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A STRING CONSTRUCTION OF A COMMUTATIVE NON-ASSOCIATIVE ALGEBRA
RELATED TO THE EXCEPTIONAL JORDAN ALGEBRA

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ABSTRACT

We demonstrate how a bosonic realization of the parastatistical operator product expansion of Fateev and Zamolodchikov relates to and extends the recent partial representation of the Jordan algebra exhibited in terms of superstring vertex operators found by Goddard, Nahm, Olive, Ruegg and Schwimmer.

Recently, Goddard, Nahm, Olive, Ruegg and Schwimmer⁽¹⁾, (GNORS), demonstrated a connection between the exceptional Jordan algebra⁽²⁾ and vertex operators defined on the weights of the three eight dimensional representations of $so(8)$. As GNORS point out, it is tempting to suppose the mathematics associated with the Jordan algebra in some way underlies the superstring. This is because some of the pieces of the puzzle are there naturally: the vertices describing the emission of a boson from a fermion, (a space-time vector in either sector, Neveu-Schwarz or Ramond⁽³⁾), and the fermion emission vertex with its conjugate, (a space-time spinor or conjugate spinor, respectively⁽⁴⁾). However, at present the connection with the Jordan algebra is certainly non-covariant, the pieces are transverse viewed from the light-cone gauge.

The work of reference(1) concentrated on the subalgebra $so(8)$ of $so(9,1)$ and as a consequence the whole of the Jordan algebra could not be realised, only that part of it involving the products of off-diagonal elements, (eq. (9b) below). Actually, the Jordan matrix is a twenty-seven dimensional representation of a non-compact E_6 , whose maximal compact algebra is F_4 , and containing $so(9,1)$ as a non-compact subalgebra. This fact, coupled with the suggestion of GNORS, supports the hint of a connection between the superstring and E_6 already conjectured some time ago⁽⁵⁾, and for which we have given, obliquely, a trace of supporting evidence from a rather different direction⁽⁶⁾. Evidently, a serious test of this idea is to be able to demonstrate a connection between the BRS ghosts needed⁽⁷⁾ in the covariant formalism and the weights of E_6 , or at least with some aspect of the Jordan algebra.

Our aim in this note is modest. We shall complete the GNORS algebra in a

natural way, but find it is not quite a Jordan algebra. Our construction is based on the weights of E_6 , but there is always the possibility a more subtle trick might be found to complete the algebra as originally envisaged.

The exceptional Jordan algebra⁽⁸⁾ is the set of hermitian, three dimensional, matrices whose entries are octonians. It is closed under the symmetric product rule

$$A \bullet B = \frac{1}{2} (AB + BA), \quad (1)$$

which is required to satisfy the 'power' associativity relation,

$$A \bullet (B \bullet A^2) = (A \bullet B) \bullet A^2. \quad (2)$$

Because the matrix entries are octonians, the product (1), although commutative, is not associative, which is why (2) is a non-trivial requirement.

The twenty-six non-compact generators of the E_6 correspond to the transformations,

$$A \rightarrow A' = \varepsilon \bullet A, \quad (3)$$

where ε is a traceless Jordan matrix. The other fifty-two compact generators, forming the compact subalgebra F_4 , are derivations corresponding to pairs of transformations (3) via,

$$A \rightarrow A' = \eta \bullet (\varepsilon \bullet A) - \varepsilon \bullet (\eta \bullet A). \quad (4)$$

There is an E_6 invariant, but to construct it it is convenient to introduce another (Freudenthal) symmetric product⁽⁸⁾,

$$A * B = A \bullet B - \frac{1}{2} A \text{Tr}(B) - \frac{1}{2} B \text{Tr}(A) + \frac{1}{2} (\text{Tr}(A) \text{Tr}(B) - \text{Tr}(A \bullet B)), \quad (5)$$

which, although symmetric, does not itself satisfy (2). However, it is possible to invert the Freudenthal product, expressing the Jordan product in terms of it,

$$A \bullet B = A * B + \frac{1}{2} A \text{Tr}(B) + \frac{1}{2} B \text{Tr}(A) - \text{Tr}(A * B)$$

