Q_- Q) = Q_+ Q_- S(Q)$$

where $Q_{\pm}$ acts only on $\underline{a}_{\pm}$, respectively.

While the form of $S_{j\ell}$ for $\alpha_j = \alpha_{\ell}$ is fixed by the requirement that the CBA states have the right discontinuities to cancel the δ -functions in h [Eq. (3.10); we have assumed the convention $\delta(x)\theta(x) = \frac{1}{2}\delta(x)$], the form of $S_{j\ell}$ for $\alpha_j \neq \alpha_{\ell}$ is dictated by factorizability alone. In a sense, it contradicts the naive expectation that non-interacting equal-chirality pseudoparticles should have a unit S -matrix. Actually, any discontinuity in $x_j - x_{\ell}$ for $\alpha_j = \alpha_{\ell}$ is possible, since the kinetic operator in Eq. (3.10) is completely blind with respect to them.

Several comments are in order before we proceed any further. Step i) might appear as a sacrilege to a relativistic field theorist. The state $|0\rangle$ certainly does not belong to the physical Hilbert space based on the true vacuum state. However, as soon as the original theory (3.2) is given an explicit UV cut-off Λ (which at any rate is always implicit when writing down bare Lagrangians or Hamiltonians), then the state $|0\rangle$ and the true ground state (the physical vacuum at fixed Λ and L) live in the same cut-off Hilbert space. When $\Lambda \rightarrow \infty$ they separate in a Λ -dependent way and act as reference states of two orthogonal Fock spaces supporting inequivalent representations of the canonical anticommutation rules. The real problem is how to introduce the UV cut-off Λ without spoiling integrability, which at this stage coincides with the possibility of exactly diagonalizing h , the "first-quantized" operator emerging at step ii). Indeed, the presence of a UV regularization must change h , by, for example, smearing the δ -functions in (3.10), introducing higher derivatives, or replacing them by finite differences, and so on. The crucial point is that step iii) requires h to have precisely the singular form (3.10). The presence of δ -functions in h implies the presence of discontinuities in the CBA wave functions (3.11). In turn this means that the CBA states have support extending to infinite momentum in Fourier space. They are not physically cut off. The conventional UV regularization in the CBA approach is only superficially a true one. It consists of putting a sharp cut-off on the pseudomomenta $k_1, \dots, k_N$ in order to make the energy eigenvalues (3.14a) bounded from below. Clearly, however, this does not correspond to a regularization of the fields $\phi(x)$, $\phi^+(x)$ themselves, which still anticommute in a singular way. This very unconventional cut-off procedure can be a source of deep trouble, as we shall shortly see.

For the moment, let us make another observation on step iii). Its validity does not put any constraint whatsoever on the interacting matrix V . This is because only discrete factorizability is required on the two-body S -matrices (3.13), and this is guaranteed by the choice (3.12a) alone, independently of the particular form of V . Continuous factorizability is required only at a second stage, when the allowed values of the pseudomomenta are explicitly necessary. Let us now turn to this second stage.

This consists of the imposition of periodic boundary conditions on the CBA states. Thanks to the factorizability relations (3.14), this imposition determines the following, finite-dimensional, eigenvalue equations for the state Φ of Eq. (3.11):

$$Z_j \Phi = e^{ik_j L} \Phi \quad ; \quad j = 1, 2, \dots, N \quad (3.15)$$

where

$$Z_j = S_{jj-1} \dots S_{j1} S_{jN} \dots S_{jj+1} \quad (3.16)$$

The system (3.15) is consistent since $[Z_j, Z_\lambda] = 0$, again thanks to Eq. (3.13). Actually, one easily verifies that there are only two distinct Z 's, since

$$Z_j = Z_{\alpha_j} = Z_{\pm} \quad (3.17)$$

From Eqs. (3.15) and (3.14), we now obtain

$$H = \frac{2\pi}{L} \sum_{j=1}^N \alpha_j n_j + \frac{i}{L} (N_+ \log Z_+ - N_- \log Z_-) \quad (3.18a)$$

$$P = \frac{2\pi}{L} \sum_{j=1}^N n_j + \frac{i}{L} (N_+ \log Z_+ + N_- \log Z_-) \quad (3.18b)$$

where the branches of the logarithms are fixed, so that the integers n_j ($n_j \neq n_\lambda$ for $\alpha_j = \alpha_\lambda$) are free BA eigenvector parameters. Since $Z_{\pm}$ satisfy also the identity $Z_+^{N_+} Z_-^{N_-} = 1$, we see that the momentum is properly quantized in multiples of $2\pi/L$.

Even before an explicit solution of the eigenvalue problem (3.15) is available, we see that the CBA has split the energy-momentum eigenvalues into two distinct and independent contributions: the first, given by integers n_j , is exactly the same as a system of free fermions without any internal degree of freedom. The second one depends only on the internal state Φ . This is a general feature of the CBA in relativistic models.

Up to this point, no restriction on the interaction matrix V has been imposed. The possibility of completely integrating the discrete eigenvalue problem (3.15) imposes some restrictions now. In particular, we demand that the two-body matrices S_{ij} satisfy a much stronger version of Eqs. (3.13), which is continuous factorizability. This means that S_{ij} should be the value at $\theta = \frac{1}{2}(\alpha_i - \alpha_j)$ of some unitary matrix $S_{ij}(\theta)$, with θ an arbitrary complex parameter, satisfying the YB algebra

$$S_{jk}(\theta) S_{je}(\theta + \theta') S_{ke}(\theta') = S_{ke}(\theta') S_{je}(\theta + \theta') S_{jk}(\theta) \quad (3.19)$$

