CERN-TH.4928/87
HUTP-87/A078

GAUGING $N = 2$ SIGMA MODELS IN HARMONIC SUPERSPACE

J.A. Bagger
Lyman Laboratory of Physics
Harvard University
Cambridge, MA 02138, U.S.A.

A.S. Galperin
Institute of Nuclear Physics, Academy of Science
Tashkent, USSR
and
Laboratory of Theoretical Physics
Joint Institute for Nuclear Research
Dubna, USSR

E.A. Ivanov^{*)}
CERN - Geneva

and

V.I. Ogievetsky
Laboratory of Theoretical Physics
Joint Institute for Nuclear Research
Dubna, USSR

ABSTRACT

We construct the general minimal coupling of $N = 2$ Yang-Mills fields to $N = 2$ sigma models in flat and curved harmonic superspace. We show that the complete symmetry group of the matter action can always be gauged in curved space. In flat space, we find that the gauging can be blocked by a local obstruction. We give a number of examples based on homogeneous quaternionic spaces and on quaternionic generalizations of the Taub-NUT and Eguchi-Hanson metrics.

^{*)} Permanent address: Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, USSR.

1. Introduction

Matter couplings in supergravity theories are most easily constructed with the help of unconstrained off-shell superfields. For $N = 1$ supergravity, such a superfield formalism has been known for many years. For $N = 2$, however, the appropriate superspace was discovered just three years ago [1]. It is now known as harmonic superspace.

In this paper we use harmonic superspace to construct the general minimal coupling of $N = 2$ Yang-Mills fields to $N = 2$ matter, in an arbitrary, curved four-dimensional background. We use the geometric language of the nonlinear sigma model, in which the scalar fields parametrize a manifold $\mathcal{M}$. The isometries of $\mathcal{M}$ correspond to rigid symmetries of the sigma model.

In flat space, the gauging of $N = 2$ sigma models has been discussed by Sierra and Townsend in terms of component fields and constrained $N = 2$ superfields [2]. It has also been studied with $N = 1$ superfields by Hull, Karlhede, Lindström and Roček [3]. Both groups find that the full isometry group cannot always be gauged.

In this paper we use unconstrained harmonic superfields to address this question in flat and curved space. Harmonic superfields allow us to write the general matter self-coupling in a simple form, where essential use is made of an unconstrained hypermultiplet compensator. This permits us to explain the obstruction to gauging the full flat-space isometry group, and show that in curved space, the symmetries can always be gauged (ignoring topological issues). In the process, we construct the complete minimal coupling of $N = 2$ Yang-Mills and matter fields in flat and curved space.

The rest of this paper is organized as follows. In sect. 2 we review the basics of harmonic superspace, following the lines of ref. [1]. In sect. 3 we gauge the $N = 2$ model in flat space. We start with the general flat-space matter coupling and show that the full isometry group cannot always be gauged. In sect. 4 we pass to curved space. We write down the complete off-shell matter coupling, find its invariances and show how to gauge the full isometry group. In sect. 5 we consider a simple example, based on the coset space $\text{Sp}(n, 1)/\text{Sp}(n) \times \text{U}(1)$. Sect. 6 contains further examples, based on the series of noncompact quaternionic spaces described in [4], and on quaternionic generalizations of the Taub-NUT and Eguchi-Hanson metrics. We conclude in sect. 7 with some comments and open questions.

2. Harmonic Superspace

As mentioned in the introduction, the appropriate arena for $N = 2$ supersymmetry is harmonic superspace. To fix notation, in this section we provide a brief summary of the main ideas underlying this approach [1].

Harmonic superspace allows all physical $N = 2$ multiplets to be described in terms of unconstrained off-shell superfields. The superspace includes four spacetime coordinates x^m as well as Grassmann coordinates θ_i^α and $\bar{\theta}^{\dot{\alpha}i} \equiv (\theta_i^\alpha)^\dagger$, where α and $\dot{\alpha}$ are spinor indices and $i = 1, 2$ refer to the $SU(2)$ automorphism group of the $N = 2$ algebra. Harmonic superspace also contains coordinates $u_i^\pm$ that parametrize the sphere $SU(2)/U(1)$. The $u_i^\pm$ are constrained by the condition $u^{i+}u_i^- = 1$, and are defined up to $U(1)$ rotations with charge ± 1 . The full theory must respect this $U(1)$ symmetry.

The $N = 2$ harmonic superspace contains an analytic subspace with only half of the Grassmann variables. The analytic subspace is defined as follows,

$$\{\zeta^M, u_i^\pm\} \equiv \{x_A^m, \theta^{\alpha+}, \bar{\theta}^{\dot{\alpha}+}, u_i^\pm\}, \quad (2.1)$$

where $x_A^m = x^m - 2i\theta^{(i}\sigma^m\bar{\theta}^{j)}u_j^+u_i^-$, $\theta^{\alpha+} = \theta^{\alpha i}u_i^+$ and $\bar{\theta}^{\dot{\alpha}+} = \bar{\theta}^{\dot{\alpha} i}u_i^+$. The charge -1 projections of the Grassmann variables are not contained in the subspace (2.1). Remarkably, this subspace is closed under $N = 2$ supersymmetry transformations. It plays a role in $N = 2$ theories analogous to that of the chiral subspace in $N = 1$. In contrast to the chiral subspace, however, the analytic space is real with respect to an appropriately defined conjugation [1]: $\bar{x}_A^m = x_A^m$, $\bar{\theta}^+ = \bar{\theta}^+$, $\bar{\theta}^+ = -\theta^+$, $\bar{u}^{i\pm} = u_i^\pm$, $\bar{u}_i^\pm = -u^{i\pm}$. From now on, we shall drop the subscript A in the analytic basis.

In harmonic superspace, $N = 2$ matter multiplets are described by unconstrained analytic superfields $Q^+(\zeta, u)$, where

$$Q^+(\zeta, u) = \phi^+(x, u) + \theta^{\alpha+}\psi_\alpha(x, u) + \bar{\theta}_\alpha^+\bar{\phi}^{\dot{\alpha}}(x, u) + \dots, \quad (2.2)$$

and

$$\phi^+(x, u) = \phi^i(x)u_i^+ + \phi^{(ijk)}(x)u_i^+u_j^+u_k^- + \dots, \quad (2.3)$$

and likewise for the other fields. The physical bosons and fermions are described by the first terms in the harmonic expansions of $\phi^+(x, u)$, $\psi_\alpha(x, u)$ and $\bar{\phi}^{\dot{\alpha}}(x, u)$. All the other fields are auxiliary.

The $N = 2$ Yang-Mills multiplets are described by Lie-algebra valued analytic superfields $V^{++}(\zeta, u)$. These superfields have $U(1)$ charge $+2$. Under gauge transformations, they shift as follows,

$$\delta V^{A++} = \mathcal{D}^{++} \epsilon^A - f^{BCA} \epsilon^B V^{C++} . \quad (2.4)$$

The ϵ^A are $U(1)$ -neutral analytic superparameters. The index A runs over the adjoint representation of the gauge group,¹ and $\mathcal{D}^{++}$ is the full harmonic derivative. On an analytic superfield in flat space, $\mathcal{D}^{++}$ reduces to D^{++} , where

$$D^{++} = u^{i+} \frac{\partial}{\partial u^{i-}} - 2i\theta^+ \sigma^m \bar{\theta}^+ \partial_m . \quad (2.5)$$

The corresponding expression for $\mathcal{D}^{++}$ in curved space contains vielbeins and connections; it will be given where needed.