The symmetric trilinear form⁽⁸⁾,

$$\text{Tr}(A * B \bullet C) \equiv [A, B, C] \quad (6)$$

is an E_6 invariant. In fact there is no quadratic E_6 invariant which can be constructed out of Jordan matrices with octonian entries. The quantity $\text{tr}(A \bullet B)$ is invariant under F_4 , (and therefore $\text{so}(8)$), but not $\text{so}(9,1)$. Our main aim is to demonstrate how an extension of the vertex operators considered by GNORS satisfies naturally operator product expansions based on (5).

It is convenient to make the definitions,

$$A_1(a) = 2 \begin{pmatrix} 0 & 0 & \bar{a} \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}, \quad A_2(a) = 2 \begin{pmatrix} 0 & a & 0 \\ \bar{a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3(a) = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & \bar{a} & 0 \end{pmatrix} \quad (7)$$

where a is an octonian, and

$$e_1 = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_3 = 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (8)$$

In terms of these we have what we shall call the Freudenthal algebra:

$$A_i(a) * A_i(b) = -2 \text{Re}(a \bar{b}) e_i \quad i=1,2,3 \quad (9a)$$

$$A_i(a) * A_j(b) = A_k(\bar{a} b) \quad i \neq j \neq k \quad (9b)$$

$$e_i * A_j(a) = \begin{cases} 0 & i \neq j \\ -A_j(a) & i = j \end{cases} \quad (9c)$$

$$e_i * e_j = \begin{cases} 0 & i = j \\ e_k & i \neq j \neq k. \end{cases} \quad (9d)$$

We notice (9b) is the same as the Jordan product (1), and constitutes the piece of the algebra represented by GNORS.

It is now easy to see that the cubic form is indefinite, simply compute

$$\begin{aligned} \text{Tr} \left((A_2(a) + \underline{\alpha} \cdot \underline{e}) * (A_2(a) + \underline{\alpha} \cdot \underline{e}) \cdot e_2 \right) \\ = 2(\alpha_1 + \alpha_3)^2 - 2(\alpha_1 - \alpha_3)^2 - 8|a|^2. \end{aligned} \quad (10)$$

We also see that it is possible to regard the set of components $(\alpha_1 + \alpha_3, \alpha_1 - \alpha_3, 2a)$ as a ten-vector of $so(9,1)$. It is the algebra (9a-d) which we intend to represent in terms of vertex operators.

The weights of a twenty-seven dimensional representation of E_6 are conveniently represented by,

$$(0, -2\lambda_i) \quad i = 1, 2, 3 \quad (11a)$$

$$(\omega_1, \lambda_1), (\omega_2, \lambda_2), (\omega_3, \lambda_3), \quad (11b)$$

where the two-vectors, λ_i , are a (scaled) triplet of $su(3)$, and the four-dimensional quantities, ω_i , are chosen among the vector, spinor and conjugate spinor weights of $so(8)$, for $i=1, 2, 3$, respectively. Thus, the inner products amongst the various vectors are,

$$\lambda_i \cdot \lambda_j = \begin{cases} -1/6 & i \neq j \\ 1/3 & i = j \end{cases} \quad (12)$$

and,

$$\omega_i \cdot \omega'_j = \begin{cases} \pm 1, 0 & i = j \\ \pm 1/2 & i \neq j \end{cases} \quad (13)$$

The squared length of each of the twenty-seven weights (11a,b) is 4/3. The weights of the conjugate twenty-seven are obtained by replacing λ_i by $-\lambda_i$ in each of (11a,b).

To match the weights, we define a six-dimensional bosonic string field,

$$X^a(z) = q^a - i p^a \ln z + i \sum \frac{\alpha_n^a}{n} z^{-n}, \quad a=1, \dots, 6 \quad (14)$$

for which the first four components belong to the space in which the $so(8)$ weights lie, and the other two belong to the space in which the $su(3)$ (scaled) weights lie. For some purposes, to make contact with GNORS, for example, it is convenient to separate the two pieces and write,

$$X = (Y, Z), \quad (15)$$

to parallel the notation in (11a,b).