Of course, for $\theta = \frac{1}{2}(\alpha_j - \alpha_k)$ and $\theta' = \frac{1}{2}(\alpha_k - \alpha_\lambda)$, (3.19) implies (3.13c). Equation (3.13a) is now replaced by $S_{jk}(\theta) S_{jk}(-\theta^*) = 1$, that is, $S_{jk}(\theta)^\dagger = S_{jk}(-\theta^*)$, while Eq. (3.13b) needs no generalization since it is true for any two-body matrix. Notice that (3.12a), $S_{ij} = P_{ij}$ for $\alpha_i = \alpha_j$, implies that $S_{ij}(\theta)$ must be a regular solution of the YB algebra:

$$S_{ij}(0) = P_{ij} \quad (3.20)$$

With the request that V be such that an integrable $S_{j\ell}(\theta)$ can be found, the problem (3.15) is solved along the usual lines of the Quantum Inverse Scattering Method (QISM) [1]. The matrix $S_{j\ell}(\theta)$ is identified with the local weights $t_{ab}(\theta)$ of Section 2

$$S_{j\ell}(\theta) \begin{matrix} a_1 \dots a_N \\ a'_1 \dots a'_N \end{matrix} = [t_{a'_\ell a_\ell}(\theta)]_{a'_j a_j} \prod_{k \neq j, \ell} \delta_{a'_k}^{a_k} \quad (3.21)$$

so that the formulation (3.19) of the YBA is mapped into the formulation (2.2) (with $R = SP$, $P =$ exchange operator). One then extends the rapidities α_j to arbitrary complex values θ_j (the inhomogeneities of Section 2) and explicitly verifies that

$$Z_{\pm} = \tau(\theta_0 = \pm ; \{\theta_j = (-)^{j+1}\}) \quad (3.22)$$

where $\tau(\theta_0; \{\theta_j\})$ is the transfer matrix of Eq. (2.1). We see that, by choosing $N_+ = N_-$ and appropriately scaling the spectral parameter θ one can map $Z_{\pm}$ into the light-cone transfer matrices U_R and U_L , and precisely

$$Z_+ = U_R ; Z_- = U_L^\dagger \quad (3.23)$$

This relation can also be derived directly from the definition (3.16) of $Z_{\pm}$ and that of $U_{R,L}$ of Ref. [2] (see Fig. 4), and holds in any case, even if continuous factorizability (3.19) cannot be established. Indeed, it is sufficient to make a precise choice for the arbitrary chirality configuration $\alpha_1, \dots, \alpha_N$, that is, $\alpha_{2j-1} = -\alpha_{2j} = +$, $j = 1, \dots, N_+$ (recall $N_+ = N_-$), and exploit the regularity condition $S_{ij} = P_{ij}$ when $\alpha_i = \alpha_j$.

Thanks to Eq. (3.23) we can identify the energy-momentum operators defined on the light-cone lattice by Eq. (2.15) with the chiral-less sector (i.e., the n_j -independent sector) of the energy-momentum of the general chiral fermionic model Eq. (3.1). Comparing Eqs. (3.18) (in the case $N_+ = N_-$), (2.15) and (3.23), we need only set $L = N_+ a$, where a is the lattice spacing. Of course, from the point of view of the continuum field theory (3.1), this identification is not quite obvious. It becomes natural only after a cut-off Λ is imposed on the integers n_j by, say,

$$\frac{2\pi}{L} \alpha_j n_j \geq -\Lambda \quad (3.24)$$

Then one should minimize H for fixed Λ , determining the cut-off ground state $|\Omega\rangle_\Lambda$. In the integrable cases when this can be explicitly performed, one finds that, in

the ground state, $N_+ = N_- = N_0 \sim \Lambda L$ and therefore $\Lambda \sim a^{-1}$ indeed plays the rôle of a UV cut-off.

In conclusion, from the point of view of complete integrable models, we see that, given a solution of the YB relations (2.2) or (3.19) satisfying the regularity condition, one can either construct a light-cone vertex system characterized by the diagonal-to-diagonal transfer matrices U_R and U_L or a fermionic QFT of type (3.1) having an interaction matrix V related to the solution $S_{jk}(\theta)$ of the YBA by Eq. (3.12b). In both cases, the BA equations governing the eigenvalue-eigenvector structure of $U_{R,L}$ or $Z_{\pm}$ are identical. Of course, the QFT model has the extra freedom of independently varying N_+ and N_- away from their ground state value N_0 . This, however, does not affect in any way the mass spectrum or the S-matrix.

Actually, there is another, more serious difference between the two above-mentioned procedures. While the light-cone lattice approach is a perfectly well-defined means of constructing relativistic, non-trivial 2D QFT in the scaling limit^{*)}, the fermionic approach suffers from the cut-off subtleties previously mentioned. In other words, it is not obvious that the CBA solution of the theory (3.1), even in the full integrable case characterized by Eq. (3.19), is the correct one, lying in the same universality class, at infinite UV cut-off, of conventional perturbation theory and other more traditional field theoretic methods. From a purely technical point of view, let us observe that the CBA state completeness, in the field theoretic sense, is very difficult to ascertain (on the contrary, completeness of BA states for fixed $N_{\pm}$ is well understood, just like in spin systems [1]). This is because the cut-off procedure (3.24) breaks completeness in each fixed (N_+, N_-) sector, and when Λ is removed to ∞ , also N_+, N_- tend to ∞ in any physical state. From another point of view, let us observe that the chiral currents

$$J_{R,L}(x) \equiv \bar{J}_{\pm}(x) = \psi_{a\pm}^{\dagger} \psi_{b\pm} \quad (3.25)$$

are replaced by multiplication operators

$$\bar{J}_{\pm}(x) \longrightarrow \sum_{j=1}^{N_{\pm}} \delta(x - x_{j\pm}) \quad , \quad x_{j\pm} \in \mathcal{X}_{\pm} \quad (3.26)$$

in the CBA Fock representation (3.6) based on the unphysical vacuum $|0\rangle$. Hence the commutators $[J_{\pm}(x), J_{\pm}(y)]$ vanish identically for any $N_{\pm}$ and will never give rise to a Schwinger term, even when $\Lambda \rightarrow \infty$. However, physical chiral currents, as is well known, must have anomalous commutators, by Schwinger's theorem [21]. The only