As in the $N = 1$ case, $N = 2$ supersymmetry invariants are obtained as integrals over harmonic superspace. In flat space, the general form of an invariant is

$$\int d\zeta^{-4} du \mathcal{L}^{+4}(Q, u) , \quad (2.6)$$

where $d\zeta^{-4} du = d^4 x d^2 \theta^+ d^2 \bar{\theta}^+ du$ and $\mathcal{L}^{+4}$ is an arbitrary expression with $U(1)$ charge $+4$. The harmonic integral $\int du$ is defined as follows,

$$\int du = 1 , \quad \int du u_{(i_1}^+ \dots u_{i_k}^+ u_{i_{k+1}}^- \dots u_{i_{k+n}}^-) = 0 . \quad (2.7)$$

The curved space generalization of (2.6) includes a superspace density. It will be introduced in sect. 4.

3. $N = 2$ Sigma Models in Flat Space

The general on-shell coupling of $N = 2$ matter fields is described by a nonlinear sigma model where the scalar fields parametrize a hyperkähler manifold $\mathcal{M}$ [5][6]. A hyperkähler manifold is a $4n$ -dimensional real riemannian manifold whose holonomy group is contained

¹ The structure constants f^{ABC} are defined by the standard relation $[T^A, T^B] = f^{ABC} T^C$. They are manifestly antisymmetric in their first two indices. For semisimple groups, they can always be chosen to be totally antisymmetric. However, in sects. 5 and 6 we will use a basis in which the f^{ABC} are not totally antisymmetric, so the order of indices is important.

in $\text{Sp}(n)$. It has three globally-defined complex structures I^A ($A = 1, 2, 3$). The complex structures are covariantly constant and obey the clifford algebra relation

$$I^A I^B + I^B I^A = -2\delta^{AB} . \quad (3.1)$$

The $N = 2$ sigma model takes a simple form in flat space. In components it can be written as follows

$$\begin{aligned} \mathcal{L} = & -g_{\mu\nu} \partial^m \phi^\mu \partial_m \phi^\nu - \frac{1}{2} \bar{\chi}^a \gamma^m \nabla_m \chi_a \\ & + \frac{1}{4} \Omega_{abcd} (\bar{\chi}^a \gamma_m \chi^b) (\bar{\chi}^c \gamma^m \chi^d) , \end{aligned} \quad (3.2)$$

where $\mu, \nu = 1, \dots, 4n$ are coordinate indices, and $a, b, c, d = 1, \dots, 2n$ are $\text{Sp}(n)$ frame indices. In (3.2), ∇_m is the $\text{Sp}(n)$ covariant derivative and Ω_{abcd} is the hyperkähler curvature. The action is invariant under reparametrizations of $\mathcal{M}$, under $\text{Sp}(n)$ frame rotations and under $N = 2$ supersymmetry transformations.

The lagrangian (3.2) can be obtained as an integral over $N = 2$ superspace. The general superspace action is

$$S = \frac{1}{2\xi^2} \int d\zeta^{-4} du [H^{a+}(Q, u) D^{++} Q_a^+ + \mathcal{V}^{++}(Q, u)] . \quad (3.3)$$

Here H^{a+} and $\mathcal{V}^{++}$ are arbitrary functions of the matter superfields Q^{a+} and the harmonics $u_i^\pm$. The superfield Q^{a+} contains the physical scalar ϕ^{ai} as its lowest component,

$$Q^{a+} = \phi^{ai} u_i^+ + \dots , \quad (3.4)$$

where $a = 1, \dots, 2n$ and $i = 1, 2$ are $\text{Sp}(n)$ and $\text{Sp}(1)$ coordinate indices. The indices a and i are raised and lowered with the help of the symplectic metrics Ω_{ab} and ϵ_{ij} . The superfields Q^{a+} are pseudoreal with respect to conjugation, $\tilde{Q}^{a+} = Q_a^+ \equiv \Omega_{ab} Q^{b+}$, so they contain a total of $4n$ real scalar physical fields.

The action (3.3) is the most general superspace action consistent with dimensional analysis and $U(1)$ charge conservation [7]. The superspace potential $\mathcal{V}^{++}$ is called the hyperkähler potential [8][9] in analogy to the Kähler potential of $N = 1$ supersymmetry [10]. Any $\mathcal{V}^{++}$ leads to an $N = 2$ sigma model where the ϕ^{ai} parametrize a hyperkähler manifold $\mathcal{M}$.

The action (3.3) enjoys a rich set of invariances. It is, of course, invariant under $N = 2$ supersymmetry transformations. It is also invariant under reparametrizations of $\mathcal{M}$, provided

$$\begin{aligned}\delta Q^{a+} &= \lambda^{a+}(Q, u) \\ \delta H^{a+} &= -H^{b+} \frac{\partial \lambda_b^+}{\partial Q_a^+} \\ \delta \mathcal{V}^{+4} &= -H^{a+} \partial^{++} \lambda_a^+, \end{aligned} \tag{3.5}$$

where $\partial^{++} = u^{i+} \partial / \partial u^{i-}$ is a partial harmonic derivative acting on the harmonics that appear explicitly in λ_a^+ . Note that these transformations are not those of $\text{Sp}(n)$ tensors. The objects H^{a+} and $\mathcal{V}^{+4}$ have a geometric interpretation as vielbeins characterizing the derivatives with respect to the coordinates of an extended manifold which includes the harmonics $u_i^\pm$ and a triplet of central charge coordinates,² along with the usual coordinates ϕ^{ai} of the hyperkähler manifold [9].

The third invariance of (3.3) is a generalization of the $N = 1$ Kähler transformations. It is called a hyperkähler transformation because the lagrangian varies into a total harmonic derivative, $\delta(H^{a+} D^{++} Q_a^+ + \mathcal{V}^{+4}) = D^{++} \Lambda^{++} = \partial^{++} \Lambda^{++} + D^{++} Q^{a+} (\partial \Lambda^{++} / \partial Q^{a+})$. The transformation laws are as follows,

$$\begin{aligned}\delta Q^{a+} &= 0 \\ \delta H^{a+} &= \frac{\partial \Lambda^{++}}{\partial Q_a^+} \\ \delta \mathcal{V}^{+4} &= \partial^{++} \Lambda^{++}, \end{aligned} \tag{3.6}$$

where Λ^{++} is an arbitrary analytic superfield. Such a transformation leaves the action (3.3) invariant. Equation (3.6) is consistent with the vielbein interpretation of H^{a+} and $\mathcal{V}^{+4}$, provided Λ^{++} describes the shift of an extra central charge coordinate [9]. Note that neither (3.5) nor (3.6) act on the coordinates of harmonic superspace.

In general, isometries are transformations that leave the lagrangian form-invariant. For the case at hand, they are combinations of coordinate and hyperkähler transformations. It is easily checked that

$$\delta Q^{a+} = \epsilon^A \lambda^{Aa+} \tag{3.7}$$

² These central charges are not related to the central charge associated with the $N = 2$ supersymmetry algebra.

generates an isometry, provided

$$\begin{aligned}\frac{\partial \Lambda^{A++}}{\partial Q_a^+} &= \frac{\partial H^{a+}}{\partial Q_b^+} \lambda_b^{A+} - \frac{\partial H^{b+}}{\partial Q_a^+} \lambda_b^{A+} \\ \frac{\partial \mathcal{V}^{+4}}{\partial Q_a^+} \lambda_a^{A+} &= \partial^{++}(\Lambda^{A++} + H^{a+} \lambda_a^{A+}) - H^{a+} \partial^{++} \lambda_a^{A+}.\end{aligned}\tag{3.8}$$

In what follows, λ^{A+} will be called a superspace Killing vector. In analogy to the $N = 1$ case [11], Λ^{A++} will be called a superspace Killing potential.