We are now in a position to make a correspondence between vertex operators defined on the weights, and the matrices $\{A_k(\alpha), e_k\}$ of eqs.(7,8). We start with the nilpotents e_k , $k=1,2,3$, and set

$$e_k \longrightarrow e_k(z) = : \exp -2i \lambda_k \cdot Z(z) : d(-2\lambda_k), \quad (16)$$

where the factor $d(-2\lambda_k)$ is a function of momentum which must be designed carefully^(9,10) to produce certain phases, as we shall see. Note first, the operator product $e_k(z) e_k(z')$ is non-singular, and therefore accords with the first part

of the eqs.(9d). On the other hand, we have

$$e_k(z) e_j(z') \sim (z-z')^{-2/3} : \exp -2i(\lambda_k + \lambda_j) \cdot Z(z') : \times d(-2\lambda_k - 2\lambda_j), \quad k \neq j \quad (17)$$

provided,

$$\hat{d}(-2\lambda_k) \hat{d}(-2\lambda_j) = \hat{d}(-2\lambda_k - 2\lambda_j) \quad k \neq j, \quad (18)$$

where,

$$\hat{d}(\rho) = e^{i\rho \cdot q} d(\rho), \quad (19)$$

Structurally, eq.(17) is almost the same as eq.(9d), second part. However, the vertex operator on the right hand side of the operator product (17) is defined on a conjugate twenty-seven weight. Actually, this is what we might have expected given that, for E_6 , the tensor product of two twenty-sevens contains a conjugate twenty-seven. Thus, we really have a relation

$$e_k(z) e_j(z') \sim (z-z')^{-2/3} e_l^*(z') \quad k \neq j \neq l, \quad (20)$$

where the $*$ denotes a similar definition to (16), but with λ replaced by $-\lambda$. Eq.(20) is reminiscent of the Fateev-Zamolodchikov 'parastatistical' operator products⁽¹¹⁾, and indeed, the two combinations,

$$\psi = \frac{1}{2} \sum_i e_i, \quad \psi^* = \frac{1}{2} \sum_i e_i^*, \quad (21)$$

satisfy precisely their algebra for fields of conformal weight $2/3$. The energy-momentum tensor associated with the conformal fields ψ and ψ^* satisfies the operator products appropriate to a Virasoro algebra with central charge $4/5$.

In order to represent the other elements of the algebra we can make use of

GNORS. They explain how to set up a representation of eight 'units' (not really octonians) in terms of sums and differences of vertex operators defined on the elementary vector weights of $so(8)$. With their conventions, a typical real pair would be

$$\begin{aligned} V^{\pm}(\omega, z) &= \sigma_{\pm} \left(: e^{i\omega \cdot Y(z)} : c(\omega) \pm : e^{-i\omega \cdot Y(z)} : c(-\omega) \right) \\ &\equiv \sigma_{\pm} \left(\psi_{\omega}(z) \pm \psi_{-\omega}(z) \right), \end{aligned} \quad (22)$$

where,

$$\sigma_{+} = 1, \quad \sigma_{-} = i,$$

and an 'octonion', a , is an appropriate linear combination of these. The main result of GNORS is based on the operator product:

$$\begin{aligned} \psi_{\omega_1}(z) \psi_{\tau\omega'_1}(z') &\sim \frac{e^{i\phi}}{\sqrt{z-z'}} \left\{ e^{-i\phi} \varepsilon(\omega_1, \tau(\omega'_1)) \psi_{\tau^2\omega''_1}(z') \right\} \\ &\quad (\text{if } \omega_1 + \tau(\omega'_1) = \tau^2(\omega''_1)), \end{aligned} \quad (23)$$