^{*)} Strictly speaking, there is also here a price to be paid: up to now, a canonical lattice field structure has not been generally available.

way out is that the canonical $J_{\pm}$ of (3.25) are not the physical $U(1) \times U(1)$ currents; not just because they are not VEV-subtracted (a c-number subtraction will not change the commutator), but because the fields ψ^+ and ψ are evaluated at the same point before the UV cut-off is sent to ∞ . In a theory like the CGN of Eq. (3.6), where the interaction is dominated by the $SU(n)$ current-current coupling (we recall that the trajectory $g' = 0$ is stable under renormalization up to the second loop), this problem with the $U(1) \times U(1)$ currents is harmless. Indeed, the $SU(n)$ currents are obviously not mapped into multiplication operators by the CBA (although their commutator is still free of central terms), and to get the wanted Schwinger terms in the commutator of physical $U(1) \times U(1)$ currents, it is presumably sufficient to split $\psi^+(x)\psi(x) \rightarrow \psi^+(x+\epsilon)\psi(x)$ and take $\epsilon \rightarrow 0$ only after the UV cut-off Λ is sent to ∞ in the CBA construction.

Quite different is the situation in the Massless Thirring Model [the $g = 0$ case of Eq. (3.6)]. Here the CBA is bound to fail. By mapping the $U(1) \times U(1)$ currents (which alone enter the interaction) into diagonal operators [Eq. (3.26)], the CBA implies a complete decoupling of physical excitations. As the exact solution shows, both in operatorial [22] and functional language [23], this decoupling is actually prevented only by the axial anomaly, which is just a consequence of Schwinger terms (the crucial rôle of Schwinger terms could already be seen in the old-fashioned non-covariant perturbation theory). For a correct BA lattice construction of the Massless Thirring Model, see, e.g., Ref. [24].

In the case of chiral fermionic models with any non-Abelian simple symmetry group in the fundamental representation, it is possible to perform a non-trivial check that the CBA solution should indeed lie in the same universality class of standard perturbative approaches. For all these models, the S-matrices $S_{j\lambda}(\theta)$ satisfying the YB equations (3.19) are known [3,4]. They all enjoy the semiclassical expansion (1.4) for large θ (recall that the S- and R-matrices are connected by the exchange operator $P: S = PR$). Thus

$$S(\theta) \simeq 1 + \frac{i}{\theta}(\pi + \lambda) + O(1/\theta^2) \quad (3.27)$$

Due to Eq. (3.12b), the large θ form of the interaction matrix V reads

$$V(\theta) \simeq \frac{i}{\theta}(\pi + \lambda) + O(1/\theta^2) = \frac{i}{\theta} \left\{ \sum_{\alpha} t^{\alpha} \otimes t^{\alpha} + \lambda \right\} + O(1/\theta^2)$$

and, with the natural identification $\theta = 1/g$ (compare this relation with the "bare" scaling limit of Section 2), the Lagrangian (3.1) becomes

$$\mathcal{L} = i\bar{\psi} \not{\partial} \psi - \frac{1}{2}g \left\{ \sum_{\alpha} J_{\mu}^{\alpha} J^{\mu\alpha} + \lambda J_{\mu} J^{\mu} \right\} + O(g^2) \quad (3.28)$$

that is, the typical G-invariant current-current interacting theory, up to $O(g^2)$ [for special cases, like $G = SU(n)$, it is possible to obtain the form (3.28), without higher-order corrections, by choosing $\theta = \theta(g) = 1/g + \alpha_0/g^2 + \dots$ and an overall phase factor for $S_{j\ell}(\theta)$, see Section 4].

We can now study the model (3.28) in conventional perturbation theory, up to first loop. The $O(g^2)$ terms are then irrelevant as far as divergences are concerned. Calculations are simplified by introducing auxiliary, non-propagating fields A_μ^α , to split each four-fermion vertex into two $A_\mu^\alpha J_\mu^\alpha$ vertices. Due to the two-dimensional property $\gamma_\mu \gamma_\nu \gamma^\mu = 0$, one can easily realize that only the two ladder diagrams of Fig. 6 are logarithmically divergent. The coefficient of this divergence is given by

$$\begin{aligned} & -\frac{g^2}{2\pi} \sum_{\alpha, \beta} [t^\alpha, t^\beta] \otimes [t^\alpha, t^\beta] = \\ & = +g^2 \frac{C_2}{2\pi} \sum_{\alpha} t^\alpha \otimes t^\alpha \end{aligned} \quad (3.29)$$

where C_2 is the quadratic Casimir operator of the adjoint representation. Notice that the Abelian J_μ^μ coupling cannot contribute to this divergence exactly because of its Abelian nature.

The result (3.29) demonstrates the asymptotic freedom of the model and provides the value $-(C_2/2\pi)$ for the first coefficient of the β -function. On the other hand, this same coefficient can be calculated from the scaling form of the dynamically-generated mass in the BA construction, Eq. (2.18). It equals κ^{-1} . A look at the Table then confirms that $\kappa = 2\pi/C_2$ for all simple Lie algebras (we have chosen the overall normalization of the Lie algebra such that the shortest root has unit length). The very different origin of the same numbers κ provides a rather non-trivial check of the validity of the BA for the CFM's in the fundamental representations.