The action (3.3) can be written in a gauge where it takes a simple form

$$S = \frac{1}{2\xi^2} \int d\zeta^{-4} du [Q^{a+} D^{++} Q_a^+ + \mathcal{V}^{+4}].\tag{3.9}$$

Here $H^{a+}(Q, u) = Q^{a+}$, so the kinetic term is purely quadratic. In this gauge, the conditions (3.8) simplify greatly,

$$\begin{aligned}\frac{\partial \Lambda^{A++}}{\partial Q^{a+}} &= -2 \lambda_a^{A+} \\ \partial^{++} \Lambda^{A++} &= -\frac{1}{2} \frac{\partial \mathcal{V}^{+4}}{\partial Q_a^+} \frac{\partial \Lambda^{A++}}{\partial Q^{a+}}.\end{aligned}\tag{3.10}$$

The expression for λ_a^{A+} shows that Λ^{A++} deserves the name superspace Killing potential.³

The transformations (3.7) and (3.10) define the isometry group H of the action (3.9). They commute with the $N = 2$ supersymmetry algebra and with its $SU(2)$ automorphism group. Note that the full isometry group of the bosonic manifold $\mathcal{M}$ can sometimes be larger than H , for it can include transformations that do not commute with supersymmetry. For example, the symmetry induced by $SU(2)$ variations of the harmonics $u_i^\pm$ is not contained in H because it acts as a spacetime symmetry in harmonic superspace. It cannot be gauged in flat space because it would entail gauging the full supersymmetry group, something that can only be done in the context of supergravity.

³ Note that the change of variables $Q^{a+} \rightarrow Q^{ai} u_i^+$, where $\partial^{++} Q^{ai} = 0$, simplifies the second equation, $\partial^{++} \Lambda^{A++} = 0 \Rightarrow \Lambda^{A++} = \Lambda^{A(ij)} u_i^+ u_j^+$ [9]. The u -independent $N = 2$ superfield $\Lambda^{A(ij)}(x^m, \theta^k, \bar{\theta}^l)$ can be identified with the superfield $X^{A(ij)}$ of Sierra and Townsend [2]. These authors found an expression for $\Lambda^{A(ij)}$ as a bilinear in the Killing vectors. In harmonic superspace, their expression becomes $\Lambda^{A++} f^{BCA} = \lambda^{Ba+} \lambda_a^{C+}$. Similar triplets of Killing potentials were introduced in ref. [3] in $N = 1$ superfield notation.

We would now like to gauge the isometries H . For such transformations, the gauge invariant superspace lagrangian can be written with the help of the Killing potential Λ^{A++} , giving

$$S = \frac{1}{2\xi^2} \int d\zeta^{-4} du [Q^{a+} D^{++} Q_a^+ + \mathcal{V}^{++} + \Lambda^{A++} V^{A++}] . \quad (3.11)$$

This action is fully supersymmetric and invariant under local symmetries H . Gauge invariance is ensured by the transformation of the Killing potential Λ^{A++} :

$$\delta \Lambda^{A++} = f^{BAC} \epsilon^B \Lambda^{C++} . \quad (3.12)$$

If the isometry group can be realized linearly on the fields ϕ^{ai} by an appropriate choice of coordinates, then⁴

$$\delta \phi^{ai} = \epsilon^A T^{Aa}{}_b \phi^{bi} , \quad (3.13)$$

where the $T^{Aa}{}_b$ are the generators of H . The superfields Q^{a+} rotate in a similar fashion,

$$\delta Q^{a+} = \epsilon^A T^{Aa}{}_b Q^{b+} , \quad (3.14)$$

and the corresponding Killing potentials take a simple form,

$$\Lambda^{A++} = Q_a^+ T^{Aa}{}_b Q^{b+} . \quad (3.15)$$

The potentials (3.15) give the following Killing vectors,

$$\lambda^{Aa+} = T^{Aa}{}_b Q^{b+} , \quad (3.16)$$

in accord with the transformation (3.14).

For $U(1)$ symmetries, the Killing potentials (3.10) are defined up to shifts [11][2][3],

$$\Lambda^{++} \rightarrow \Lambda^{++} + c^{ij} u_i^+ u_j^+ , \quad (3.17)$$

where c^{ij} is an arbitrary, symmetric constant matrix. This gives rise to an additional term in the action

$$\frac{1}{2\xi^2} \int d\zeta^{-4} du c^{++} V^{++} . \quad (3.18)$$

This is the Fayet-Iliopoulos term [12] of flat $N = 2$ supersymmetry. It breaks the rigid $Sp(1)$ symmetry down to $U(1)$, while the gauged isometries remain unbroken. Adding the Fayet-Iliopoulos term does not change the hyperkähler metric of the bosonic fields.

⁴ The isometry group can be realized linearly on all hyperkähler manifolds known to us. Whether this is true for all hyperkähler manifolds is an interesting open question.

4. $N = 2$ Sigma Models in Curved Space

A surprising fact about $N = 2$ supersymmetry is that hyperkähler models cannot be directly coupled to curved space. Instead, the hyperkähler metric must be changed by terms proportional to Newton's constant κ . This changes the holonomy group G of the manifold $\mathcal{M}$ from $G \subset \text{Sp}(n)$ to $G \subset \text{Sp}(n) \times \text{Sp}(1)$, where the $\text{Sp}(1)$ curvature is nonvanishing and proportional to κ . Such a manifold is called quaternionic [4][6][13].

The $\text{Sp}(1)$ curvature of a quaternionic manifold is very important. Locally, quaternionic manifolds have three almost complex structures that obey the clifford algebra relation (3.1). However, the complex structures are not covariantly constant, so they are not globally well-defined. As we pass from point to point, the complex structures rotate into each other under the $\text{Sp}(1)$ part of the holonomy group. The $\text{Sp}(1)$ curvature obstructs the global definition of the complex structures. We shall see that this $\text{Sp}(1)$ plays an important role in harmonic superspace.

The general supersymmetric sigma model in curved harmonic superspace takes the following form [14]

$$S = - \frac{1}{2\kappa^2} \int d\zeta^{-4} du \mathcal{E} \left\{ \mathcal{E}^{-1} q^{i+} \mathcal{D}_w^{++} q_i^+ - \frac{\kappa^2}{\xi^2} [H^{a+}(Q, w) \mathcal{D}_w^{++} Q_a^+ + \mathcal{V}^{+4}(Q, w)] \right\}, \quad (4.1)$$

where the superspace density $\mathcal{E}$ and the harmonics $w_i^\pm$ are given by

$$\begin{aligned} \mathcal{E} &= (q^{i+} u_i^-)^2 \\ w_i^- &= u_i^- \\ w_i^+ &= \left(\frac{q_i^+}{q^{j+} u_j^-} \right), \end{aligned} \quad (4.2)$$

and

$$\begin{aligned} \mathcal{D}_w^{++} &= \partial^{++} + H^{M++} \partial_M + H^{+4} \partial^{--} + w^{i+} u_i^+ \mathcal{D}^0 \\ &\equiv \mathcal{D}_u^{++} + w^{i+} u_i^+ \mathcal{D}^0. \end{aligned} \quad (4.3)$$

The index M runs over the set $(m, \alpha+, \dot{\alpha}+)$ and $\partial^{--} = u^{i-} (\partial / \partial u^{i+})$. The fields H^{M++} and H^{+4} are $N = 2$ conformal supergravity prepotentials (the $++$ components of the harmonic vielbeins) [14][15]. The superfield q_i^+ is the $N = 2$ superconformal compensator,

and the derivative $\mathcal{D}^0 = u^{i+} \partial / \partial u^{i+} - u^{i-} \partial / \partial u^{i-} + \theta^{\alpha+} \partial / \partial \theta^{\alpha+} + \theta^{\dot{\alpha}+} \partial / \partial \theta^{\dot{\alpha}+}$ is the U(1) charge operator. Note that the compensator is an unphysical field since it enters the action with the wrong sign for its kinetic term.