where τ is a cubic automorphism, $v \rightarrow s \rightarrow \bar{s}$, of the weight lattice of $so(8)$, (that is, an outer automorphism which when lifted to the Lie algebra leaves invariant a G_2 subalgebra of $so(8)$, rather than an $su(3)$ which would be the case for the other possibility, discussed in ref.(6)). It is the definition of a product ' $\circ$ ' in terms of which, and via the definitions (22), eq.(9b) may be verified. The product is defined by the relationship,

$$\Phi(\omega_1) \circ \Phi(\omega'_1) = e^{-i\phi} \varepsilon(\omega_1, \tau(\omega'_1)) \Phi(\omega''_1), \quad (24)$$

where the symbol Φ , labelled in each instance by a vector weight, has a different meaning for each factor in the product, corresponding to its meaning in a vertex operator sense in eq.(23). The phase factor $e^{i\phi}$ is defined by

$$e^{2i\phi} = \frac{\varepsilon(\omega_1, \tau(\omega'_1))}{\varepsilon(\tau(\omega'_1), \omega_1)} = i \quad (25)$$

and is necessary for the 'o' product to enjoy the correct conjugation properties. The non-trivial cocycle and phase factor do indeed allow (24) to mimic the octonian multiplication in the sense of eq.(9b). We note there is an identical 'o' product defined on spinors and conjugate spinors.

The automorphism τ has an action on the E_6 weights, simply sending $\lambda_1 \rightarrow \lambda_2 \rightarrow \lambda_3$, in addition to its action on the weights of $so(8)$. In view of this, we may adjust eqs.(22-23) defining an operator corresponding to $A_i(a)$ in terms of vertex operators using the twenty-four weights, (11b). Thus, for each 'unit', u , we will have expressions similar to (22), but defined in terms of ω_i , depending on the position in the Jordan matrix, and multiplied by extra factors involving the $su(3)$ weights. For example, we may take

$$A_i(a) \rightarrow A_i(a, z) = \sum_{\omega_i} a(\omega_i) \psi_{\omega_i}(z) : \exp i\lambda_i \cdot Z(z) : d(\lambda_i), \quad (26)$$

to represent the octonian a in each of its three possible positions. To verify an operator product representing eq.(9b) is relatively easy. Using the results of GNORS we find,

$$A_i(a, z) A_j(b, z') \sim (z - z')^{-2/3} A_k^*(\overline{ab}, z'), \quad i \neq j \neq k \quad (27)$$

provided,

$$\begin{aligned} \hat{a}(\lambda_i) \hat{a}(\lambda_j) &= e^{-i\phi} \hat{a}(\lambda_i + \lambda_j) \quad i \neq j \\ &= -i \hat{a}(\lambda_j) \hat{a}(\lambda_i). \end{aligned} \quad (28)$$

To check eq.(9a), we remark it is enough to do so for each particular unit at a time. The factor of two in (9a) comes from the pair of terms in the generalisation of (22), and the sign is again a choice of cocycle. Thus,

$$A_i(a, z) A_i(b, z') \sim - (z - z')^{-2/3} 2 R_e(a \bar{b}) e_i^*(z'), \quad (29)$$

provided,

$$\hat{d}(\lambda_i) \hat{d}(\lambda_i) = - \hat{d}(2\lambda_i). \quad (30)$$

Finally, we need to examine the operator product corresponding to eq.(9c). Clearly, if $i \neq j$ the operator product is non-singular. However, if $i = j$ we reproduce the second part of (9c) in the form,

$$e_i(z) A_i(a, z') \sim - (z - z')^{-2/3} A_i^*(a, z'), \quad (31)$$

provided,

$$\hat{d}(-2\lambda_i) \hat{d}(\lambda_i) = - \hat{d}(-\lambda_i). \quad (32)$$

To convince ourselves the operator products are precisely as claimed we must ensure the relations (18), (28), (30) and (32) are mutually consistent. To check this, we may represent them by ordinary (complex) matrices whose multiplication is assuredly associative. Because of eq. (28) such matrices must be 4x4 or bigger. A quick inspection reveals that the choice

$$\hat{d}(\lambda_1) \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad \hat{d}(\lambda_2) \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (33)$$

satisfies (28), defines, $d(\lambda_3)$ and the appropriate matrices on the other weights,

$$\hat{d}(2\lambda_1) \sim -\sigma_3 \otimes 1, \quad \hat{d}(2\lambda_2) \sim -1 \otimes \sigma_1$$

and everything else follows. Once we are satisfied the cocycles and symmetry factors are consistent we can construct the quantities $d(\lambda)$ according to a

canonical prescription^(9,10). Given all the correct factors, we see the '*' product defined by the operator products is symmetric, but not associative.