4. - THE CBA FOR MULTIFLAVOUR MODELS AND THE PRINCIPAL CHIRAL σ -MODEL

In the preceding section we have reviewed the CBA, discussed its main shortcomings, and presented an Abelian example in which it fails (the MTM), as well as a large class of non-Abelian current-current models [Eq. (3.28): the generalization to any simple Lie group of the $SU(n)$ CGN model], in which it is successful. Indeed, the CBA for this class of models passes other checks besides the one-loop universality check examined here; for instance, it agrees with the $1/N$ expansion and with semiclassical methods.

In this section, we shall present another very important example, in which the CBA of Section 3 fails. This is the multiflavour generalization of model (3.28), when the "colour" currents of the Lie group G read

$$\bar{J}_\mu^\alpha = \sum_{r=1}^{N_f} \bar{\Psi}_r \gamma_\mu t^\alpha \Psi_r \quad (4.1)$$

This class of models is important also because of its connection with the PC σ -model on the group G [25,26]. To be specific, we shall consider only the simplest case $G = \text{SU}(n)$, that is, the Multiflavour Chiral Gross-Neveu (MCGN) model. In this case, the internal space has total dimensionality $q = nN_f$.

First let us see how the CBA works for the MCGN. The flavour acts solely as a spectator of the colour interaction (though its presence has non-trivial consequences through the Pauli exclusion principle); hence the colour two-body S -matrices are unchanged. In the continuous form satisfying Eq. (3.19) they read, for $g' = 0$ [compare with Eq. (3.20)]

$$S_{je}(\theta) = \frac{2ni\tilde{g}\theta + (n-1)g}{2ni\tilde{g}\theta - (n-1)g} \cdot \frac{1+i\theta P_{je}}{1+i\theta} \cdot P_{je} \quad (4.2)$$

where

$$\tilde{g} = g \left(1 - \frac{n^2-1}{4n^2} g\right)^{-1}$$

and the discrete S_{je} of Eqs. (3.12a,b) are recovered when

$$\theta \rightarrow \frac{1}{2\tilde{g}} (\alpha_j - \alpha_e) \quad (4.3)$$

To take flavour into account in an integrable way there are two possibilities. The most natural one consists of taking a unit S -matrix in flavour space and it will be analyzed first. In this case, the CBA wave functions [Eq. (3.11)] are modified into

$$\langle x \otimes r | \underline{k}, \Phi, \Psi \rangle = \left[\Psi_r \prod_{j=1}^N e^{i\tilde{k}_j x_j} \right]_{N_+ N_-} \sum_{Q \in S_N} \Theta(x_Q) [S(Q) \Phi]_Q \quad (4.4)$$

where $\Psi = (\Psi_{r_1}, \dots, \Psi_{r_N})$ is the flavour state. [...] $N_+ N_-$ now denotes complete antisymmetrization in the pair sets $(\underline{x}r)_+$ and $(\underline{x}r)_-$. Hence the pseudomomenta $\underline{k}_+$ and $\underline{k}_-$ must obey an exclusion principle of order N_f , i.e., the same entry can appear in $\underline{k}_\pm$ at most N_f times. Although colour and flavour are decoupled in the CBA states (3.30), the global internal state being just a direct product $\Phi_{\underline{x}, \underline{k}} = \Phi_{\underline{x}} \Psi_{\underline{k}}$, it should be clear that one can always find a complete basis in each (N_+, N_-) sector of the decoupled form (3.30). As we previously observed, however, "first-quantization" completeness in each (N_+, N_-) sector is not enough; one needs a (much more difficult

even to define) field theoretic, i.e., "second-quantization" completeness. To clarify this question, suppose we proceed for (3.30) like in the general case. After periodic boundary conditions are imposed, we would arrive at Eqs. (3.18a,b), which express the energy-momentum eigenvalues in terms of arbitrary integers $\underline{n} = (n_1, \dots, n_N)$ (now satisfying an exclusion principle of order N_f), and the eigenvalues $z_{\pm}$ of the $Z_{\pm}$ operators [Eqs. (3.16) and (3.17)]. For fixed $N_{\pm}$, the number $z_{\pm}$ has a lower bound and so does $\log z_{\pm}$, since the branch is fixed. The unboundedness from below of the relativistic Hamiltonian h is due to the integers $\underline{n}$ which can be arbitrarily large. The standard BA procedure is to cut them off as in Eq. (3.24). It thus becomes possible to find the ground state, for fixed cut-off, and excited (physical) particle states. In this way, when the cut-off Λ on $\underline{n}$ is sent to ∞ , one reconstructs the physical Fock space.

Is this the right one, in the sense that all physical states can be obtained in this way? For the MCGN, which has $N_f > 1$, the answer is "No" (it is "Yes" only for $N_f = 1$, i.e., for the standard CGN). There are basically two independent arguments to see this. First, in the ground state and excited states, colour and flavour are fully decoupled for any value of cut-offs Λ and L ; hence colour and flavour are decoupled in the "physical" Fock space. This is due to the peculiar form of the CBA states (4.4): they achieve complete antisymmetry in $(\underline{xar})_{\pm}$ by multiplying a factor completely antisymmetric in $(\underline{xr})_{\pm}$, $[\Psi_{\underline{r}} e^{ik \cdot x}]_{N_+, N_-}$, and a factor completely symmetric in $(\underline{xq})_{\pm}$, $\sum_Q \theta(x_Q) (S(Q)\Phi)_{\underline{q}}$. However, this colour-flavour decoupling in the physical sector contradicts simple perturbation theory. In case one starts invoking non-perturbative effects, there is the second argument, which shows that this decoupling is not due to the interaction, but is rather a CBA pathology. Consider the non-interacting case, $g = 0$, and, since in this case the (+) and (-) worlds do not see each other, let us focus our attention on, say, the (+) pseudoparticles. The CBA states (4.4) apply equally well to the $g = 0$ case, and lead to a bookkeeping of particle states very different from the conventional bookkeeping based on a Fermi sea filled by nN_f negative energy particles per level. Recall that, by Eq. (3.12), $S_{j\lambda}$ at $V = 0$ is 1 for $\alpha_j \neq \alpha_{\lambda}$, but is $P_{j\lambda} \neq 1$ for $\alpha_j = \alpha_{\lambda}$. The BA bookkeeping is based on the energy formula [Eq. (3.18a)]:

$$E_+ = \frac{2\pi}{L} \sum_{j=1}^{N_+} n_j + i \frac{N_+}{L} \log Z_+ \quad (4.5)$$

where Z_+ turns out to be the cyclic operator on colour indices

$$(Z_+ \Phi)_{a_1 \dots a_{N_+}} = \Phi_{a_2 \dots a_{N_+} a_1} \quad (4.6)$$

as is evident from (3.16) and $S_{j\lambda} = P_{j\lambda}$. Since $Z_+^{N_+} = 1$, E_+ is properly quantized

in multiples of $2\pi/L$. The eigenvectors of Z_+ are very well known (see, for example, the review in Ref. [27] for a very clear and simple treatment), hence given a cut-off Λ such that $2\pi n_j > \Lambda L$, the ground state can be determined and all the degeneracies of excited levels can be calculated (the strength of the BA approach lies in the fact that one can do the same in the interacting case, with essentially no extra complications). In such computations, one must "fold" the degeneracies of the first term in (4.5), which are proper to a Fermi system with N_f internal degrees of freedom, with those of the second term, which follow the rather complicated rules of the BA equations. In Ref. [27] the calculations are performed in the case $N_f = 1$ (and $n = 2$), and complete agreement is found with the conventional bookkeeping. (This is one of the most important reasons why the answer for the one-flavour CGN is yes, the BA construction is complete.) For this very reason, however, no agreement is possible in the $N_f > 1$ case; that is, the degeneracy of each energy level $2\pi n/L$ is much larger in the conventional bookkeeping than in the one based on Eq. (4.5). Indeed, the contribution of the colour part, $(iN_+/L)\log Z_+$, is the same regardless of N_f and of the way the n_j levels are filled; in other words, colour and flavour are completely decoupled in contrast with conventional bookkeeping. The latter is really based on the group $U(nN_f)$, whose irreducible representations (irreps) break into irreps of $U(N_f) \times SU(n)$ which are no longer degenerate under the energy formula (4.5). The fermion packing is much tighter in the standard approach, so that the density of states is larger. The BA treatment of free chiral fermions with separated colour and flavour [i.e., a $U(N_f) \times SU(n)$ bookkeeping] misses a large number of states, and thus happens although the CBA states (4.4) are complete in each N_+ pseudoparticle sector at infinite Λ , that is, when the integers n_j are unconstrained. When the interaction between left and right movers is turned on, the symmetry is indeed reduced from $U(nN_f) \times U(nN_f)$ down to $U(N_f) \times U(N_f) \times SU(n)$, and part of the free system degeneracies are lifted. Nevertheless, this cannot make complete at $g > 0$ a vector basis which was incomplete at $g = 0$. This is why the results of Ref. [28], where the CBA (4.4) is assumed, are necessarily wrong (besides being, by the way, in contradiction with general field theoretic expectations).

We mentioned beforehand that there exists another way to take flavour into account while preserving integrability. The analysis of the free chiral fermions performed above suggests that the two-body S-matrices should not be trivial in flavour space. Rather, one should take CBA states (3.11) with

$$S_{je} = S_{je}^{(colour)} \otimes S_{je}^{(flavor)}, \quad \Phi = (\Phi_{2x}) \quad (4.7)$$

where $S_{j\ell}^{(colour)}$, given by Eqs. (4.2) and (4.3), acts only in colour space, while

$$S_{je}^{(\text{flavor})} = \begin{cases} P_{je}^{(\text{flavor})} & , \alpha_j = \alpha_e \\ 1 & , \alpha_j \neq \alpha_e \end{cases} \quad (4.8)$$

Equations (3.33) and (3.34) imply that S_{je} is the total exchange operator P_{je} ,

$$P_{je} = P_{je}^{(\text{color})} \otimes P_{je}^{(\text{flavor})} \quad (4.9)$$

when $\alpha_j = \alpha_e$. Now the free system can be properly described by the CBA; the energy formula (4.5) is modified into

$$E_+ = \frac{2\pi}{L} \sum_{j=1}^{N_+} n_j + i \frac{N_+}{L} \log Z_+^{(\text{color})} Z_+^{(\text{flavor})} \quad (4.10)$$

with the integers n_j all distinct. The only subtlety is in the way $Z_+ \equiv Z_+^{(\text{colour})} Z_+^{(\text{flavour})}$ is diagonalized: we can use an $SU(nN_f)$ bookkeeping, as well as an $SU(n) \times SU(N_f)$ one. The former corresponds, in the language of the QISM, to the identification

$$Z_+ = \tau_+(\theta_0 = 0 ; \{\theta_j = 0\}) \quad (4.11)$$

where $\tau_+(\theta_0; \{\theta_j\})$ is the transfer matrix for right movers constructed according to Eqs. (2.1) and (3.21) from the "continuous" S-matrices

$$S_{je}(\theta) = \frac{i\theta + P_{je}}{i\theta + 1} \quad (4.12)$$

The $SU(n) \times SU(N_f)$ bookkeeping corresponds instead to the choice

$$Z_+ = \tau_+^{(\text{color})}(0; \{\theta_j\}) \otimes \tau_+^{(\text{flavor})}(0; \{\theta_j\}) \quad (4.13a)$$

$$S_{je}(\theta) = \frac{i\theta + P_{je}^{(\text{color})}}{i\theta + 1} \otimes \frac{i\theta + P_{je}^{(\text{flavor})}}{i\theta + 1} \quad (4.13b)$$

and implies a colour-flavour factorization: $\Phi_{\underline{a}_+, \underline{e}_+} = \Phi_{\underline{a}_+} \Phi_{\underline{e}_+}$. The $SU(nN_f)$ bookkeeping leads to the correct degeneracies of the free system, that is, the BA $U(1) \times SU(nN_f)$ bookkeeping is equivalent to the standard $U(nN_f)$ bookkeeping, as mentioned above. The $SU(n) \times SU(N_f)$ approach, on the other hand, is equivalent to the $U(N_f) \times SU(n)$ bookkeeping based on Eq. (4.5): in other words, it is wrong.