In the above expressions, the potentials H^{a+} and $\mathcal{V}^{+4}$ are arbitrary functions of the superfields Q^{a+} and of the harmonics $w_i^\pm$. The prepotentials H^{M++} and H^{+4} , the matter superfields Q^{a+} and the compensator q_i^+ are all defined on the analytic superspace $(\zeta^M, u_i^\pm)$. When rewritten in components, with the auxiliary fields eliminated, the action (4.1) (together with that of the $N = 2$ Maxwell-Weyl compensator H^{5++} [14][15]) reproduces the general component action given in [13].

In the $N = 2$ superspace, the supergravity transformations of the coordinates, vielbeins and the harmonic derivatives are as follows:

$$\begin{aligned}
\delta x^m &= \lambda^m(\zeta, u) \\
\delta \theta^{\hat{\alpha}+} &= \lambda^{\hat{\alpha}+}(\zeta, u) \\
\delta u_i^+ &= \lambda^{++}(\zeta, u) u_i^- \\
\delta u_i^- &= 0 \\
\delta H^{m++} &= \mathcal{D}^{++} \lambda^m \\
\delta H^{\hat{\alpha}+++} &= \mathcal{D}^{++} \lambda^{\hat{\alpha}+} - \theta^{\hat{\alpha}+} \lambda^{++} \\
\delta H^{+4} &= \mathcal{D}^{++} \lambda^{++} \\
\delta \mathcal{D}_u^{++} &= - \lambda^{++} \mathcal{D}^0 \\
\delta \mathcal{D}^0 &= 0 ,
\end{aligned} \tag{4.4}$$

where $\hat{\alpha}$ denotes α or $\dot{\alpha}$. Note the local SU(2) symmetry that acts on the coordinate $u_i^\pm$. This transformation will play a crucial role in gauging the sigma model isometries.

The supergravity transformation of the compensator q_i^+ is fixed by invariance of the measure $d\zeta^{-4} du \mathcal{E}$:

$$\delta q_i^+ = - \frac{1}{2} (\partial_m \lambda^m + \partial^{--} \lambda^{++} - \partial_{\hat{\alpha}}^- \lambda^{\hat{\alpha}+}) q_i^+ . \tag{4.5}$$

This in turn implies

$$\delta \left(\frac{q^{i+} u_i^+}{q^{j+} u_j^+} \right) = \delta(w^{i+} u_i^+) = \lambda^{++} , \tag{4.6}$$

so

$$\begin{aligned}
\delta w_i^+ &= 0 \\
\delta \mathcal{D}_w^{++} &= 0 .
\end{aligned} \tag{4.7}$$

The harmonics $w_i^\pm$ are inert under $N = 2$ supergravity transformations, so they can appear explicitly in the action (4.1).

Because of the compensator q_i^+ , the action (4.1) is invariant under the full $N = 2$ superconformal group. By a general superconformal transformation, it is always possible to gauge $q_i^+ \rightarrow u_i^+$, where $w_i^\pm = u_i^\pm$. This breaks the $N = 2$ superconformal symmetry down to the $N = 2$ Einstein supergravity group. The reduction to flat space is obtained by setting $q_i^+ = u_i^+$ and taking $\kappa \rightarrow 0$.

The superfields Q^{a+} , H^{a+} and $\mathcal{V}^{+4}$ can be chosen to have weight zero under supergravity transformations:

$$\begin{aligned}\delta Q^{a+} &= 0 \\ \delta H^{a+} &= 0 \\ \delta \mathcal{V}^{+4} &= 0.\end{aligned}\tag{4.8}$$

We stress that this is only possible because the compensator q_i^+ is analytic and unconstrained, and gives rise to an analytic density $\mathcal{E}$. Other versions of $N = 2$ supergravity [16][17] use constrained compensators, and consequently do not have analytic densities. In the constrained theories, matter superfields have nonzero weights, so the matter couplings are severely restricted [18].

As in flat space, it is possible to define general coordinate transformations of the superfields Q^{a+} . Now, however, the compensator transforms as well,

$$\begin{aligned}\delta Q^{a+} &= \lambda^{a+}(Q, w) \\ \delta H^{a+} &= -H^{b+} \frac{\partial \lambda_b^+}{\partial Q_a^+} - \frac{\kappa^2}{\xi^2} (H^{b+} \partial_w^{--} \lambda_b^+) H^{a+} \\ \delta \mathcal{V}^{+4} &= -H^{a+} \partial_w^{++} \lambda_a^+ - \frac{\kappa^2}{\xi^2} (H^{b+} \partial_w^{--} \lambda_b^+) \mathcal{V}^{+4} \\ \delta q_i^+ &= \frac{\kappa^2}{2\xi^2} (H^{a+} \partial_w^{--} \lambda_a^+) q_i^+ \\ \delta u_i^\pm &= \delta w_i^\pm = 0 \\ \delta \mathcal{D}_w^{++} &= \delta \mathcal{D}_u^{++} = 0 \\ \delta \mathcal{D}^0 &= 0.\end{aligned}\tag{4.9}$$

Note that the flat-space limit of these expressions reproduces (3.5).

Similarly, it is also possible to define curved-space generalizations of the hyperkähler transformations (3.6). We call these quaternionic transformations in recognition of the

fact that $\mathcal{M}$ is now a quaternionic manifold. The crucial difference from the flat case is that the matter action is now *invariant* because of the transformations of the compensator and the harmonics,

$$\begin{aligned}
\delta Q^{a+} &= 0 \\
\delta H^{a+} &= \frac{\partial \Lambda^{++}}{\partial Q_a^+} + \frac{\kappa^2}{\xi^2} (\partial_w^- \Lambda^{++}) H^{a+} \\
\delta \mathcal{V}^{+4} &= \partial_w^{++} \Lambda^{++} + \frac{\kappa^2}{\xi^2} (H^{a+} Q_a^+) \Lambda^{++} + \frac{\kappa^2}{\xi^2} (\partial_w^- \Lambda^{++}) \mathcal{V}^{+4} \\
\delta q_i^+ &= \frac{\kappa^2}{\xi^2} \left[(q^{j+} w_j^-) \Lambda^{++} w_i^- - \frac{1}{2} (\partial_w^- \Lambda^{++}) q_i^+ \right] \\
\delta w_i^+ &= \frac{\kappa^2}{\xi^2} \Lambda^{++} w_i^- \\
\delta w_i^- &= 0 \\
\delta u_i^+ &= \delta u_i^- = 0 \\
\delta \mathcal{D}_w^{++} &= - \frac{\kappa^2}{\xi^2} \Lambda^{++} \mathcal{D}^0 \\
\delta \mathcal{D}_u^{++} &= 0 \\
\delta \mathcal{D}^0 &= 0 .
\end{aligned} \tag{4.10}$$

Note that there is a remarkable similarity between the $N = 2$ conformal supergravity transformations (4.4) and the quaternionic transformations (4.10). It is not hard to establish the correspondence $u_i^\pm \leftrightarrow w_i^\pm$, $\mathcal{D}_u^{++} \leftrightarrow \mathcal{D}_w^{++}$, and $\lambda^{++} \leftrightarrow (\kappa^2/\xi^2) \Lambda^{++}$. Both sets of transformations contain $SU(2)$ subgroups, with parameters $\lambda^{ij} u_i^+ u_j^+$ and $\Lambda^{ij} w_i^+ w_j^+$, respectively. The former are contained in the $N = 2$ superconformal group, while the latter are directly related to the $Sp(1)$ part of the isometry group. The harmonics $u_i^\pm$ are invariant under the quaternionic transformations, but transform under the supergravity group. For the $w_i^\pm$ the situation is reversed. The only object that transforms under both groups is the compensator q_i^+ . The gauge $q_i^+ = u_i^+$ is preserved by simultaneous rotations of the two $SU(2)$'s.