The operator products of the starred operators clearly close onto the unstarred operators. The products of starred and unstarred operators is more complicated. The most singular parts of these products will contain operators of conformal weight zero, (that is, central-type terms), as well as currents pertaining to the non-compact E_6 , and energy-momentum tensors. It remains to be seen how these new pieces relate to the old, and if the whole forms a sensible algebra. It certainly seems to provide an interesting generalisation of the Fateev-Zamolodchikov algebra and may, as we have said, relate to the mysterious structure of superstrings.

In conclusion, we emphasise that it is important to see if the role of ghosts can be understood within this context, and if there is a natural niche for the bosonic part of the string. There is interesting geometry associated with the exceptional Jordan algebra⁽⁸⁾ which it may well be profitable to explore. In this context it is perhaps worth mentioning that the set of complex 3×3 Jordan matrices describe a vector and a Majorana spinor of $sl(2, \mathbb{C})$ reminiscent of twistor theory⁽¹²⁾, (the analogue of the non-compact E_6 is $sl(3, \mathbb{C})$, with $su(3)$ its maximal compact subalgebra). It was these considerations which prompted ref.5, together with the fact that the standard string action may be regarded as part of a triple product (6); but not the superstring action, unfortunately, even though some of the ingredients are there. Finally, we should remark that Jordan algebras have been considered before, in the contexts of colour and supergravity⁽¹³⁾.

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References

1. P. Goddard, W. Nahm, D. Olive, H. Ruegg and A. Schwimmer, *Comm. Math. Phys.* 112 (1987) 385
2. N. Jacobsen, *Basic Algebra I*. San Francisco, CA. (Freeman 1974)
3. P. Ramond, *Phys. Rev.* D3 (1971) 2415
A. Neveu and J. H. Schwarz, *Nucl. Phys.* B31 (1971) 86
4. M. B. Green and J. H. Schwarz, *Nucl. Phys.* B181 (1981) 502
5. E. Corrigan and D. B. Fairlie, unpublished contribution to the International Summer Institute on Superstrings, Argonne National Laboratory, June 1986.
D. B. Fairlie and C. Manogue, *Phys. Rev.* D34 (1986) 1832
6. E. Corrigan and T. J. Hollowood, CERN preprint TH 4869/87, to appear in *Nucl. Phys. B*.
7. D. Friedan, E. Martinec and S. Shenker, *Nucl. Phys.* B271 (1986) 93
J. Cohn, D. Friedan, Z. Qiu and S. Shenker, *Nucl. Phys.* B278 (1986) 577
8. P. Jordan, J. von Neumann and E. Wigner, *Ann. Math.* 36 (1934) 29
H. Freudenthal, *Adv. Math.* 1 (1964) 145
9. I. Frenkel and V. G. Kac, *Inv. Math.* 62 (1980) 23
G. Segal, *Comm. Math. Phys.* 80 (1981) 301
10. P. Goddard and D. Olive, *Int. J. Mod. Phys.* A1 (1986) 303
11. V. A. Fateev and A. B. Zamolodchikov, *Sov. Phys. JETP* 62 (1985) 215

12. M.A. H. MacCallum and R. Penrose, Phys. Reports 6C (1973) 243

13. M. Gunaydin and F. Gursev, J. Math. Phys. 14 (1973) 1651

Phys. Rev. D9 (1974) 33

M. Gunaydin, G. Sierra and P. Townsend, Phys. Letts. B133 (1983) 72

Nucl. Phys. B242 (1984) 244