While for the free system we can choose the right bookkeeping, for the interacting case the dangerous $U(N_f) \times U(N_f) \times SU(n)$ bookkeeping is forced upon us by the symmetry reduction. It corresponds to the choice

$$\Phi_{\underline{a}\underline{r}} = \Phi_{\underline{a}} \Psi_{\underline{r}+}^{(+)} \bar{\Psi}_{\underline{r}}^{(-)} \quad (4.14a)$$

$$Z_{\pm} = \tau^{(\text{colour})}(\theta_0 = \pm \tilde{g}^{-1}; \{\theta_j = (-)^{j+1} \tilde{g}^{-1}\}) \otimes \tau_{\pm}^{(\text{flavour})}(\theta_0 = 0, \{\theta_j = 0\}) \quad (4.14b)$$

where $\tau^{(\text{colour})}(\theta_0; \{\theta_j\})$ is constructed with $S_{j\lambda}^{(\text{colour})}(\theta)$ [Eq. (4.2) ; observe also that θ is scaled by $\tilde{g}$ with respect to the general discussion of Eqs. (3.19)-(3.22)] and $\tau_{\pm}^{(\text{flavour})}$ is constructed with $(i\theta + p_{j\lambda}^{(\text{flavour})})(i\theta + 1)^{-1}$. Quite naturally, this faulty approach leads to BA equations with no coupling whatsoever among colour, flavour and chiral labels n_j . At present, it seems very hard to find a successful cut-off procedure, more sophisticated than the simple bound (3.24) on the n_j 's, while retaining the singular CBA form (3.11). It is very likely, after all, that the source of trouble is just in the unregularized nature of the first-quantized Hamiltonian h .

To circumvent all these difficulties, it has been proposed [25,26,17] that the MCGN model (as well as any other multiflavour model symmetric under a classical Lie group in the fundamental representation) is actually equivalent to one-flavour CFM's of the general type (3.1), with the interaction matrix V taken to be

$$V = 2i \frac{S_T(\theta) - 1}{S_T(\theta) + 1} \Big|_{\theta = 1/g} \quad (4.15)$$

[that is, the standard relation (3.12b) for $\alpha_j = \alpha_{\lambda} = +$], where S_T is a two-body S-matrix for the scattering of symmetric tensors of rank N_f . In other words, the idea is to replace the internal space of a multiflavour model, which is obviously reducible under the symmetry group G , by an irreducible internal space corresponding to the highest irrep. contained in the original reducible space. The tensor S-matrix in (4.15) is not quite arbitrary; it is obtained by "fusion" [29,15] from the S-matrix for "particles" in the fundamental representation. As such, it is a regular solution of the YB equations (3.19). Let us briefly illustrate the fusion procedure for the case $G = \text{SU}(n)$. Consider the factorized scattering of N_f $\text{SU}(n)$ vectors upon other N_f such vectors and assign to the p -th vector, $1 \leq p \leq N_f$, a complex valued rapidity θ_p . The total scattering matrix is a suitable product of matrices like (4.12), each evaluated at argument $\theta = \theta_p - \theta_{p'}$, if the corresponding two-body scattering involves the vectors p and p' . The many different possible product structures are all equivalent, thanks to the YB algebra (3.19). For special values of the rapidities θ_j (the so-called "string values"), the full S-matrix is projected upon the channel in which two symmetric tensors of rank N_f scatter upon

each other. This provides the wanted two-body tensor S-matrix $S_T(\theta)$ of Eq. (4.15). It is the irreducibility under the symmetry group $[SU(n)]$ in the case at hand of the internal space of a CFM which ensures that the BA bookkeeping of excited states is the correct one.

The physically intuitive argument supporting the above-mentioned substitution is the following (no explicit proof exists in the published literature). The factorizable interaction between $SU(n)$ vectors in the MCGN is antiferromagnetic, being most attractive in the singlet channel. The ground state of the MCGN must contain an infinity of these vector-like pseudoparticles and, due to the extra flavour degeneracy, up to N_f equal chirality pseudoparticles can be located at the same point in space with "parallel" colours, i.e., in the symmetric tensor representation of rank N_f . The singlet channel of the fused tensor S-matrix is the most attractive, leading to the largest phase shift among all singlet channels of the scattering of N_f vectors upon other N_f vectors. Hence the vector pseudoparticles in all the physical states near the true vacuum will tend to form bound states which are symmetric tensors of rank N_f . Within the standard cut-off procedure (3.24), the CBA states (4.4) or the equivalent ones constructed according to Eqs. (4.14) (with decoupled colour and flavour) certainly do not allow such bound-state formation, since the pseudoparticle rapidities are "frozen" to the discrete values $\alpha_j = \pm 1$. On the other hand, the model with the tensor interaction (4.15) describes only that sector of the MCGN where the bound-state formation is maximum; from this point of view, it can hardly be "equivalent" to the full MCGN in the field-theoretic sense (see also Ref. [30] on this point). Nevertheless, the coloured, massive and chiral-less sectors of the two theories are probably equivalent^{*)}, at least in the $N_f \rightarrow \infty$ limit relevant for the connection to the PC σ -model pointed out in the pioneering works [25] and extensively used in Ref. [26] also for Lie groups of other than $SU(n)$. Therefore, given that the $SU(n)$ PC σ -model is obtained from the $N_f \rightarrow \infty$ limit of the MCGN model; that the massive sector of the latter can be constructed by solving the subsidiary irreducible tensor CF model; and that the PC σ -model exhibits dynamical mass generation just like the MCGN but without any massless sector, one could claim, as in Ref. [26], to have solved the PC σ -model via CBA.