The harmonics $w_i^\pm$ rotate under quaternionic transformations. They play an important role in describing the quaternionic geometry of the matter fields. In curved space it is natural to define the physical components of Q^{a+} by decomposing the superfield with respect to the harmonics $w_i^\pm$,

$$Q^{a+} = \phi^{ai} w_i^+ + \dots \tag{4.11}$$

As we will see later, the scalar fields ϕ^{ai} have the correct properties to be the coordinates of a quaternionic manifold. They are related to the fields defined in terms of $u_i^\pm$ by an field transformation involving the compensator.

Using the gauge freedom (4.9) and (4.10), it is possible to go to a gauge where the kinetic action is bilinear in Q^{a+} . In this gauge, the action takes the following form:

$$S = -\frac{1}{2\kappa^2} \int d\zeta^{-4} du \mathcal{E} \left\{ \mathcal{E}^{-1} q^{i+} \mathcal{D}_w^{++} q_i^+ - \frac{\kappa^2}{\xi^2} [Q^{a+} \mathcal{D}_w^{++} Q_a^+ + \nu^{+4}] \right\}. \quad (4.12)$$

This simplified action is left form-invariant under a generalized set of isometries. The isometry transformations are as follows,

$$\begin{aligned} \delta Q^{a+} &= \epsilon^A \lambda^{Aa+}(Q, w) \\ \delta q^{i+} &= \frac{\kappa^2}{\xi^2} \left[\left(1 - \frac{1}{2} Q^{a+} \frac{\partial}{\partial Q^{a+}} \right) \Lambda^{A++} (q^{j+} w_j^-) w_i^- - \frac{1}{2} \partial_w^{--} \Lambda^{A++} q_i^+ \right] \epsilon^A \\ \delta w_i^+ &= \frac{\kappa^2}{\xi^2} \left(1 - \frac{1}{2} Q^{a+} \frac{\partial}{\partial Q^{a+}} \right) \Lambda^{A++} \epsilon^A w_i^-, \end{aligned} \quad (4.13)$$

where

$$\lambda^{Aa+} = \frac{1}{2} \left[\frac{\partial \Lambda^{A++}}{\partial Q_a^+} + \frac{\kappa^2}{\xi^2} Q^{a+} \partial_w^{--} \Lambda^{A++} \right], \quad (4.14)$$

and

$$\begin{aligned} \partial_w^{++} \Lambda^{A++} + \frac{1}{2} \left[\frac{\partial}{\partial Q_a^+} + \frac{\kappa^2}{\xi^2} Q^{a+} \partial_w^{--} \right] \nu^{+4} \frac{\partial}{\partial Q^{a+}} \Lambda^{A++} \\ + \frac{\kappa^2}{\xi^2} \left(1 - \frac{1}{2} Q^{a+} \frac{\partial}{\partial Q^{a+}} \right) \nu^{+4} \partial_w^{--} \Lambda^{A++} \\ - \frac{\kappa^2}{\xi^2} \partial_w^{--} \nu^{+4} \Lambda^{A++} = 0. \end{aligned} \quad (4.15)$$

As before, λ^{Aa+} is a superfield Killing vector and Λ^{A++} is a superfield Killing potential. Note that (4.14) and (4.15) reduce to (3.10) in the limit of flat space.⁵

⁵ The flat-space formula of [2] expressing the Killing potential as a bilinear in Killing vectors is modified in curved space. The new result is $\Lambda^{A++} f^{BCA} = 2\lambda^{Ba+} \lambda_a^{C+} - \frac{\kappa^2}{\xi^2} (\Lambda^{B++} \partial_w^{--} \Lambda^{C++} - \Lambda^{C++} \partial_w^{--} \Lambda^{B++})$.

It is interesting to note that in curved space it is no longer possible to shift Λ^{++} leaving λ^{a+} invariant. This is easily seen from equations (4.14) and (4.15). Therefore it is not in general possible to include a Fayet-Iliopoulos term in curved space without modifying the lagrangian and/or the Killing vector.

The action (4.12) is invariant under the isometries (4.13). It is simple to gauge these isometries. One just adds the term $(1/2\xi^2)\Lambda^{A++}V^{A++}$ to the matter lagrangian. It is not hard to show that the variation of this term precisely cancels the $D_w^{++}\epsilon^A$ terms in the variation of (4.12). The new multiplicative terms are cancelled among themselves because of the transformation law of Λ^{A++} :

$$\delta\Lambda^{A++} = f^{BAC}\epsilon^B\Lambda^{C++} + \frac{\kappa^2}{\xi^2}\Lambda^{A++}(\partial_w^{--}\Lambda^{B++}\epsilon^B). \quad (4.16)$$

Thus the full gauge invariant action of superspace sigma model is given by

$$S = -\frac{1}{2\kappa^2} \int d\zeta^{-4} du \mathcal{E} \left\{ \mathcal{E}^{-1} q^{i+} \mathcal{D}_w^{++} q_i^+ \right. \\ \left. - \frac{\kappa^2}{\xi^2} [Q^{a+} \mathcal{D}_w^{++} Q_a^+ + \mathcal{V}^{++} + \Lambda^{A++} V^{A++}] \right\}. \quad (4.17)$$

This action is invariant under all local isometries, both linear and nonlinear.

Before turning to the examples, let us briefly discuss how to introduce potential terms for scalar fields in the present framework. Such terms always appear in the component form of (4.17) whenever Λ^{A++} depends on Q^{a+} . They can also be induced via the Scherk-Schwarz mechanism [19], in which the generator of some one-parameter isometry of the matter action is identified with the central charge of the $N = 2$ supersymmetry algebra. In harmonic superspace, there is a natural setting for the central charge because the Maxwell-Weyl compensator H^{5++} serves as a superspace vielbein gauging central charge transformations. The matter superfields Q^{a+} inherit a central charge after the central charge transformations are identified with the one-parameter isometry group. The original matter lagrangian must then be modified by adding the term $m(\kappa^2/2\xi^2)H^{5++}\Lambda^{++}$, where $\dim[m] = (\text{mass})$. The new term gives rise to a potential for the physical scalars ϕ^{ai} . The compensator H^{5++} plays a role similar to that of a $U(1)$ gauge superfield V^{++} , with a nonzero vacuum expectation value for the physical scalar fields.

5. The Simplest Case: $\text{Sp}(n,1)/\text{Sp}(n) \times \text{Sp}(1)$

To illustrate the above construction, let us now consider the simplest matter coupling, with $2n$ matter superfields Q^{a+} , and no hyperkähler potential, $\mathcal{V}^{++} = 0$. The complete superspace action is given by

$$S = - \frac{1}{2\kappa^2} \int d\zeta^{-4} du \mathcal{E} \left\{ \mathcal{E}^{-1} q^{i+} \mathcal{D}_w^{++} q_i^+ - \frac{\kappa^2}{\xi^2} Q^{a+} \mathcal{D}_w^{++} Q_a^+ \right\}. \quad (5.1)$$

The action (5.1) has a manifest $\text{Sp}(n)$ symmetry,

$$\begin{aligned} \delta Q^{a+} &= \lambda^{ab} Q_b^+ \\ \delta q_i^+ &= 0 \\ \delta \mathcal{E} &= 0, \end{aligned} \quad (5.2)$$

where $\lambda^{ab} = \lambda^{ba}$ is a symmetric $\text{Sp}(n)$ generator. This symmetry does not affect the superspace coordinates or the harmonics $w_i^\pm$. It acts as a simple rotation on the component fields ϕ^{ai} ,

$$\delta \phi^{ai} = \lambda^{ab} \phi_b^i. \quad (5.3)$$

The corresponding superspace Killing potential is just $\Lambda^{ab++} = Q^{a+} Q^{b+}$. Thus the $\text{Sp}(n)$ symmetry can be gauged in flat or curved space by adding

$$\frac{1}{2\xi^2} \int d\zeta^{-4} du \mathcal{E} \Lambda^{ab++} V_{ab}^{++} \quad (5.4)$$

to the action (5.1).

The above action is also invariant under an $\text{Sp}(1)$ symmetry

$$\begin{aligned} \delta Q^{a+} &= \frac{\kappa^2}{\xi^2} \beta^{ij} (w_i^+ w_j^-) Q^{a+} \\ \delta q_i^+ &= \frac{\kappa^2}{\xi^2} \beta_{ij} q^{j+} \\ \delta w_i^+ &= \frac{\kappa^2}{\xi^2} \beta^{jk} (w_j^+ w_k^+) w_i^- \\ \delta w_i^- &= 0 \\ \delta \mathcal{E} &= -2 \frac{\kappa^2}{\xi^2} \beta^{ij} (w_i^+ w_j^-) \mathcal{E}. \end{aligned} \quad (5.5)$$

These transformations do not affect the analytic superspace coordinates, so they manifestly preserve $N = 2$ analyticity. The transformation law for the component fields can be found from (5.5) and the expansion

$$Q^{a+}|_{\theta=\bar{\theta}=0} = \phi^{ai} w_i^+ + \phi^{a(ijk)} w_i^+ w_j^+ w_k^- + \dots \quad (5.6)$$

The physical fields ϕ^{ai} transform linearly off-shell, although in a somewhat unusual way,

$$\delta\phi^{ai} = -\frac{\kappa^2}{\xi^2} \beta^{ij} \phi_j^a - \frac{\kappa^2}{2\xi^2} \beta^{jk} \phi_{(jk}^a{}^{i)} . \quad (5.7)$$

The equations of motion imply $\phi^{a(ijk)} = 0$, so (5.7) reduces to a standard $\text{Sp}(1)$ rotation after the auxiliary fields are eliminated,

$$\delta\phi^{ai} = -\frac{\kappa^2}{\xi^2} \beta^{ij} \phi_j^a . \quad (5.8)$$

Equation (5.5) shows that the $\text{Sp}(1)$ transformations act on the superconformal compensator q_i^+ . In the gauge $q_i^+ = u_i^+$, the isometry (5.5) induces an $\text{SU}(2)$ transformation of the harmonic coordinate u_i^+ ,

$$\delta u_i^+ = \frac{\kappa^2}{\xi^2} \beta^{jk} (u_j^+ u_k^+) u_i^- . \quad (5.9)$$

This is the reason why the $\text{SU}(2)$ symmetry cannot be gauged in flat space. If $\beta_{ij} \sim \kappa^{-2}$, the $\text{Sp}(1)$ transformations induce a rotation of the harmonic coordinate u_i^+ . If β_{ij} is independent of κ , the variations (5.5) and (5.9) vanish.

The Killing potentials for the $\text{Sp}(1)$ transformations are not hard to find. They are given in terms of the harmonics w_i^+ , $\Lambda^{ij++} = w^{i+} w^{j+}$. Therefore the linearly realized $\text{Sp}(n) \times \text{Sp}(1)$ symmetries are readily gauged in curved space. One simply adds

$$\frac{1}{2\xi^2} \int d\zeta^{-4} du \mathcal{E} (\Lambda^{ab++} V_{ab}^{++} + \Lambda^{ij++} V_{ij}^{++}) \quad (5.10)$$

to the action (5.1).

Actually, the action (5.1) has a larger isometry group, $\text{Sp}(n,1)$. This can be seen by the field redefinition

$$\begin{aligned} \hat{Q}^{a+} &= \frac{\kappa}{\xi} (q^{i+} w_i^-) Q^{a+} \\ \hat{q}_i^+ &= q_i^+ . \end{aligned} \quad (5.11)$$

In terms of the variables $\hat{Q}^{a+}$ and $\hat{q}_i^+$, the original action can be written as follows,

$$S = - \frac{1}{2\kappa^2} \int d\zeta^{-4} du \left\{ \hat{q}^{i+} \mathcal{D}_w^{++} \hat{q}_i^+ - \hat{Q}^{a+} \mathcal{D}_w^{++} \hat{Q}_a^+ \right\}. \quad (5.12)$$

The lagrangian now has a manifest $\text{Sp}(n,1)$ symmetry.⁶ The coset generators $\text{Sp}(n,1)/\text{Sp}(n) \times \text{Sp}(1)$ rotate $\hat{Q}^{a+}$ into $\hat{q}_i^+$ and vice versa:

$$\begin{aligned} \delta \hat{q}^{i+} &= - \lambda^{ai} \hat{Q}_a^+ \\ \delta \hat{Q}^{a+} &= + \lambda^{ai} \hat{q}_i^+. \end{aligned} \quad (5.13)$$

Of course, this symmetry is easily gauged.

The coset transformations also act on the original variables Q^{a+} and q_i^+ :

$$\begin{aligned} \delta Q^{a+} &= \lambda^{ai} w_i^+ + \frac{\kappa^2}{\xi^2} \lambda^{bi} (Q_b^+ w_i^-) Q^{a+} \\ \delta q^{i+} &= - \frac{\kappa^2}{\xi^2} \lambda^{ai} Q_a^+ (w^{i-} q_i^+) \\ \delta w_i^+ &= \frac{\kappa^2}{\xi^2} (\lambda^{aj} Q_a^+ w_j^+) w_i^- \\ \delta \mathcal{E} &= - 2 \frac{\kappa^2}{\xi^2} \lambda^{ai} w_i^- Q_a^+ \mathcal{E}. \end{aligned} \quad (5.14)$$

These are *nonlinear* isometries of the action (5.1). The coset transformations can be derived from the following Killing potential,

$$\Lambda_{ai}^{++} = 2 w_i^+ Q_a^+. \quad (5.15)$$

In the flat space limit, $\kappa \rightarrow 0$, the transformations (5.14) contract into pure translations of the superfields Q^{a+} ,

$$\delta Q^{a+} = \lambda^{ai} u_i^+. \quad (5.16)$$

These are the standard abelian isometries of the free hypermultiplet action in the flat case [8].

Thus we see that in curved space the quadratic action has isometry group $\text{Sp}(n,1)$. After elimination of the auxiliary fields, these transformations are realized on scalar fields

⁶ The symmetry would be $\text{Sp}(n+1)$ if the sign were changed for the kinetic term of the matter multiplets. This would give a theory where the matter fields are all ghosts.

ϕ^{ai} in the standard nonlinear fashion. The scalar fields ϕ^{ai} are the coordinates of the quaternionic space $\text{Sp}(n,1)/\text{Sp}(n) \times \text{Sp}(1)$.