We close this rather lengthy section with a careful investigation of the link between the MCGN model and the PC σ -model. We shall show that the $N_f \rightarrow \infty$ limit of the MCGN does not reproduce the entire Hilbert space of the PC σ -model, but only a subspace of it. Details of this derivation can be found in Ref. [14].

*) A similar situation is encountered in the BA solution of the multichannel Kondo model [31]: there a suitable regularization is found that yields a complete tensor bound-state formation, leading to the correct and full description of the electron-impurity magnetic interaction; the electron gas itself, on the other hand, including its charge degrees of freedom, is only partially accounted for.

Consider the action functionals of the $SU(n)$ PC σ -model,

$$\mathcal{A} = \frac{1}{g} \int d^2x \operatorname{tr} (\partial_\mu U) (\partial^\mu U^\dagger) \quad ; \quad U \in SU(n) \quad (4.16)$$

and that of the MCGN model,

$$\mathcal{A}_{N_f} = \int d^2x \left\{ i \bar{\psi} \not{\partial} \psi - \frac{1}{2} g J_\mu^\alpha J^{\mu\alpha} \right\} \quad (4.17)$$

where

$$J_\mu^\alpha = \sum_{j=1}^{N_f} \bar{\psi}_j \gamma_\mu T^\alpha \psi_j \quad ; \quad \operatorname{tr} T^\alpha T^\beta = \frac{1}{2} \delta^{\alpha\beta}$$

Using a functional integral argument, Polyakov and Wiegmann claimed [25] that $\mathcal{A}_\infty = \mathcal{A}$ (in the sense, of course, that expectation values calculated with $\mathcal{A}_{N_f}$ tend to those calculated with $\mathcal{A}$ as $N_f \rightarrow \infty$). Now consider the partition function $Z_{N_f}(\beta)$ at inverse temperature β for model (4.17). It can be written

$$Z_{N_f}(\beta) = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A \exp \left\{ \int_0^\beta dx_2 \int d^2x_1 \left[i \bar{\psi} \not{\partial} \psi - \frac{1}{g} \operatorname{tr} A_\mu A_\mu \right] \right\} \quad (4.18)$$

where $D_\mu = \partial_\mu - iA_\mu$, $A_\mu = A_\mu^\alpha T^\alpha$ and the space-time signature is the Euclidean one. The boundary conditions in the Euclidean time x_2 are periodic for A_μ and antiperiodic for ψ , $\bar{\psi}$, as is appropriate for finite-temperature QFT.

Integration over A_μ gives back the action (4.17). Integration over ψ , $\bar{\psi}$ gives the effective action (in gauge-invariant regularizations)

$$\Gamma[A] = \frac{1}{g} \int d^2x \operatorname{tr} A_\mu A_\mu - N_f \Gamma_{\text{WZW}}[MM^\dagger] \quad (4.19)$$

where M is a $SL(n, \mathbb{C})$ -valued field related to A_μ by

$$A \equiv A_1 - iA_2 = iM^{-1} \partial M \quad ; \quad \partial \equiv \partial_1 - i\partial_2 \quad (4.20)$$

and $\Gamma_{\text{WZW}}[\Omega]$, for any invertible Ω , is the (Euclidean) WZW functional [18,23]. The argument now is that for $N_f \rightarrow \infty$, only "pure gauge" fields $A_\mu = iU^{-1} \partial_\mu U$, corresponding to $M = U \in SU(n)$, will contribute to the residual functional integration over A_μ , since they make $\Gamma_{\text{WZW}}[MM^\dagger]$ vanish. All other configurations give a strictly negative Euclidean Γ_{WZW} and will be suppressed by the $N_f \rightarrow \infty$ limit. The first term in (4.19), quadratic in A_μ , then gives precisely the PC σ -model action (4.16).

The missing point in Ref. [25] is that the change of variables from A to U allows for a twisted boundary condition (b.c.) on U . To see this properly it is convenient to change variables from A to $M \in \text{SL}(n, \mathbb{C})$ at fixed N_f . Then the periodic b.c. on A

$$A(x_1, \beta) = A(x_1, 0) \quad (4.21)$$

allows for a twisted b.c. on M

$$M(x_1, \beta) = L_\beta M(x_1, 0) \quad (4.22)$$

where L_β is an arbitrary constant matrix of $\text{SL}(n, \mathbb{C})$. Moreover, it is necessary to integrate over all possible twists L_β in order to reproduce in full the A -integration. Also the Jacobian of this change of variables should be properly taken into account [14]. The net result is that, when $N_f \rightarrow \infty$ and $M \rightarrow U \in \text{SU}(n)$, also $L_\beta \rightarrow U_\beta \in \text{SU}(n)$ and one is left with

$$\lim_{N_f \rightarrow \infty} Z_{N_f}(\beta) = \int dU_\beta \int_{(U_\beta)} dU \exp(-\mathcal{A}[U]) \quad (4.23)$$

where $\mathcal{A}[U]$ is given by Eq. (4.16) and the subscript (U_β) indicates that U satisfies the twisted b.c. $U(x_1, \beta) = U_\beta U(x_1, 0)$. We now recognize in Eq. (4.23) not just the partition function of the PC σ -model, but rather the following trace [14]

$$\lim_{N_f \rightarrow \infty} Z_{N_f}(\beta) = \text{Tr} (e^{-\beta H} P_0) \quad (4.24)$$

where H is the Hamiltonian of the PC σ -model and P_0 is the projector on singlet states of $\text{SU}(n)_L$. It is clear that an analogous expression can be obtained with the projector on the $\text{SU}(n)_R$ singlet, by just setting $A = i(\partial M)M^{-1}$ rather than $A = iM^{-1}\partial M$ as in Eq. (4.20). What one cannot obtain are the physical states which transform non-trivially under both $\text{SU}(n)_L$ and $\text{SU}(n)_R$.

Our result (4.24) explains why the careful counting of BA states of Ref. [13] did not reproduce all the states of the $\text{SU}(2)$ PC σ -model, but only the singlets under either $\text{SU}(2)_R$ or $\text{SU}(2)_L$. In Ref. [13], the setting is essentially that of a light-cone vertex model; the R -matrix is the $S \rightarrow \infty$ limit of the R -matrix (2.36) for symmetric tensors of rank $2S$ under $\text{SU}(2)$ (i.e., states of spin S). We have seen in Section 3 that such a vertex model is equivalent, in the scaling limit, to the massive sector of a CFM with spin S internal space. In turn, this should be equivalent to the massive sector of the multiflavour $\text{SU}(2)$ CGN with $N_f = 2S$. But as $N_f \rightarrow \infty$ this fails to reproduce the entire Hilbert space, giving only the $\text{SU}(2)_L$ singlets.

In the $SU(2)$ case it is possible to reconstruct the entire spectrum and S-matrix from the knowledge of their restriction to the $SU(2)_L$ singlets. For a generic classical group G , or even exceptional Lie groups, this remains to be fully established. The situation appears even less convincing if Green functions are to be considered. The construction of an explicit, regularized version of the full PC σ -model, with a regularized Hamiltonian still exactly diagonalizable, remains an open challenge.

Lie Algebra	Dynkin's Diagram	κ	m_k
A_n	$1-2-3-\dots-0_n$	$2\pi/n+1$	$\sin(\pi k/n+1), 1 \leq k \leq n$
B_n	$1-2-3-\dots-0_{n-1} \rightleftharpoons 0_n$	$\pi/2n-1$	$\sin(\pi k/2n-1), 1 \leq k \leq n-1; m_n = \frac{1}{2}$
C_n	$1-2-3-\dots-0_{n-1} \leftrightharpoons 0_n$	$\pi/n+1$	$\sin(\pi k/2(n+1)), 1 \leq k \leq n$
D_n	$1-2-3-\dots-0_{n-2} \begin{matrix} \nearrow 0_{(-)} \\ \searrow 0_{(+)} \end{matrix}$	$\pi/n-1$	$\sin(\pi k/2(n-1)), 1 \leq k \leq n-2; m_{\pm} = \frac{1}{2}$
E_6	$0_6 \begin{matrix} \\ 1-2-3-4-5 \end{matrix}$	$\pi/6$	$m_1 = m_5 = \frac{m_6}{2} = \frac{\sqrt{3}}{2}; m_2 = m_4 = \frac{3+\sqrt{3}}{2}; m_3 = \frac{3+\sqrt{3}}{\sqrt{2}}$
E_7	$0 \begin{matrix} \\ 0-0-0-0-0-0 \end{matrix}$	$\pi/9$	(*)
E_8	$0 \begin{matrix} \\ 0-0-0-0-0-0-0 \end{matrix}$	$\pi/15$	(*)
F_4	$0-0 \rightleftharpoons 0-0$	$\pi/9$	(*)
G_2	$0 \rightleftharpoons 0$	$\pi/6$	(*)

(*) These values can be (very laboriously) extracted from Refs. [4] and [6].

T A B L E

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FIGURE CAPTIONS

- Fig. 1 : Statistical weight associated to the depicted configuration, $1 \leq \alpha, \beta, a, b, \leq q$.
- Fig. 2 : The row-to-row transfer matrix, Eq. (2.1).
- Fig. 3 : Graphical representation of the regularity condition Eq. (2.5) and the transfer matrix $\tau(-\theta, \theta)$.
- Fig. 4 : Diagonal-to-diagonal transfer matrices (N is even).
- Fig. 5 : The lattice current T_n^α associated to the nth link.
- Fig. 6 : The two logarithmically divergent ladder diagrams. The wiggled lines represent the contraction of two auxiliary fields A_μ^α .

Fig. 1

Fig. 2

$$[t_{ab}(0)]_{\alpha\beta} = a \begin{array}{c} \beta \\ | \\ \text{---} \\ | \\ \alpha \end{array} b \quad ; \quad \begin{array}{c} 1 \quad 2 \quad 3 \quad \dots \quad N-1 \quad N \\ | \quad | \quad | \quad \dots \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \quad \dots \quad \text{---} \quad \text{---} \\ | \quad | \quad | \quad \dots \quad | \quad | \end{array}$$

Fig. 3

$$U_R(\theta) = \begin{array}{c} \beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4 \quad \dots \quad \beta_{N-1} \quad \beta_N \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \dots \quad \diagdown \quad \diagup \\ \alpha_N \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \dots \quad \alpha_{N-2} \quad \alpha_{N-1} \end{array}$$

$$U_L(\theta) = \begin{array}{c} \beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4 \quad \dots \quad \beta_{N-1} \quad \beta_N \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \dots \quad \diagup \quad \diagdown \\ \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \dots \quad \alpha_N \quad \alpha_1 \end{array}$$

Fig. 4

Fig. 5

Fig. 6