In our construction, essential use was made of the compensator q_i^+ . The compensator plays a role analogous to the σ -field of the Gell-Mann–Levy sigma model. It allows one to linearize the full isometry group. The supergravity gauge choice $q_i^+ = u_i^+$ is analogous to the condition $\sigma = 1$. In this gauge, after elimination of the auxiliary fields, the $\text{Sp}(n,1)$ transformations are realized nonlinearly on the fields ϕ^{ai} . Note that in the flat space limit, $\kappa \rightarrow 0$, the gauged isometry group reduces to the direct product of the local $\text{Sp}(n)$ and a group with the following Lie algebra,

$$[T_i^a, T_j^b] = i\Omega^{ab} Z_{ij} . \quad (5.17)$$

In this expression, the T_i^a generate the remnants of the coset transformations (5.14), and the Z_{ij} form a triplet under the automorphism $\text{SU}(2)$. The Z_{ij} commute with all the other generators. The transformations they induce are found by taking the $\kappa \rightarrow 0$ limit of the above $\text{Sp}(1)$ transformations. It is not hard to show that the Z_{ij} do not act on the Q^{a+} , but do transform the V_{ij}^{++} via the superfields V_i^{a++} . Their role in hyperkähler geometry is discussed in [9].

6. Further Examples

In the previous section, we showed how to gauge the isometry group of the simplest quaternionic space, $\text{Sp}(n,1)/\text{Sp}(n) \times \text{Sp}(1)$. In this section we will consider a few more complicated examples. We will find the lagrangians and Killing potentials, and show that in curved space it is always possible to gauge the complete isometry group.

We start by considering the following homogeneous quaternionic spaces [4],

$$\frac{\text{SU}(n,2)}{\text{U}(n) \times \text{Sp}(1)} \quad \frac{\text{SO}(n,4)}{\text{SO}(n) \times \text{SU}(2) \times \text{Sp}(1)} \quad \frac{\tilde{\text{G}}_2}{\text{SU}(2) \times \text{Sp}(1)} , \quad (6.1)$$

where $\tilde{\text{G}}_2$ denotes a noncompact version of G_2 . To construct the corresponding sigma models, we must first specify how the matter fields transform with respect to the linear subgroups of the full isometry groups. This is easy to do using the branching rules for group decompositions. The representation labels follow immediately from the fact that the lowest components of Q^{a+} parametrize the coset spaces (6.1).

To find the superfield actions, we need the quaternionic potentials $\mathcal{V}^{+4}$. The potentials are easy to find because the above spaces contain linearly realized $\text{Sp}(1)$ symmetries. This tells us that the potentials $\mathcal{V}^{+4}$ do not depend explicitly on the harmonics $w_i^\pm$. Charge conservation then implies that the quaternionic potentials must be quartic polynomials of the Q^{a+} .

The Killing potentials are solutions to equation (4.15). For the case at hand, this simplifies to

$$\partial^{++}\Lambda^{++} - \frac{1}{2} \frac{\partial \mathcal{V}^{+4}}{\partial Q^{+a}} \frac{\partial \Lambda^{++}}{\partial Q_a^+} - \frac{\kappa^2}{\xi^2} \mathcal{V}^{+4} \partial^{--}\Lambda^{++} = 0. \quad (6.2)$$

Equation (6.2) is not hard to solve. The representation labels, potentials $\mathcal{V}^{+4}$ and Killing potentials for the spaces (6.1) are listed in Table 1. The Killing potentials allow us to gauge the full isometry group in the manner of sect. 4.

The remaining noncompact homogeneous quaternionic spaces

$$\begin{array}{cccc} \frac{\tilde{F}_4}{\text{Sp}(3) \times \text{Sp}(1)} & \frac{\tilde{E}_6}{\text{SU}(6) \times \text{Sp}(1)} & \frac{\tilde{E}_7}{\text{SO}(12) \times \text{Sp}(1)} & \frac{\tilde{E}_8}{\text{E}_7 \times \text{Sp}(1)} \end{array} \quad (6.3)$$

can be treated in a similar fashion. The $\text{Sp}(1)$ symmetry ensures that the potentials $\mathcal{V}^{+4}$ are quartic polynomials of the Q^{a+} , so the Killing potentials obey (6.2). The Killing potentials exist, so there is no local obstruction to gauging the isometry groups of these manifolds.

Our final two examples are based on quaternionic extensions of the Taub-NUT and Eguchi-Hanson hyperkähler spaces. The sigma model actions in flat harmonic superspace were given in [7][20]. The Taub-NUT model in a curved background is described by the action

$$\begin{aligned} S_{TN} = & - \frac{1}{2\kappa^2} \int d\zeta^{-4} du \mathcal{E} \left\{ \mathcal{E}^{-1} q^{i+} \mathcal{D}_w^{++} q_i^+ \right. \\ & \left. - \frac{\kappa^2}{\xi^2} [\bar{Q}^+ \mathcal{D}^{++} Q^+ - Q^+ \mathcal{D}^{++} \bar{Q}^+ + g(\bar{Q}^+ Q^+)^2] \right\}. \end{aligned} \quad (6.4)$$

It is invariant under the rigid $\text{Sp}(1) \times \text{U}(1)$ transformations generated by the following potentials

$$\begin{aligned} \Lambda^{++} &= \bar{Q}^+ Q^+ \\ \Lambda_{ij}^{++} &= w_i^+ w_j^+ + g \frac{\kappa^2}{\xi^2} (\bar{Q}^+ Q^+)^2 w_i^- w_j^- . \end{aligned} \quad (6.5)$$

As above, these transformations can be gauged in curved space.

The story for the Eguchi-Hanson model is a little more complicated. We start by considering the following action,

$$S = -\frac{1}{2\kappa^2} \int d\zeta^{-4} du \mathcal{E} \left\{ \mathcal{E}^{-1} q^{i+} \mathcal{D}_w^{++} q_i^+ - \frac{\kappa^2}{\xi^2} [Q^{Ia+} \mathcal{D}^{++} Q_a^{I+}] \right\}. \quad (6.6)$$

Here Q^{Ia+} is an $SU(2)$ spinor ($a = 1, 2$) and an $O(2)$ vector ($I = 1, 2$). The physical scalars ϕ^{ai} parametrize an eight-dimensional manifold. The action is invariant under the following rigid $SU(2) \times U(1) \times U(1)$ transformations

$$\begin{aligned} \Lambda_{ab}^{++} &= Q_a^{I+} Q_b^{I+} \\ \Lambda^{++} &= Q^{Ia+} Q_a^{J+} \epsilon_{IJ} + \alpha c^{ij} w_i^+ w_j^+ \\ \Lambda'^{++} &= c^{ij} w_i^+ w_j^+. \end{aligned} \quad (6.7)$$

Here α is a constant, and c^{ij} is a constant matrix that breaks $Sp(1)$ down to its $U(1)$ subgroup. Note that the groups generated by (6.7) are mutually commuting.

The Eguchi-Hanson model is constructed by gauging the first $U(1)$ with an auxiliary gauge superfield A^{++} . The action is as follows,

$$S = -\frac{1}{2\kappa^2} \int d\zeta^{-4} du \mathcal{E} \left\{ \mathcal{E}^{-1} q^{i+} \mathcal{D}_w^{++} q_i^+ - \frac{\kappa^2}{\xi^2} [Q^{Ia+} \mathcal{D}^{++} Q_a^{I+} + A^{++} (Q^{Ia+} Q_a^{J+} \epsilon_{IJ} + \alpha c^{++})] \right\}. \quad (6.8)$$

The superfield A^{++} acts as a Lagrange multiplier that reduces the dimension of component sigma model from eight to four. The remaining $SU(2) \times U(1)$ symmetry can be gauged by introducing dynamical vector fields, as shown in sect. 4.

7. Conclusion

In this paper we have constructed the most general minimal coupling of supersymmetric $N = 2$ matter to $N = 2$ Yang-Mills, in an arbitrary, curved four-dimensional background. We found that in curved space, continuous rigid symmetries can always be gauged. Essential use was made of the compensator q_i^+ . In flat space, we found that the full isometry group cannot always be gauged. This follows from the fact that the superconformal gauge $q_i^+ = u_i^+$ ties the $Sp(1)$ of the isometry group to the $SU(2)$ automorphism

of the $N = 2$ supersymmetry algebra. The gauge choice breaks $\text{Sp}(1) \times \text{SU}(2)$ down to the diagonal $\text{SU}(2)$ subalgebra. Gauging the $\text{Sp}(1)$ of the isometry group implies gauging supersymmetry, something that can be done only in the context of curved space.

In this paper, we have implicitly assumed that the vector superfields V^{++} describe physical, dynamical gauge bosons. For the noncompact isometry groups of the curved-space models, the minimal gauge-invariant kinetic terms contain ghosts. Nonminimal kinetic energies are needed, and such terms give rise to additional interactions between the matter and Yang-Mills fields. It would be of interest to construct the ghost-free kinetic terms for the gauge superfields considered here.

Other interesting problems include realizing the coset space models as constrained linear systems. In sect. 5 we showed how to represent the quaternionic space $\text{Sp}(n,1)/\text{Sp}(n) \times \text{Sp}(1)$ linearly using the compensator q_i^+ . One would like to know how this generalizes for the coset models of sect. 6. Also, the entire construction presented here was local from the point of view of the manifold $\mathcal{M}$. It would be interesting to know if there are any global topological obstructions to gauging the isometry groups of hyperkähler and quaternionic spaces.

The authors are grateful to O. Ogievetsky for stimulating discussions. J.B. would like to thank Prof. V.G. Kadyshevsky and the members of the Laboratory of Theoretical Physics at Dubna for their kind hospitality while this work was performed. E.I. would like to thank B. de Wit for discussions on quaternionic manifolds in harmonic superspace, and C.A. Lütken for his help with the computer, and the CERN Theory Division for its kind hospitality. This work was supported in part by the Alfred P. Sloan Foundation and by the U.S. National Science Foundation, grants PHY82-15249 and PHY86-57291.

References

- [1] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, *Class. Quant. Grav.* 1 (1984) 469.
- [2] G. Sierra and P. Townsend, *Nucl. Phys.* B233 (1984) 289.
- [3] C. Hull, A. Karlhede, U. Lindström and M. Roček, *Nucl. Phys.* B266 (1986) 1.
- [4] S. Salamon, *Invent. Math.* 67 (1982) 143;
J. Wolf, *J. Math. Mech.* 14 (1965) 1033.
- [5] L. Alvarez-Gaumé and D. Freedman, *Comm. Math. Phys.* 80 (1981) 443.
- [6] See, for example, J. Bagger, in *Supersymmetry*, eds. K. Dietz, R. Flume, G. Gehlen and V. Rittenberg (Plenum, New York) 1985.
- [7] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, *Comm. Math. Phys.* 103 (1986) 515.
- [8] A. Galperin, E. Ivanov and V. Ogievetsky, *Nucl. Phys.* B282 (1987) 74.
- [9] A. Galperin, E. Ivanov, V. Ogievetsky, and E. Sokatchev, Dubna preprints E2-87-263, E2-87-264 (1987).
- [10] B. Zumino, *Phys. Lett.* 87B (1979) 203.
- [11] J. Bagger and E. Witten, *Phys. Lett* 118B (1982) 103;
J. Bagger, *Nucl. Phys.* B211 (1983) 302.
- [12] P. Fayet and J. Iliopoulos, *Phys. Lett.* 51B (1974) 461.
- [13] J. Bagger and E. Witten, *Nucl. Phys.* B222 (1983) 1.
- [14] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, *Class. Quant. Grav.* 4 (1987) 1255.
- [15] A. Galperin, N. Ky and E. Sokatchev, *Class. Quant. Grav.* 4 (1987) 1235.
- [16] E. Fradkin and M. Vasiliev, *Phys. Lett.* 85B (1979) 47;
B. de Wit, J. van Holten and A. van Proeyen, *Nucl. Phys.* B167 (1980) 186; B184 (1981) 77.
- [17] M. Muller, *Nucl. Phys.* B282 (1987) 329; B289 (1987) 557.
- [18] B. de Wit, P. Lauwers and A. van Proeyen, *Nucl. Phys.* B255 (1985) 569;
K. Galicki, *Nucl. Phys.* B271 (1986) 402; B289 (1987) 573.
- [19] J. Scherk and J. Schwarz, *Nucl. Phys.* B153 (1979) 61.
- [20] A. Galperin, E. Ivanov, V. Ogievetsky and P. Townsend, *Class. Quant. Grav.* 3 (1986) 625.

Table 1

Coset space	Representation Labels	Potential $\mathcal{V}^{++}$	Killing Potential Λ^{++}
$\frac{\text{SU}(n,2)}{\text{U}(n) \times \text{Sp}(1)}$	$Q_a^+ , \bar{Q}^{a+}$ $a = 1 \dots n$ SU(n) spinor	$-\frac{\kappa^2}{\xi^2} (\bar{Q}^{a+} Q_a^+)^2$	$\Lambda_a^{b++} = Q_a^+ \bar{Q}^{b+}$ $\Lambda_{ij}^{++} = w_i^+ w_j^+ + \frac{\kappa^2}{\xi^2} \mathcal{V}^{++} w_i^- w_j^-$ $\Lambda_i^{a++} = \bar{Q}^{a+} (w_i^+ + \frac{\kappa^2}{\xi^2} w_i^- \bar{Q}^{b+} Q_b^+)$
$\frac{\text{SO}(n,4)}{\text{SO}(n) \times \text{SU}(2) \times \text{Sp}(1)}$	Q^{ak+} $a = 1, \dots, n, \quad k = 1, 2$ SO(n) vector SU(2) spinor	$\frac{\kappa^2}{2\xi^2} (Q^{ak+} Q_k^{b+})^2$	$\Lambda^{ab++} = Q^{ak+} Q_k^{b+}$ $\Lambda_{kl}^{++} = Q_k^{a+} Q_{al}^+$ $\Lambda_{ij}^{++} = w_i^+ w_j^+ + \frac{\kappa^2}{\xi^2} \mathcal{V}^{++} w_i^- w_j^-$ $\Lambda_{ik}^{a++} = w_i^+ Q_k^{a+} - \frac{\kappa^2}{\xi^2} w_i^- Q_l^{a+} Q_k^{b+} Q^{bl+}$
$\frac{\tilde{\text{G}}_2}{\text{SU}(2) \times \text{Sp}(1)}$	$Q^{(abc)+}$ $a, b, c = 1, 2$ SU(2) $\underline{4}$	$-\frac{\kappa^2}{3\xi^2} (Q^{abc+} Q_{abd}^+)$ $\times (Q^{efd+} Q_{efc}^+)$	$\Lambda_{ab}^{++} = Q_a^{cd+} Q_{bcd}^+$ $\Lambda_{ij}^{++} = w_i^+ w_j^+ + \frac{\kappa^2}{\xi^2} \mathcal{V}^{++} w_i^- w_j^-$ $\Lambda_{abc}^{++} = w_i^+ Q_{abc}^+ + \frac{2\kappa^2}{3\xi^2} w_i^- Q_{(ab}^{c+} Q_{c)df} Q_e^{df}$